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Local Strong Convexity and Local Lipschitz Continuity of the Gradient of Convex Functions

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Given a pair of convex conjugate functions f and f^*, we investigate the relationship between local Lipschitz continuity of ∇f and local strong convexity properties of f^*.

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1. Introduction

It is known that differentiability of a convex function is closely related to strict convexity of its conjugate. Similarly, Lipschitz continuity of the gradient of a differentiable convex function is related to strong convexity of the conjugate. (We make these statements precise in Section 2.)

To our knowledge, an equivalent characterization of local Lipschitz continuity of the gradient of a differentiable convex function in terms of a property of the conjugate function has not been studied. Carrying out such a study is the purpose of this paper.

It will turn out that the sought after property is not quite what a natural guess suggests, i.e., it is not "local strong convexity" defined via a local combination of definitions of strict and strong convexity. Rather, it will require that a "strong" subdifferential inequality hold on all compact – but not necessarily convex – subsets of the domain of the subdifferential of the conjugate function on which that subdifferential has a bounded selection. However, such property will reduce to the expected "local strong convexity" in many cases.

2. Background

Let $f : I\!R^n \to \overline{I\!R}$ be a proper (never $-\infty$ and finite somewhere), lower semicontinuous (lsc), and convex function. The *effective domain* of f is the set $\operatorname{dom} f = \{x \in I\!R^n \mid f(x) < \infty\}$. Given f, its *conjugate function* is

$$f^*(y) = \sup_{x \in R^n} \{y \cdot x - f(x)\}.$$

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It is also a proper, lsc, and convex function, and $(f^*)^* = f$. We have $f^*(y) < \infty$ for all $y \in I\!R^n$ if and only if f is *coercive*: $\lim_{|x|\to\infty} f(x)/|x| = \infty$.

The *subdifferential* of f at x, denoted $\partial f(x)$, is the set of all *subgradients* y of f at x:

$$\partial f(x) = \{y \in I\!R^n \mid f(x') \geq f(x) + y \cdot (x' - x) \ \forall x' \in I\!R^n\}$$

The domain of ∂f is the set $\operatorname{dom} \partial f = \{x \in I\!R^n \mid \partial f(x) \neq \emptyset\}$. In general, $\operatorname{dom} \partial f$ need not be convex. The subdifferential mappings of f and f^* are inverses of one another: $y \in \partial f(x)$ if and only if $x \in \partial f^*(y)$. Either of these inclusions is also equivalent to $f(x) + f^*(y) = x \cdot y$.

The subdifferential mapping is a *monotone mapping*: for all $x_1, x_2 \in \operatorname{dom} \partial f$, all $y_1 \in \partial f(x_1)$, $y_2 \in \partial f(x_2)$, $(x_1 - x_2) \cdot (y_1 - y_2) \geq 0$. It is *strictly monotone* if $(x_1 - x_2) \cdot (y_1 - y_2) > 0$ as long as $x_1 \neq x_2$ and *strongly monotone* with constant $\sigma > 0$ if $(x_1 - x_2) \cdot (y_1 - y_2) \geq \sigma |x_1 - x_2|^2$.

The function f is *essentially differentiable* if $C = \operatorname{int} \operatorname{dom} f$ is nonempty, f is differentiable on C, and $\lim_{i\to\infty} |\nabla f(x_i)| = \infty$ for any sequence $\{x_i\}_{i=1}^\infty$ converging to a boundary point of C. The function f is *essentially strictly convex* if f is strictly convex on every convex subset of $\operatorname{dom} \partial f$; that is, if for each convex $S \subset \operatorname{dom} \partial f$, we have $(1-\lambda) f(x_1) + \lambda f(x_2) > f((1-\lambda)x_1 + \lambda x_2)$ for all $\lambda \in (0,1)$, all $x_1, x_2 \in S$ with $x_1 \neq x_2$. The following are equivalent:

- f is essentially differentiable;
- ∂f is single-valued on $\operatorname{dom} \partial f$;
- ∂f^* is strictly monotone;
- f^* is essentially strictly convex.

The function f is *strongly convex* if there is a constant $\sigma > 0$ such that

$$(1-\lambda) f(x_1) + \lambda f(x_2) \geq f((1-\lambda)x_1 + \lambda x_2) + \frac{1}{2}\sigma\lambda(1-\lambda)|x_1 - x_2|^2 \tag{1}$$

for all $\lambda \in [0,1]$, all $x_1, x_2 \in I\!R^n$. A manipulation of (1) shows that strong convexity of f with constant σ is equivalent to $f - \frac{1}{2}\sigma|\cdot|^2$ being convex. The following are equivalent:

- f is differentiable and ∇f is Lipschitz continuous with constant $1/\sigma$;
- f^* is strongly convex with constant σ;
- ∂f^* is strongly monotone with constant σ.

