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Preservation of the Range of a Vector Measure under Shortenings of the Domain

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Let $\boldsymbol{\mu}$ be a non-zero, non-atomic vector measure on the measurable space $(\boldsymbol{X}, \boldsymbol{\Omega})$ taking values in $\boldsymbol{R}^n$. Liapounov's convexity theorem gives that the range $R(\mu)$ is convex, an immediate consequence of this is that there exist uncountably many smaller collections $S \subset \Omega$ with preservation of the range, that is $\boldsymbol{R}(\boldsymbol{\mu}) = \boldsymbol{R}(\boldsymbol{\mu}/S)$. In case $\boldsymbol{X}$ is a topological space and $\boldsymbol{\Omega}$ the class of Borel sets, such reductions consisting of open sets or other sets related to continuous functions on $\boldsymbol{X}$ have been obtained by H. Render and H. Stroetmann [15], D. Wulbert ([20], [21], [22] and [23]) and H. G. Kellerer ([8] and [9]). One can see a unified version in [23]. In [6] and [7] José M.Gouweleeuw gave a decomposition of an $\boldsymbol{R}^n$-valued vector measure $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ on the measurable space $(\boldsymbol{X}, \boldsymbol{\Omega})$, where each μ_i is a non-negative real measure on $(\boldsymbol{X}, \boldsymbol{\Omega})$, based on the atoms of the measure $\boldsymbol{\mu}$. She also characterized those $\boldsymbol{\mu}$ which have a convex range. Jan van Mill and André Ran [14] gave various interesting variants and generalizations of the Gouweleeuw decomposition and convexity results. It is the purpose of this paper to apply these decompositions and make attempts to shorten the domain $\boldsymbol{\Omega}$ to various minimal subsets of $\boldsymbol{\Omega}$ called shortenings, which preserve the range of $\boldsymbol{\mu}$. We also make use of the Rényi criteria [15] and the work on interval filling sequences, particularly by Z. Daróczy, A. Járai, I. Katái and T. Szabó ([1], [2], [3]).

Keywords: Vector measure, preservation of the range, shortening of the domain, Gouweleeuw decomposition, van Mill and Ran decomposition, interval filling sequence

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1. Preliminaries and introduction

In this paper $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots \mu_n)$ will denote an $\boldsymbol{R}^n$-valued measure on the measurable space $(\boldsymbol{X}, \boldsymbol{\Omega})$ unless otherwise stated. We shall say that $\boldsymbol{\mu}$ is non-negative if each μ_i is non-negative. The range of $\boldsymbol{\mu}$ will be denoted by $\boldsymbol{R}(\boldsymbol{\mu})$. A measurable set E is said to be an *atom* if $\boldsymbol{\mu}(E) \neq 0$ and, for any measurable $F \subset E$, either $\boldsymbol{\mu}(F) = 0$ or $\boldsymbol{\mu}(E) = \boldsymbol{\mu}(F)$. A non-zero vector measure $\boldsymbol{\mu}$ is said to be *atomic* if it has atoms, otherwise *non-atomic*. It is called *purely atomic* if every set of non-zero measure contains an atom.

For measurable sets E and F in $\boldsymbol{\Omega}$ we say $E \cong F$ if and only if $E \Delta F$ is $\boldsymbol{\mu}$-null. For each atom E, let $[E]$ denote its equivalence class. The number of such equivalence classes is countable. We pick up one and only one set from each equivalence class. Let $\boldsymbol{A}$ be their

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union, then the measure $\boldsymbol{\mu}_A$ defined by $\boldsymbol{\mu}_A(E) = \boldsymbol{\mu}(E \cap A)$ is called the atomic part of $\boldsymbol{\mu}$ and the measure $\boldsymbol{\mu}_N$ defined by $\boldsymbol{\mu}_N(E) = \boldsymbol{\mu}(E \backslash A)$ is called the non-atomic part of $\boldsymbol{\mu}$. We shall say that $\boldsymbol{\mu}$ has atomic part if its atomic part is non-zero and similarly for the non-atomic part.

1.1. Background, Purpose and Techniques

We begin with the background. For each $E \in \boldsymbol{\Omega}$, let $\widetilde{E}$ denote the equivalence class of E with respect to the equivalence relation $\sim$ given by: $A \sim B$ if and only if $\boldsymbol{\mu}(A) = \boldsymbol{\mu}(B)$. Further, let $\widetilde{\boldsymbol{\Omega}} = \{\widetilde{E} : E \in \boldsymbol{\Omega}\}$ be the decomposition of $\boldsymbol{\Omega}$ into mutually disjoint equivalence classes. Let $\Phi : \boldsymbol{R}(\boldsymbol{\mu}) \to \widetilde{\boldsymbol{\Omega}}$ be the mapping defined by $\Phi(\alpha) = \widetilde{E}$ with $\boldsymbol{\mu}(F) = \alpha$ for all $F \in \widetilde{E}$.

