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Banach Spaces with an Infinite Number of Smooth Faces in their Unit Ball

F. J. García-Pacheco
Department of Mathematical Sciences,
Kent State University, Kent, OH 44242, USA
fgarcia@math.kent.edu

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In this paper we study Banach spaces having smooth faces in their unit ball. In particular, we show that if the unit ball of a finite dimensional Banach space has an infinite number of smooth faces then their interiors relative to the unit sphere approach the empty set in a certain way. We also show that this situation does not hold in infinite dimensions since we prove that every infinite dimensional Banach space can be equivalently renormed to have infinitely many smooth faces with interior relative to the unit sphere of the same "size". This fact characterizes having infinite algebraic dimension.

Keywords: Smooth face, interior relative to the unit sphere, Hausdorff metric

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1. Introduction and background

Remember that a subset C of the unit sphere S_X of a real Banach space X is said to be a face of the unit ball B_X of X if C is convex and verifies the extremal condition: If $x, y \in \mathsf{B}_X$ and $\alpha \in (0,1)$ are so that $\alpha x + (1-\alpha)x \in C$ then $x, y \in C$. Remember also that C is said to be a smooth face of B_X if C is a face of B_X with non-empty interior relative to S_X. Among others, it is well known that if C is a smooth face then its interior relative to unit sphere is dense in C, and if C and D are different smooth faces then their interior relative to the unit sphere are disjoint. As a consequence, the unit ball of a separable Banach space cannot have uncountably many smooth faces. We refer the reader to [2] for a wider perspective about the concept of face, and to [1] for a wider perspective about the concept of smooth face including topological applications to the set of norm-attaining functionals. On the other hand, in [4] it is proved that, given any real Banach space X, any norm-attaining functional $f \in \mathsf{S}_{X^*}$ on X, and any real number $t \in (0,1)$, the set $\mathsf{B}_X \cap f^{-1}([-t,t])$ defines a new equivalent norm on X. This new equivalent norm has the property that its unit ball has at least two smooth faces: $\mathsf{B}_X \cap f^{-1}(\{-t\})$ and $\mathsf{B}_X \cap f^{-1}(\{t\})$. In the same paper it is also proved that the unit ball of any complex Banach space is always free of smooth faces.

Along this manuscript we will make use of the following standard notation. If X denotes a Banach space then B_X, S_X, and U_X denote respectively the closed unit ball, the unit sphere, and the open unit ball. When we refer to usual balls or spheres of center x and radius r we will add (x,r) to the above expressions. Finally, if X denotes a topological space and M is any subset of it, then $\mathrm{cl}(M)$, $\mathrm{bd}(M)$, and $\mathrm{int}(M)$ denote respectively the closure of M, the boundary of M, and the interior of M.

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To conclude the introduction, we will make a brief review on the Hausdorff metric (see [3]). Let S be a metric space. The Hausdorff distance between two bounded closed subsets C and D of S is defined as

$$\mathrm{d}_{\mathcal{H}}(C,D) = \sup\left\{\mathrm{dist}(c,D), \mathrm{dist}(C,d) : c \in C, d \in D\right\}.$$

Among other properties, this metric verifies the following:

(1) The set of bounded closed subsets of S endowed with the Hausdorff metric is complete if and only if S is complete.
(2) The set of bounded closed subsets of S endowed with the Hausdorff metric is compact if and only if S is compact.
(3) If S is a normed space, then the set of bounded closed convex subsets of S is $\mathrm{d}_{\mathcal{H}}$-closed.

2. Main results

In this section we will state and prove the main results of this paper. We will begin by the finite dimensional case.

Remark 2.1. Let X be a finite dimensional real Banach space. If $(C_n)_{n\in\mathbb{N}}$ is a sequence of smooth faces of B_X, then there exists a subsequence of it converging in the Hausdorff metric. Indeed, we have that S_X is a compact metric space, therefore the set of bounded closed convex subsets of S_X endowed with the Hausdorff metric is compact.

Theorem 2.2. *Let X be a real Banach space. If $(C_n)_{n\in\mathbb{N}}$ is an infinite sequence of smooth faces converging in the Hausdorff metric to the bounded closed convex subset C of S_X, then* $\mathrm{int}_{\mathsf{S}_X}(C) = \varnothing$.