Finally, if f is smooth, then strong convexity of f with constant σ is equivalent to the eigenvalues of the Hessian Hf of f (that are real and nonnegative, by convexity of f) being bounded below by σ.

For details of the facts summarized above see [1] and [2].

3. Examples

In what follows, we will say that $f : I\!R^n \to \overline{I\!R}$ is *essentially locally strongly convex* if for any compact and convex $K \subset \operatorname{dom} \partial f$, f is strongly convex on K, in the sense that there exists $\sigma > 0$ such that (1) is satisfied for all $x_1, x_2 \in K$ and all $\lambda \in (0,1)$.

Example 3.1. Consider $f(x) = (1/p)|x|^p$ and $f^*(y) = (1/q)|y|^q$ where $1/p + 1/q = 1$ and $p, q > 1$. Then ∇f is locally Lipschitz continuous if $p \geq 2$ while f^* is essentially locally strongly convex (one could omit the term "essentially" here as $\operatorname{dom} f^* = \mathbb{R}^n$) if $q \leq 2$. Note that ∇f is globally Lipschitz continuous only when $p = 2$ and then also $q = 2$, $f^* = f$, f^* is strongly convex, and in fact f^* is (globally) strongly convex only in this case.

Example 3.2. Consider $f(x) = e^x$ and

$$f^*(y) = \begin{cases} y \log y - y & \text{if } y > 0 \\ 0 & \text{if } y = 0 \\ \infty & \text{if } y < 0. \end{cases}$$

Then f is differentiable, what is reflected in f^* being essentially strictly convex. Furthermore, ∇f is is locally Lipschitz continuous, while f^* is essentially strongly convex. (This can be seen by looking at the second derivative of f^*, which is bounded below by a positive number on each compact subset of $\operatorname{dom} \partial f^* = (0, \infty)$.)

Similarly, f is strictly convex, what is reflected in f^* being essentially differentiable. (Note that $|\nabla f^*(y)| = -\log y$ blows up as $y \searrow 0$ and $\partial f^*(0) = \emptyset$.) Furthermore, f is locally strongly convex (which can be seen by noting that the second derivative of f is locally bounded below by positive numbers), while ∇f^* is locally Lipschitz continuous on $\operatorname{int} \operatorname{dom} f = \operatorname{dom} \partial f^* = (0, \infty)$.

Note that in the example above, ∇f is locally Lipschitz continuous on $\mathbb{R}^n$ but f^* is not strongly convex. Such a situation can occur even if $\operatorname{dom} f^*$ is compact, as the next example shows.

Example 3.3. Consider a convex function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ that is even, differentiable, with f' piecewise differentiable, and such that, for $n = 1, 2, \ldots$, $f''(x) = 0$ if $x \in (n-1, n-1/n^4]$, $f''(x) = n$ if $x \in (n - 1/n^4, n]$. In other words, f' is constant on $(n-1, n-1/n^4]$ and increases at the rate n on $(n-1/n^4, n]$ (so f' increases on $(n-1/n^4, n]$ by $1/n^3$). Then f' is locally Lipschitz continuous. However, f^* is not strongly convex on compact subsets of $\mathbb{R}$; in fact it is not strongly convex on compact subsets of $\operatorname{dom} f^*$. Indeed, we have $\operatorname{dom} f^* = [-c, c]$ where $c = \sum_{n=1}^{\infty} 1/n^4$, while $(f^*)''(p) = 1/n$ on the interval $\left(\sum_{k=1}^{n-1} 1/k^4, \sum_{k=1}^{n} 1/k^4\right)$. In other words, there are intervals of arbitrarily slow growth of ∂f^* in the compact $\operatorname{dom} f^*$, which means that ∂f^* is not strongly monotone, which in turn means that f^* is not strongly convex.

A natural conjecture to make is that the conjugate of an essentially differentiable convex function f with locally Lipschitz continuous gradient will be an essentially locally strongly convex. This turns out to be false, as the next, more complicated, example shows.