By Axiom of Choice, we can have a function $\varphi : \boldsymbol{R}(\boldsymbol{\mu}) \to \boldsymbol{\Omega}$ with $E_\alpha = \varphi(\alpha)$ in $\Phi(\alpha)$ for each α. Then the range $\boldsymbol{R}(\boldsymbol{\mu})$ can be given by $\boldsymbol{R}(\boldsymbol{\mu}) = \{\boldsymbol{\mu}(E_\alpha) : \alpha \in \boldsymbol{R}(\boldsymbol{\mu})\}$. If at least one of $\Phi(\alpha)$ has two or more elements then the range $S = \boldsymbol{\Omega}_\varphi$ of any such φ is a non-trivial *shortening* (*or reduction*) of $\boldsymbol{\Omega}$ with preservation of the range $\boldsymbol{R}(\boldsymbol{\mu})$ of $\boldsymbol{\mu}$. Let $\hat{S}_\mu$, or simply $\hat{S}$, if no confusion can arise, denote the collection of all such shortenings S. The paper is devoted to finding different φ, S and studying how large $\hat{S}$ can be.

In this paper # will denote the cardinality of sets and $\boldsymbol{c}$ the cardinal number of the set $\boldsymbol{R}$ of real numbers. Also N will denote the set of natural numbers and $\aleph_o$ its cardinality. If infinitely many $\Phi(\alpha)$ have two or more elements then $\#\hat{S} \geq \boldsymbol{c} = 2^{\aleph_o}$. Further, if at least one of $\Phi(\alpha)$ is infinite then so is $\#\hat{S}$.

Definition 1.1. A *reduction* of a vector measure $\boldsymbol{\mu}$ is a subset S of $\boldsymbol{\Omega}$ such that $\boldsymbol{R}(\boldsymbol{\mu}) = \boldsymbol{R}(\boldsymbol{\mu}/S)$. A minimal reduction will be called a *shortening*.

Obviously $S = \Omega$ is a trivial reduction and $\hat{S}$ above constitutes all shortenings of $\boldsymbol{\mu}$. Further, for $E \in \boldsymbol{\Omega}$ any reduction S_1 of $\boldsymbol{\mu}/E$ gives rise to a reduction $S = \{A \cup B : A \in S_1, B \subseteq X \backslash E, B \in \boldsymbol{\Omega}\}$ of $\boldsymbol{\mu}$ simply because $\boldsymbol{R}(\boldsymbol{\mu}) = \boldsymbol{R}(\boldsymbol{\mu}/E) + \boldsymbol{R}(\boldsymbol{\mu}/(X \backslash E))$. As a consequence $\#\hat{S}_\mu \geq \#\hat{S}_{\mu/E}$

Liapounov's Theorem states that the range $\boldsymbol{R}(\boldsymbol{\mu})$ of the vector measure $\boldsymbol{\mu}$ is compact and, in case $\boldsymbol{\mu}$ is non-atomic, the range is also convex. One immediate consequence of Liapounov's theorem is the following theorem

Theorem 1.2.

(i) *If μ is a non-negative, non-atomic vector measure, then $\#\hat{S} \geq 2^{\boldsymbol{c}}$.*

(ii) *If μ is a non-atomic vector measure, then $\#\hat{S} \geq 2^{\boldsymbol{c}}$.*

Proof. In view of the remarks above, it is enough to prove (i). Since $\boldsymbol{\mu}$ is non-atomic, by Liapounov's theorem the range $\boldsymbol{R}(\boldsymbol{\mu})$ is convex, and therefore it contains the line segment $[0, \boldsymbol{\mu}(\boldsymbol{X})]$ in $\boldsymbol{R}^n$. Thus it is enough to give the details for a non-negative real measure μ.

Let $0 < x < \mu(\boldsymbol{X})$, then there exists a measurable set $E \in \boldsymbol{\Omega}$ such that $\mu(E) = x$. Now either $\mu(\boldsymbol{X} \setminus E) < \mu(E)$ or $\mu(\boldsymbol{X} \setminus E) > \mu(E)$ or $\mu(\boldsymbol{X} \setminus E) = \mu(E)$.

Case I. When $\mu(\boldsymbol{X} \setminus E) < \mu(E)$.

Let $y = \mu(\boldsymbol{X} \backslash E)$. Since $0 < y < \mu(E)$, and μ is non-atomic, there exists a measurable subsets $D \subset E$ such that $\mu(D) = y$. Let $F = (\boldsymbol{X} \setminus E) \cup (E \setminus D)$. Since $(\boldsymbol{X} \setminus E) \cap (E \setminus D) = \phi$, we have $\mu(F) = \mu(\boldsymbol{X} \setminus E) + \mu(E \setminus D) = y + x - y = x$.

Thus F is a measurable set such that $\mu(E \Delta F) \neq 0$ and $\mu(E) = x = \mu(F)$.

The proof for *Case II*, when $\mu(\boldsymbol{X} \setminus E) > \mu(E)$ is similar.

Case III. When $\mu(\boldsymbol{X} \setminus E) = \mu(E)$.

Take $F = \boldsymbol{X} \setminus E$, then F is a measurable set such that $\mu(E \Delta F) \neq 0$ and $\mu(E) = x = \mu(F)$.

Thus in all the three cases, we can find a measurable set F such that $\mu(E \Delta F) \neq 0$ and $\mu(E) = x = \mu(F)$.