Proof. Assume that $\mathrm{int}_{\mathsf{S}_X}(C) \neq \varnothing$ and take any $c \in \mathrm{int}_{\mathsf{S}_X}(C)$. Since $\mathrm{d}_{\mathcal{H}}(C, C_n)$ tends to 0 as n goes to ∞, we can consider a sequence $(c_n)_{n\in\mathbb{N}}$ converging to c such that $c_n \in C_n$ for every $n \in \mathbb{N}$. By taking into account that $\mathrm{cl}(\mathrm{int}_{\mathsf{S}_X}(C_n)) = C_n$ for all $n \in \mathbb{N}$, we can suppose without any loss of generality that $c_n \in \mathrm{int}_{\mathsf{S}_X}(C_n)$ for all $n \in \mathbb{N}$. Then, there is $n_0 \in \mathbb{N}$ such that $c_n \in \mathrm{int}_{\mathsf{S}_X}(C)$ for all $n \geq n_0$, which means that $C_n = C$ for all $n \geq n_0$. This contradicts the fact that the sequence $(C_n)_{n\in\mathbb{N}}$ is infinite. $\square$

In accordance to the Remark 2.1 and the Theorem 2.2, we deduce that, in the unit sphere of finite dimensional real Banach spaces, infinte sequences of smooth faces approach the empty set in a certain way. To finish, we will take care of the infinite dimensional case, where a complete different situation occurs.

Theorem 2.3. *Let X be an infinite dimensional real Banach space. Then, for every $t \in (0,1)$ there exists an equivalent renorming of X whose unit ball possesses an infinite sequence $(C_n)_{n\in\mathbb{N}}$ of smooth faces such that:*

(1) For every $n \neq m \in \mathbb{N}$, $\mathrm{d}_{\mathcal{H}}(C_n, C_m) \geq t$.
(2) For every $n \in \mathbb{N}$, $\mathrm{diam}(C_n) \geq \delta$, for some $\delta > 0$.

Proof. Let $(e_n)_{n\in\mathbb{N}} \subset \mathsf{S}_X$ be a basic sequence in X, and let $(e_n^*)_{n\in\mathbb{N}} \subset \mathsf{S}_{X^*}$ be its dual basis sequence. Let $0 < t < 1$. We will show that

$$\mathcal{B} := \mathsf{B}_X \cap \bigcap_{n=1}^{\infty} (e_n^*)^{-1}([-t,t])$$

is the new unit ball of the new equivalent norm that we are looking for. We will follow several steps.

(1) The set $\mathcal{B}$ is the unit ball of an equivalent norm $\|\cdot\|_{\mathcal{B}}$ on X such that $\|\cdot\| \leq \|\cdot\|_{\mathcal{B}}$. Indeed, $\mathcal{B}$ is a closed absolutely convex set with non-empty interior. Furthermore, $\mathcal{B} \subset \mathsf{B}_X$ and hence $\|\cdot\| \leq \|\cdot\|_{\mathcal{B}}$. As a consequence, for every $n \in \mathbb{N}$ we have that $\|e_n^*\|_{\mathcal{B}} = t$ and thus $(e_n^*)^{-1}(\{t\}) \cap \mathrm{bd}(\mathcal{B})$ defines a face of $\mathcal{B}$.

(2) Let $\rho > 0$ such that $\rho < t$ and $t + \rho < 1$. For every $n \in \mathbb{N}$ we have that $\mathsf{B}_X(te_n, \rho) \cap \mathrm{bd}(\mathcal{B}) \subseteq (e_n^*)^{-1}(\{t\}) \cap \mathrm{bd}(\mathcal{B})$. Indeed, let $x \in \mathsf{B}_X(te_n, \rho) \cap \mathrm{bd}(\mathcal{B})$ and suppose that $e_n^*(x) \neq t$. Let us see first that $|e_n^*(x)| < t$ for which it will be enough to show that $e_n^*(x) \neq -t$. Otherwise,