Example 3.4. Consider the pair of convex conjugate functions on $\mathbb{R}^2$:

$$f(x_1, x_2) = \begin{cases} \dfrac{1}{2}\left(\dfrac{1}{2}x_1^2 + x_2\right)^2 & \text{if } \dfrac{1}{2}x_1^2 + x_2 \geq 0 \\ 0 & \text{if } \dfrac{1}{2}x_1^2 + x_2 < 0, \end{cases}$$

$$f^*(y_1,y_2) = \begin{cases} \frac{1}{2}\frac{y_1^2}{y_2} + \frac{1}{2}y_2^2 & \text{if } y_2 > 0 \\ 0 & \text{if } y_1 = y_2 = 0 \\ \infty & \text{otherwise.} \end{cases}$$

Note that $\operatorname{dom} f^* = \{y \mid y_2 > 0 \text{ or } y = 0\}$ and $\operatorname{dom} \partial f^* = \operatorname{dom} f$. (At $y = 0$, we have $0 \in \partial f^*(0)$.) Also note that f^* is smooth on $\operatorname{int} \operatorname{dom} f^* = \{y \mid y_2 > 0\}$.

The function f is differentiable and ∇f is locally Lipschitz continuous. However, f^* is not strongly convex on the compact and convex set $K = \{(y_1,y_2) \mid 2 \geq 2y_2 \geq y_1^2\}$. There are several ways to see that. For example, note that for $y_2 > 0$,

$$\nabla f^*(y) = \begin{pmatrix} \dfrac{y_1}{y_2} \\ -\dfrac{1}{2}\dfrac{y_1^2}{y_2^2} + y_2 \end{pmatrix}, \qquad H f^*(y) = \begin{pmatrix} \dfrac{1}{y_2} & -\dfrac{y_1}{y_2^2} \\ -\dfrac{y_1}{y_2^2} & \dfrac{y_1^2}{y_2^3} + 1 \end{pmatrix},$$

and the smaller of the two real eigenvalues of $Hf^*(y)$ equals

$$\gamma(y) = 2\left(1 + \frac{y_1^2}{y_2^2} + y_2 + \sqrt{\left(1 + \frac{y_1^2}{y_2^2} + y_2\right)^2 - 4y_2}\right)^{-1}.$$

In particular, $\gamma(y) < 2\left(1 + \dfrac{y_1^2}{y_2^2}\right)^{-1}$, and if $y_2 = y_1^2$, $\gamma(y) < 2y_1^2$. Thus the eigenvalues of Hf^* are not uniformly bounded below by a positive number on the interior of K (where f^* is smooth), and so f^* is not strongly convex on K.

We need to note though that the function f^* from the example above is strongly convex on each compact and convex $K \subset \operatorname{dom} \partial f^* \cap \{y \mid y_2 > 0\}$, and more generally, on each compact and convex $K \subset \operatorname{dom} \partial f^*$ such that ∂f^* is uniformly bounded on $K \setminus \{0\}$. Indeed, if ∂f^* is uniformly bounded on $K \setminus \{0\}$ and K is compact, then for some $M > 0$, for all $y \in K \setminus \{0\}$ we have $|y_1| \leq M y_2$ and $y_2 \leq M$. At the same time

$$\gamma(y) \geq \left(1 + \frac{y_1^2}{y_2^2} + y_2\right)^{-1} \geq \left(1 + M^2 + M\right)^{-1},$$

and so the eigenvalues of Hf^* are uniformly bounded below by a positive number on $K \setminus \{0\}$. It is easy to check that f^* is continuous relative to the set K under discussion (but it is not continuous at 0 in general, only lsc). This is enough to conclude that f^* is strongly convex on K. Note that while $\partial f^*(0)$ is not bounded, and thus ∂f^* is not bounded on K, there exists a bounded selection from ∂f^* on K. This to an extent illustrates the condition on f^* that turns out to be equivalent to local Lipschitz continuity of ∇f; more precisely, condition (d) in Theorem 4.1.

4. Main results

Theorem 4.1. *For a proper, lower semicontinuous, and convex function* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, *the following are equivalent:*

(*a*) *f is essentially differentiable and ∇f is locally Lipschitz continuous on* $\operatorname{int}\operatorname{dom} f$*;*