Therefore $\#\Phi(x) \geq 2$ for all $0 < x < \mu(\boldsymbol{X})$. □

Render and Stroetmann [15] initiated research on certain topological versions of Liapounov's Theorem by obtaining reductions consisting of open sets. Kellerer [8] provided a complete characterization for such a possibility: given a non-atomic finite-dimensional vector measure on a topological space, he established a criterion for obtaining its full range by considering open (or closed) sets only. Related work can be seen in a series of papers by Wulbert ([20], [21], [22] and [23]), [23] being special in the sense that in this he proved the unified Liapounov's Theorem and gave reductions for various settings.

This unified Theorem together with Liapounov's original convexity Theorem was motivation for a comprehensive critical survey presented in the M. Phil. dissertation of the first-named author [10] at University of Delhi and [11].

It is the *purpose* of this paper to study shortenings of vector measures with atoms. The *techniques* that we use are:

(i) Rényi criteria [16] and the work on interval filling sequences, particularly by Z. Daróczy, A. Járai, I. Katái and T. Szabó ([1], [2] and [3]);
(ii) Decomposition of certain vector measures given by José Gouweleeuw ([6] and [7]) and later by Jan van Mill and André Ran [14].

For the sake of convenience for applying these techniques, we devote the next two subsections to collecting relevant material on interval filling sequences and decomposition results.

1.2. Interval filling sequences

The following result is due to A. Rényi [16], p. 80, Exercise 48.

Theorem 1.3. *Let μ be a non-negative real measure on $(\boldsymbol{X}, \boldsymbol{\Omega})$, then it contains at most a countable number of atoms. Let μ have atoms $E_1, E_2, E_3, \ldots$ ordered in such a way that $\mu(E_1) \geq \mu(E_2) \geq \mu(E_3) \ldots$. Then $\boldsymbol{R}(\mu)$ is convex if and only if*

- either *μ has non-atomic part and for every n,*

$$\mu(E_n) \leq \frac{1}{2}\left(\mu(\boldsymbol{X}) - \sum_{i=1}^{n-1} \mu(E_i)\right);$$

- or *the number of atoms is infinite and*

$$\text{for every } n \in \boldsymbol{N},\ \mu(E_n) \leq \mu\left(\boldsymbol{X} - \bigcup_{j=1}^{\infty} E_j\right) + \sum_{r=n+1}^{\infty} \mu(E_r).$$

The remaining part of this subsection is based on the papers [1], [2] and [3]. Some related interesting material can also be seen in [16], [17] and other sources.

Let Λ denote the set of those real sequences $\lambda = (\lambda_n)$, for which $\lambda_n > \lambda_{n+1} > 0$ for any $n \in \boldsymbol{N}$ and $L(\lambda) = \sum_{n=1}^{\infty} \lambda_n < \infty$. We call $\lambda \in \Lambda$ *interval filling* in short, *IF*, if for any $x \in [0, L(\lambda)]$ there exists a sequence $\varepsilon = (\varepsilon_n) \in \{0,1\}^{\boldsymbol{N}}$ such that

$$x = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n.$$

We now give a result on interval filling sequences.

Theorem 1.4 ([1], Satz 2.1). *The sequence $\lambda = (\lambda_n) \in \Lambda$ is interval filling if and only if*

$$\lambda_n \leq \sum_{i=n+1}^{\infty} \lambda_i \quad \text{for any } n \in \boldsymbol{N}.$$

From now onwards let $\lambda = (\lambda_n) \in \Lambda$ be an interval filling (in short, IF) sequence.

For $x \in [0, L(\lambda)]$, we define, by induction,

$$\varepsilon_n(x) = 1 \quad \text{if } \sum_{i=1}^{n-1} \varepsilon_i(x)\lambda_i + \lambda_n \leq x,$$

and

$$\varepsilon_n(x) = 0 \quad \text{if } \sum_{i=1}^{n-1} \varepsilon_i(x)\lambda_i + \lambda_n > x;$$

also,

$$\varepsilon_n^*(x) = 1 \quad \text{if } \sum_{i=1}^{n-1} \varepsilon_i^*(x)\lambda_i + \lambda_n < x,$$

and

$$\varepsilon_n^*(x) = 0 \quad \text{if } \sum_{i=1}^{n-1} \varepsilon_i^*(x)\lambda_i + \lambda_n \geq x.$$

Theorem 1.5 ([1], Satz 2.1, Beweis (ii); [2], Theorem 5.1). *For any $x \in [0, L(\lambda)]$, we have*

$$x = \sum_{n=1}^{\infty} \varepsilon_n(x)\lambda_n$$

and

$$x = \sum_{n=1}^{\infty} \varepsilon_n^*(x)\lambda_n.$$

The first expression is known as the *regular expansion* for x and the latter one is known as the *quasiregular expansion* for x.

For $x \in [0, L(\lambda)]$, we call the number x to be *finite* (with respect to λ) if there exists $N \in \boldsymbol{N}$ and $\varepsilon_i = 0$ or 1 for $i = 1, \ldots, N$ such that $x = \sum_{i=1}^{N} \varepsilon_i\lambda_i$; otherwise, we call x non-finite.

Theorem 1.6 ([2], Lemma 5.3 and Lemma 5.4, [1], Satz 3.3).