$$2t = |e_n^*(x - te_n)| \leq \|x - te_n\| \leq \rho < t < 2t,$$

which is impossible. Now, let us take $\gamma > 0$ such that $t + \rho + \gamma < 1$, $|e_n^*(x)| + \gamma < t$, and $\rho + \gamma < t$. We will show that $\mathsf{B}_X(x, \gamma) \subseteq \mathcal{B}$ which will contradict the fact that $x \in \mathrm{bd}(\mathcal{B})$. So, let $y \in \mathsf{B}_X(x, \gamma)$. On the one hand, we have that

$$\begin{aligned} \|y\| &\leq \|te_n\| + \|te_n - x\| + \|x - y\| \\ &\leq t + \rho + \gamma \\ &< 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} |e_n^*(y)| &\leq |e_n^*(x)| + |e_n^*(y - x)| \\ &\leq |e_n^*(x)| + \|y - x\| \\ &\leq |e_n^*(x)| + \gamma \\ &< t. \end{aligned}$$

On the last hand, for every $m \neq n$,

$$\begin{aligned} |e_m^*(y)| &\leq |e_m^*(x)| + |e_m^*(y - x)| \\ &\leq |e_m^*(x - te_n)| + \|y - x\| \\ &\leq \|x - te_n\| + \gamma \\ &\leq \rho + \gamma \\ &< t. \end{aligned}$$

Finally, we have that $y \in \mathcal{B}$.

(3) For every $n \in \mathbb{N}$ we have that $\mathrm{diam}_{\mathcal{B}}((e_n^*)^{-1}(\{t\}) \cap \mathrm{bd}(\mathcal{B})) \geq \delta := 2\rho$. Indeed, let us see that $te_n - \rho e_m, te_n + \rho e_m \in \mathsf{B}_X(te_n, \rho) \cap \mathrm{bd}(\mathcal{B})$ where $n \neq m \in \mathbb{N}$. Obviously, $te_n - \rho e_m, te_n + \rho e_m \in \mathsf{B}_X(te_n, \rho)$, and since $t + \rho < 1$ and $0 < \rho < t$ we have that $te_n - \rho e_m, te_n + \rho e_m \in \mathcal{B}$. Furthermore, since

$$\mathrm{int}(\mathcal{B}) \subseteq \mathsf{U}_X \cap \bigcap_{n=1}^{\infty} (e_n^*)^{-1}((-t,t)),$$

we deduce that $te_n - \rho e_m, te_n + \rho e_m \in \mathrm{bd}(\mathcal{B})$. Next,

$$\begin{aligned} \|(te_n - \rho e_m) - (te_n + \rho e_m)\|_{\mathcal{B}} &\geq \|(te_n - \rho e_m) - (te_n + \rho e_m)\| \\ &= 2\rho. \end{aligned}$$

(4) For every $n \neq m \in \mathbb{N}$, the Hausdorff distance between $(e_n^*)^{-1}(\{t\}) \cap \mathrm{bd}(\mathcal{B})$ and $(e_m^*)^{-1}(\{t\}) \cap \mathrm{bd}(\mathcal{B})$ is greater than or equal to t. Indeed, for every $x \in (e_m^*)^{-1}(\{t\}) \cap \mathrm{bd}(\mathcal{B})$, we have that

$$\|te_n - x\| \geq |e_m^*(te_n - x)| = |e_m^*(x)| = t,$$

therefore

$$\begin{aligned} \mathrm{d}_{\mathcal{H}}((e_n^*)^{-1}(\{t\}) \cap \mathrm{bd}(\mathcal{B}), (e_m^*)^{-1}(\{t\}) \cap \mathrm{bd}(\mathcal{B})) &\geq \mathrm{dist}(te_n, (e_m^*)^{-1}(\{t\}) \cap \mathrm{bd}(\mathcal{B})) \\ &\geq t. \end{aligned}$$

To finalize, we take, for every $n \in \mathbb{N}$, $C_n := (e_n^*)^{-1}(\{t\}) \cap \mathrm{bd}(\mathcal{B})$. □

Finally, by taking into account the Remark 2.1 and the Theorem 2.3, we consequently obtain the following characterization of having infinite algebraic dimension.

Corollary 2.4. *Let X be a real Banach space. The following conditions are equivalent:*

(1) *The space X is infinite dimensional.*

(2) *The space X can be equivalently renormed to have in its unit sphere an infinite and $\mathrm{d}_{\mathcal{H}}$-closed set of smooth faces.*

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