(*i*) *Let $0 < x \leq L(\lambda)$. Then $\varepsilon_n^*(x) = 1$ for infinitely many values of n. If x is a non-finite number, then $\varepsilon_n^*(x) = \varepsilon_n(x)$ for every $n \in \boldsymbol{N}$.*

(*ii*) *If $0 \leq x < L(\lambda)$ then $\varepsilon_n(x) = 0$ for infinitely many values of n.*

We call a number $x \in [0, L(\lambda)]$ *unique* (with respect to λ) if there exists one and only one sequence $\varepsilon = (\varepsilon_n) \in \{0,1\}^{\boldsymbol{N}}$ for which $x = \sum_{n=1}^{\infty} \varepsilon_n\lambda_n$ otherwise, we call it *non-unique.*

An IF sequence $\lambda = (\lambda_n) \in \Lambda$ is said to be here *locker* (*lax*) in case, for every $N \in \boldsymbol{N}$, the remaining sequence $(\lambda_n)_{n \neq N}$ is IF. An IF sequence is called *ergiebig (plentiful)* if every number x in $(0, L(\lambda))$ is non-unique.

Theorem 1.7 ([1], Satz 2.2). *If an interval filling sequence is lax then it is plentiful.*

We define $q(k)$ to be the unique number in $(1, 2)$ which is the solution of $L\left(\left(\frac{1}{q^n}\right)\right) - 1 = \frac{1}{q^k}$. We note that $q(1) = \frac{1+\sqrt{5}}{2}$.

Theorem 1.8 ([1], Satz 2.3 and Satz 2.4). *For $1 < q < 2$ the sequence $\left(\frac{1}{q^n}\right)$ is interval filling. If $1 < q \leq q(1)$, then it is lax and, therefore, plentiful, whereas for $q(1) < q < 2$, it is not plentiful and also not lax.*

A function F on $[0, L(\lambda)]$ to $\boldsymbol{R}$ is said to be *completely additive* if

$$F\left(\sum_{n=1}^{\infty} \varepsilon_n\lambda_n\right) = \sum_{n=1}^{\infty} \varepsilon_n F(\lambda_n) \quad \text{for all } \varepsilon = (\varepsilon_n) \in \{0,1\}^{\boldsymbol{N}}.$$

There is an extensive study of such maps in the literature. The following theorem may be considered as the last word on them.

Theorem 1.9 ([3], Main Theorem). *If $\lambda = (\lambda_n)$ is IF and $F : [0, L(\lambda)] \to \boldsymbol{R}$ is a completely additive function with respect to λ, then there exists $c \in \boldsymbol{R}$ such that $F(x) = c.x$ for any $x \in [0, L(\lambda)]$.*

Remark. Let $a = (a_n)$ be any sequence in $\boldsymbol{R}$ with $\sum_{n=1}^{\infty} |a_n| < \infty$ and $\boldsymbol{\mu}$ be the purely atomic vector measure on the set $\boldsymbol{N}$ of natural numbers given by $\boldsymbol{\mu}(\{n\}) = (\lambda_n, a_n), n \in \boldsymbol{N}$. Then $\boldsymbol{R(\mu)} = \{(\sum_{n=1}^{\infty} \varepsilon_n\lambda_n, \sum_{n=1}^{\infty} \varepsilon_n a_n)\}$, $\varepsilon = (\varepsilon_n) \in \{0,1\}^{\boldsymbol{N}}$. Then in view of Theorem 1.9 above we have $\boldsymbol{R(\mu)}$ is the graph of some function F on $[0, L(\lambda)]$ if and only if there exists $c \in \boldsymbol{R}$ such that $a_n = c\lambda_n$ for all n and in this case the range $\boldsymbol{R(\mu)}$ is convex (cf. [11]).

1.3. Decomposition of $\boldsymbol{\mu}$ and convexity of $\boldsymbol{R}(\boldsymbol{\mu})$

This section gives decompositions of vector measures.

J. M. Gouweleeuw ([6], [7]) decomposes the vector measure $\boldsymbol{\mu}$ into countably many mutually singular vector measures. Jan van Mill and André Ran [14] develop a similar decomposition in a more general set-up. For the sake of convenience, we give their formulations relevant to our paper.

Theorem 1.10 ([7], Theorem 2.5). *Let $\boldsymbol{\mu}$ be a non-zero, non-negative vector measure on $(\boldsymbol{X}, \boldsymbol{\Omega})$, then there exists a unique (up to order) decomposition*

$$\boldsymbol{\mu} = \sum_i \boldsymbol{\sigma}_i$$

such that the following five conditions are satisfied (there can be a finite or a countably infinite number of $\boldsymbol{\sigma}_i$);

(*i*) $\boldsymbol{\sigma}_i$ *is non-atomic if and only if* $i = 0$;
(*ii*) $\boldsymbol{\sigma}_i \perp \boldsymbol{\sigma}_j$ *for all* $i \neq j$;
(*iii*) *dim span* $\boldsymbol{R}(\boldsymbol{\sigma}_i) = 1$ *for all* $i \geq 1$;
(*iv*) *span* $\boldsymbol{R}(\boldsymbol{\sigma}_i) \cap \operatorname{span} \boldsymbol{R}(\boldsymbol{\sigma}_j) = \{0\}$ *for all all* $i \neq j, i, j \geq 1$;
(*v*) *if dim span* $\boldsymbol{R}(\boldsymbol{\sigma}_0/D) = 1$, *for some* $D \in \boldsymbol{\Omega}$, *then span* $\boldsymbol{R}(\boldsymbol{\sigma}_0/D) \cap$ *span* $\boldsymbol{R}(\boldsymbol{\sigma}_i) = \{0\}$ *for all* $i \geq 1$.

Theorem 1.11 ([7], Theorem 3.5). *Let $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}_i'$s be as in Theorem 1.10 above, then the following statements are equivalent:*

(*i*) $\boldsymbol{R}(\boldsymbol{\mu})$ *is convex,*
(*ii*) $\boldsymbol{R}(\boldsymbol{\sigma}_i)$ *is convex for every* $i \geq 0$.

We define a variant of the convexity condition on the range of a vector measure.

Definition 1.12. A non-zero, non-negative vector measure $\boldsymbol{\mu}$ is said to be *partially convex* if for some $i \geq 0$, $\boldsymbol{\sigma_i}$ has a (non-zero) convex range.

Obviously, if $\boldsymbol{\mu}$ has a non-atomic part then it is partially convex.

Let L be a mertrizable locally convex vector space and let $\boldsymbol{\mu} : \boldsymbol{\Omega} \to \boldsymbol{L}$ be a countably additive measure. As in [14], a measurable set $E \in \boldsymbol{\Omega}$ is said to be *1-dimensional* if $\boldsymbol{\mu}(E) \neq 0$ and there exists a 1-dimensional linear subspace $\boldsymbol{L}_E$ of $\boldsymbol{L}$ such that $\boldsymbol{R}(\boldsymbol{\mu}/E) \subseteq \boldsymbol{L}_E$. In other words, for every measurable subset $F \subset E$ we have $\boldsymbol{\mu}(F) \in \boldsymbol{L}_E$. An atom of $\boldsymbol{\mu}$ is always 1-dimensional. A 1-dimensional set E is called *maximal* if for every measurable set $F \subset \boldsymbol{X}\backslash E$, with $\boldsymbol{\mu}(F) \neq \boldsymbol{0}$, the set $E \cup F$ is not 1-dimensional. Now if E and F are maximally 1-dimensional sets and if $\boldsymbol{\mu}(E \cap F) \neq \boldsymbol{0}$ then $E \Delta F$ is $\boldsymbol{\mu}$-negligible.

Here $A \in \boldsymbol{\Omega}$ is said to be $\boldsymbol{\mu}$-*negligible* if for every measurable subset B of A, we have $\boldsymbol{\mu}(B) = \mathbf{0}$.

There exists at most a countable number of maximally 1-dimensional sets, say A_1, A_2, A_3, $\ldots$, if any. We may assume that $A_i \cap A_j = \phi$ for $i \neq j$. Put $A_0 = \boldsymbol{X} \setminus \cup A_n$. Then $\{A_0, A_1, A_2, A_3, \ldots .\}$ is a partition of X. Every atom is contained in one of the A_n's, $n \geq 1$ and A_0 contains no atoms. We may denote $(\boldsymbol{\mu}/A_n)$ by $\boldsymbol{\rho}_n$. We now give a Theorem by Mill and Ran [14].

Theorem 1.13 ([14], Theorem 3.3). *Let L be a metrizable locally convex vector space and let $\boldsymbol{\mu} : \boldsymbol{\Omega} \to \boldsymbol{L}$ be a bounded countably additive measure. Finally let $A_1, A_2, A_3, \ldots$ be the maximally 1-dimensional sets of $\boldsymbol{\Omega}$. If $\boldsymbol{R}(\boldsymbol{\mu})$ is convex then $\boldsymbol{R}(\boldsymbol{\mu}/A_n)$ is compact and convex for every $n \geq 1$.*

Definition 1.14. Let $\boldsymbol{\mu}$ and $\boldsymbol{\rho}_n$ be here as above. We call $\boldsymbol{\mu}$ *partially convex* if at least one of the $\boldsymbol{\rho}_n$'s has non-zero convex range.

Theorem 1.15 ([14], Theorem 3.4). *Let $\boldsymbol{\mu} : \boldsymbol{\Omega} \to \boldsymbol{R}^n$ be a bounded countably additive measure and let $A_1, A_2, A_3, \ldots$ be the maximally 1-dimensional sets. Then the following statement are equivalent:*

(*i*) $\boldsymbol{R}(\boldsymbol{\mu})$ *is convex,*
(*ii*) $\boldsymbol{R}(\boldsymbol{\mu})$ *is compact and convex,*
(*iii*) *For every $i \geq 1, \boldsymbol{R}(\boldsymbol{\mu}/A_i)$ is convex.*

2. Shortening of domain for a non-negative, real measure $\boldsymbol{\mu}$ with convex range

This section gives the shortening techniques and estimates for non-zero, non-negative real measures μ on a measurable space $(\boldsymbol{X}, \boldsymbol{\Omega})$ with convex range $[0, \mu(\boldsymbol{X})]$. We begin with the case when the measure is purely atomic.

2.1. When $\boldsymbol{\mu}$ is purely atomic

Theorem 2.1. *If the purely atomic and non-negative real measure μ has convex range, then $\#\hat{S} \geq \boldsymbol{c}$.*

Proof. Since the range of μ is convex, there have to be countably infinitely many atoms. Let us enumerate them as $E_1, E_2, E_3, \ldots$ Since $\sum_{n=1}^{\infty} \mu(E_n)$ is finite, we have $\mu(E_n) \to 0$, so we may assume $\mu(E_n) \geq \mu(E_{n+1}) > 0$ for all $n \in \boldsymbol{N}$. We first consider the case that they are all distinct. For $n \in \boldsymbol{N}$, let $\lambda_n = \mu(E_n)$. Since the range is convex, we have that the sequence (λ_n) is an interval filling sequence. Then

$$\boldsymbol{R}(\mu) = \left[0, \sum_{n=1}^{\infty} \mu(E_n)\right] = \left[0, \sum_{n=1}^{\infty} \lambda_n\right]$$

As the sequence $\lambda = (\lambda_n)$ is an interval filling sequence, if $x \in \boldsymbol{R}(\mu)$ then there exists $\varepsilon = (\varepsilon_n) \in \{0, 1\}^N$ such that $x = \sum_{n=1}^{\infty} \varepsilon_n \mu(E_n) = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n = \mu(E_\varepsilon)$, where E_ε is

$\varepsilon^{-1}\{1\}$. So, for any x in $\boldsymbol{R}(\mu)$,

$$\Phi(x) = \left\{ E_\varepsilon \text{ with } \varepsilon = (\varepsilon_n) \in \{0,1\}^N \text{ and } x = \sum_{n=1}^{\infty} \varepsilon_n \mu(E_n) \right\}.$$

Now suppose that $0 < x \in \boldsymbol{R}(\mu)$ is finite (with respect to $\lambda = (\lambda_n)$). Then there exists $N \in \boldsymbol{N}$ such that

$$x = \sum_{i=1}^{N} \varepsilon_i(x)\lambda_i.$$

From Theorem 1.5, we can write

$$x = \sum_{n=1}^{\infty} \varepsilon_n^*(x)\lambda_n$$

Let K, $L \subset \boldsymbol{N}$ be given by $i \in K$ if and only if $\varepsilon_i(x) = 1$ and $j \in L$ if and only if $\varepsilon_j^*(x) = 1$. Since x is finite, by Theorem 1.6, $K \neq L$. Therefore,

$$\sum_{i \in K} \varepsilon_i(x)\lambda_i = x = \sum_{j \in L} \varepsilon_j^*(x)\lambda_j$$

so

$$\mu\left(\bigcup_{i \in K} E_i\right) = x = \mu\left(\bigcup_{j \in L} E_j\right).$$

Let $E = \bigcup_{i \in K} E_i$ and $F = \bigcup_{j \in L} E_j$, then $\mu(E \Delta F) \neq 0$ and $E, F \in \Phi(x)$.

Therefore $\#\Phi(x) \geq 2$ for all finite $x \in \boldsymbol{R}(\mu)$. Finally the cardinality of such numbers is countably infinite. So $\#\hat{S} \geq 2^{\aleph_o} = c$.

A little modification gives the same estimate if all but finitely many $\mu(E_n)$ are distinct. In the remaining case $\#\Phi(n) \geq 2$ for infinitely many $n \in \boldsymbol{N}$. So we may refer to the remarks on the background in the begining of Section 1.1 to obtain the same estimate. □

Theorem 2.2. *Let μ be as in Theorem 2.1 above with $(\mu(E_n))$ in the proof, all distinct. If the sequence $(\lambda_n) = (\mu(E_n))$ has c-many non-unique numbers, then $\#\hat{S} \geq 2^{\mathfrak{c}}$; in particular, it is so if the sequence (λ_n) is plentiful.*

Proof. The proof is obvious from the discussion in §1.2.

2.2. When $\boldsymbol{\mu}$ has both atomic and non-atomic parts

Theorem 2.3. *If the non-negative, real measure μ has convex range and a non-atomic part, then $\#\hat{S} \geq 2^{\mathfrak{c}}$.*

Proof. It follows from Theorem 1.2 because μ has a non-atomic part.

We give another explicit method to show the same fact when μ also has infinitely many atoms say $E_1, E_2, E_3, \ldots$ with $\mu(E_n) > 0$, $\mu(E_n) \geq \mu(E_{n+1}) > 0$. Let $C = \bigcup_{i=1}^{\infty} E_i$.

Let $\mu(X \setminus C) = \alpha$. Let, for $n \in \boldsymbol{N}$, $L_n = \sum_{r=n+1}^{\infty} \mu(E_r)$. Since the range $\boldsymbol{R}(\mu)$ is convex, we have by Rényi Criteria R1, $\mu(E_n) \leq \alpha + L_n$ for $n \in \boldsymbol{N}$. Since $L_n \to 0$, there exists a $j \in \boldsymbol{N}$ such that $L_n < \alpha$ for all $n \geq j$. Fix any $n \geq j$. Let p be such that $\mu\left(\bigcup_{r=n+1}^{\infty} E_r\right) = \sum_{r=n+1}^{\infty} \mu(E_r) = L_n < p < \alpha$. Now $p < \alpha$, therefore there exists a non-atomic set G such that $\mu(G) = p$. Also $0 < p - L_n < \alpha$, so there exists a non-atomic set H such that $\mu(H) = p - L_n$. Thus the set $K = H \bigcup \left(\bigcup_{r=n+1}^{\infty} E_r\right)$ is measurable and $\mu(K) = p = \mu(G)$ and $\mu(G \Delta K) \neq 0$.

Therefore $\#\Phi(p) \geq 2$ for all $L_n < p < \alpha$. $\square$

3. Shortening of the domain of a partially convex vector measure $\boldsymbol{\mu}$

In this Section we utilize decomposition techniques to obtain estimates for shortenings.

Remark 3.1. Let $\boldsymbol{\mu}$ be a non-negative, non-zero vector measure defined on the measurable space$(\boldsymbol{X}, \boldsymbol{\Omega})$ and let $\boldsymbol{\sigma}_0$, $\boldsymbol{\sigma}_i$ for $i = 1, 2, 3.....$ be as in Theorem 1.10. Suppose $\boldsymbol{\mu}$ has convex range. By Theorem 1.11 each of the vector measures $\boldsymbol{\sigma}_i$ has convex range. In case the measure $\boldsymbol{\sigma}_0$ is present in decomposition, its domain can be reduced by the techniques used in Theorem 1.2 and the domain of measures $\boldsymbol{\sigma}_i$ for $i = 1, 2, 3.....$, if present, each with one-dimensional range, can be reduced by the various techniques given in Section 2. The components $\boldsymbol{\sigma}_i$'s have disjoint support. So for each i, $\boldsymbol{R}(\boldsymbol{\sigma}_i) \subset \boldsymbol{R}(\boldsymbol{\mu})$. Thus, each reduction of the components $\boldsymbol{\sigma}_i$ gives rise to a reduction for the measure $\boldsymbol{\mu}$. Further, we may combine them to give more reductions.

Theorem 3.2. *Let $\boldsymbol{\mu}$ be a non-negative partially convex vector measure defined on the measurable space$(\boldsymbol{X}, \boldsymbol{\Omega})$, then it has at least c many shortenings.*

Proof. By Theorem 1.10, $\boldsymbol{\mu}$ can be decomposed as

$$\boldsymbol{\mu} = \sum_i \boldsymbol{\sigma}_i.$$

Since $\boldsymbol{\mu}$ is partially convex, there is an $i_o \geq 0$ with range $\boldsymbol{R}(\boldsymbol{\sigma}_{i_o})$ convex. By Theorems 1.2, 2.1, 2.3 and their proofs the measure $\boldsymbol{\sigma}_{i_o}$ has at least c-many shortenings. Since $\boldsymbol{R}(\boldsymbol{\mu}) = \sum_i \boldsymbol{R}(\boldsymbol{\sigma}_{i_o})$, the discussion in the background gives the desired result. $\square$

Remark 3.3. For real measure μ on $(\boldsymbol{X}, \boldsymbol{\Omega})$, let μ^+ and μ^- be the Jordan decomposition of μ, that is μ^+ and μ^- are mutually singular non-negative measures on $(\boldsymbol{X}, \boldsymbol{\Omega})$ and $\mu = \mu^+ - \mu^-$. Let Y and Z be the disjoint supports of μ^+ and μ^- respectively with $Y \cup Z = X$. Let $|\mu|$ be the total variation measure of μ. Then $\boldsymbol{R}(\mu) = \boldsymbol{R}(\mu^+) - \boldsymbol{R}(\mu^-)$. $|\mu| = \mu^+ + \mu^-$, $\boldsymbol{R}(|\mu|) = \boldsymbol{R}(\mu^+) + \boldsymbol{R}(\mu^-)$. Because $\boldsymbol{R}(\mu^-)$ is symmetric about $\frac{1}{2}\mu^-(X)$, we have $\boldsymbol{R}(|\mu|) - \mu^-(X) = \boldsymbol{R}(\mu)$. In particular, $\boldsymbol{R}(\mu)$ is convex if and only if $\boldsymbol{R}(|\mu|)$ is so. Further, if Y_j, Z_j, $j = 1, 2$ be in Ω such that $Y_j \subset Y$ and $Z_j \subset Z$ then $|\mu|(Y_1 \cup Z_1) = |\mu|(Y_2 \cup Z_2)$ if and only if $\mu(Y_1 \cup Z_2) = \mu(Y_2 \cup Z_1)$.

This *swap*-relationship gives the following useful estimates for the sets Φ for μ and $|\mu|$.

Swap Relations. The following are equivalent:

(i) $\Phi(\alpha)$ for μ is infinite for some α or $\Phi(\beta)$ for μ has cardinality bigger than or equal to 2 for infinitely many β,

(ii) $\Phi(\gamma)$ for $|\mu|$ is infinite for some γ or $\Phi(\delta)$ for $|\mu|$ has cardinality bigger than or equal to 2 for infinitely many δ.

We now give a result which combines this well with 1.13 and 1.15. We follow the notation and terminology of Subsection 1.3.

Theorem 3.4. *Let $\boldsymbol{\mu}$ be a partially convex vector measure as in Theorem* 1.13 *defined on the measurable space* $(\boldsymbol{X}, \boldsymbol{\Omega})$. *Then it has infinitely many shortenings.*

Proof. There is an n for which $\boldsymbol{\rho}_n$ has convex range. If $n \geq 1$, then we may regard $\boldsymbol{\rho}_n$ as a real measure ρ_n with convex range. Then $|\rho_n|$ is a non-negative real measure with a convex range. By Theorem 2.1 and 2.3, ρ_n satisfies (ii) of *swap* relations in with ρ_n in place of μ. So (i) is satisfied with ρ_n in place of μ. So $\#\hat{S}_{\rho_n}$ is infinite. On the other hand, if n is zero then ρ_n is non-atomic and, therefore by arguments as in the proof of Theorem 1.2, $\#\hat{S}_{\rho_n}$ is infinite (in fact, $\geq 2^c$ in this case). This, in view of background discussion, gives that mu has infinitely many shortenings. $\square$

Remark 3.5. An easy way to improve these lower estimates to at least c-many shortenings would be to use shortenings for ρ_n^+ or ρ_n^-, if any of them has a convex range. But Example 4.1 illustrates that $\boldsymbol{R}(\mu)$ can be convex even when none of $\boldsymbol{R}(\mu^+)$ and $\boldsymbol{R}(\mu^-)$ is convex! We note that $|\mu|$ has at least c-many shortenings, simply because $|\mu|$ is purely atomic with infinitely many pairs (2n, 2n-1) with common value $\frac{2}{3^n}$ at both the atoms in the pair. The interesting part is that μ has at least 2^c-many shortenings! The general phenomena working here is that $\boldsymbol{R}(\mu^+)$ and $\boldsymbol{R}(\mu^-)$ have c-many points in common and, in fact, for c-many points α, $\boldsymbol{R}(\mu^+)$ and $\boldsymbol{R}(\mu^-) + \alpha$ have c-many points in common. This gives $\#\Phi(\alpha) \geq c$ for c-many α.

4. Examples

Section 4 is to illustrate the limitation of our results.

We now give different examples to illustrate the limitations in shortening of the domain with preservation of the range of a general vector measure. They are based on the well-known counterexamples in the study of Liapounov convexity Theorems ([12], [13], [5], [19]).

4.1. Example of a real measure having non-shortenable domain with preservation of the range

We consider the non-negative measure μ defined on $(\boldsymbol{N}, P(\boldsymbol{N}))$ given by

$$\mu = \sum_{n=1}^{\infty} \frac{2}{3^n} \delta_n.$$

Then $\boldsymbol{R}(\mu) = C$, the Cantor Ternary set. Since each point of Cantor set C has unique ternary representation, the domain $P(\boldsymbol{N})$ cannot be shortened.

We note the interesting fact that the related purely atomic real measure μ on $(\boldsymbol{N}, P(\boldsymbol{N}))$ for which, for n in $\boldsymbol{N}$, $\mu\{2n\} = \frac{2}{3^n}$ and $\mu\{2n-1\} = -\frac{2}{3^n}$ has convex range $[-1, 1]$ and $\#\hat{S} \geq 2^c$.

4.2. Example of a vector measure having non-shortenable domain with preservation of the range

Let μ be a non-negative non-atomic measure on the measurable space $(\boldsymbol{X}, \boldsymbol{\Omega})$. Let $\boldsymbol{\tau}$ be the $\boldsymbol{L}^1(\mu)$-valued vector measure on $(\boldsymbol{X}, \boldsymbol{\Omega})$ defined by $\boldsymbol{\tau}(E) = \chi_E$, $E \in \boldsymbol{\Omega}$. The domain of $\boldsymbol{\tau}$ cannot be shortened.

4.3. Example of a vector measure having non-shortenable domain with preservation of the range with values in any infinite dimensional Banach space

Let Y be an infinite dimensional Banach space and $(\boldsymbol{X}, \boldsymbol{\Omega}, \mu)$ be a probability measure space possessing a bounded sequence $(\varphi_n)_{n=1}^{\infty}$ in the dual of $\boldsymbol{L}^1(\mu)$ that separates points of $\boldsymbol{L}^1(\mu)$. For instance, for $\boldsymbol{X} = [0,1]$; $\boldsymbol{\Omega}$, the class of Lebesgue measurable subsets of [0,1] and μ, the Lebesgue measure, the characteristic functions of intervals with dyadic fractions as end points serve as such a sequence. We may use the relevant preliminaries from ([4], [5], [12], [19]) to develop the required vector measure as indicated below. Let $(x_n)_{n=1}^{\infty}$ be a bounded ω-linearly independent sequence in Y and let $\beta = (\beta_n)_{n=1}^{\infty}$ be a sequence in l_1 such that $\beta_n \neq 0$ for each $n = 1, 2, \ldots$. Define mappings $\boldsymbol{\tau} : \boldsymbol{\Omega} \to \boldsymbol{L}^1(\mu)$ and $\boldsymbol{v}_\beta : \boldsymbol{L}^1(\mu) \to E$ by

$$\boldsymbol{\tau}(E) = \chi_E, E \in \boldsymbol{\Omega} \quad \text{and} \quad \boldsymbol{v}_\beta(g) = \sum_{n=1}^{\infty} \beta_n \varphi_n(g) x_n \quad \text{for } g \in \boldsymbol{L}^1(\mu).$$

The mapping $\boldsymbol{v} = \boldsymbol{v}_\beta \circ \boldsymbol{\tau}$ is a Y-valued vector measure on $(\boldsymbol{X}, \boldsymbol{\Omega})$.

As $\boldsymbol{v}_\beta$ is injective, the domain of $\boldsymbol{v}$ cannot be shortened with preservation of the range.

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