(*b*) *the mapping ∂f is single-valued on* $\operatorname{dom}\partial f$ *and locally Lipschitz continuous relative to* $\operatorname{gph}\partial f$ *in the sense that for each compact subset $S \subset \operatorname{gph}\partial f$ there exists $\rho > 0$ such that for any $(x_1, y_1), (x_2, y_2) \in S$ one has*

$$|y_1 - y_2| \le \rho |x_1 - x_2|;$$

(*c*) *the mapping ∂f^* is locally strongly monotone relative to* $\operatorname{gph}\partial f^*$ *in the sense that for each compact subset $S \subset \operatorname{gph}\partial f^*$ there exists $\sigma > 0$ such that for any $(y_1, x_1), (y_2, x_2) \in S$ one has*

$$(y_1 - y_2) \cdot (x_1 - x_2) \ge \sigma |y_1 - y_2|^2;$$

(*d*) *for each compact subset $K \subset \operatorname{dom}\partial f^*$ and each compact $K' \subset I\!R^n$ such that $K' \cap \partial f^*(y) \neq \emptyset$ for all $y \in K$ there exists $\sigma > 0$ such that*

$$f^*(y') \ge f^*(y) + x \cdot (y' - y) + \frac{1}{2}\sigma |y' - y|^2 \tag{2}$$

for all $y, y' \in K$, all $x \in \partial f^*(y) \cap K'$.

Proof. (*a*) $\Longrightarrow$ (*d*): Pick any compact $K \subset \operatorname{dom}\partial f^*$ and a compact $K' \subset I\!R^n$ such that $K' \cap \partial f^*(y) \neq \emptyset$ for all $y \in K$. (If no such K' exists for a chosen K there is nothing to prove.) For all $y \in K$ there exists $x \in K'$ with $x \in \partial f^*(y)$, and so with $y = \nabla f(x)$. Thus $K \subset \nabla f(K')$ and $K' \subset C$, where $C = \operatorname{dom}\nabla f = \operatorname{int}\operatorname{dom} f$. As C is convex and open, $\operatorname{con} K'$ is a compact subset of C. Let $\varepsilon > 0$ be such that $K'_\varepsilon = \operatorname{con} K' + \varepsilon I\!B \subset C$, and let $\rho > 0$ be a Lipschitz constant for ∇f on K'_ε. Then for all $x, x' \in K'_\epsilon$,

$$f(x') \le f(x) + \nabla f(x) \cdot (x' - x) + \frac{1}{2}\rho |x' - x|^2.$$

This is shown in the proof of [2, Proposition 12.60]. Take any $y \in K$, $x \in \partial f^*(y) \cap K'$. Then $f^*(y) = y \cdot x - f(x)$ while for any $y' \in K$,

$$\begin{aligned}
f^*(y') &= \sup_{x'} \{y' \cdot x' - f(x')\} \ge \sup_{x' \in K'_\varepsilon} \{y' \cdot x' - f(x')\} \\
&\ge \sup_{x' \in K'_\varepsilon} \left\{ y' \cdot x' - f(x) - \nabla f(x) \cdot (x' - x) - \frac{1}{2}\rho \|x' - x\|^2 \right\} \\
&= y \cdot x - f(x) + \sup_{x' \in K'_\varepsilon} \left\{ (y' - y) \cdot x' - \frac{1}{2}\rho \|x' - x\|^2 \right\} \\
&= f^*(y) + x \cdot (y' - y) + \sup_{x' \in K'_\varepsilon} \left\{ (y' - y) \cdot (x' - x) - \frac{1}{2}\rho \|x' - x\|^2 \right\}.
\end{aligned}$$

Now pick any $\alpha > 0$ such that $\sigma = 2(\alpha - \frac{1}{2}\rho\alpha^2) > 0$ and $\alpha \|y' - y\| \le \varepsilon$ for all $y, y' \in K$. Considering x' in the supremum above so that $x' - x = \alpha(y' - y)$ yields the desired inequality.

(*d*) $\Longrightarrow$ (*c*): Let S be any compact subset of $\operatorname{gph}\partial f^*$. Let

$$K = \{y \,|\, \exists\, x \text{ such that } (y, x) \in S\},$$

$$K' = \{x \,|\, \exists\, y \text{ such that } (y, x) \in S\}.$$

Both K and K' are compact and for all $y \in K$, $\partial f^*(y) \cap K' \neq \emptyset$. Let $\sigma > 0$ be as in (d). Then for any $(y_1, x_1), (y_2, x_2) \in S$,

$$f^*(y_2) \geq f^*(y_1) + x_1 \cdot (y_2 - y_1) + \frac{1}{2}\sigma |y_2 - y_1|^2,$$

$$f^*(y_1) \geq f^*(y_2) + x_2 \cdot (y_1 - y_2) + \frac{1}{2}\sigma |y_1 - y_2|^2.$$

Adding the two inequalities shows the needed strong monotonicity.

$(c) \Longrightarrow (b)$: Given any compact S in $\operatorname{gph} \partial f$, let $S' = \{(y, x) \,|\, (x, y) \in S\}$ and let $\sigma > 0$ be as in (c). One can then take $\rho = 1/\sigma$, since for any $(x_1, y_1), (x_2, y_2) \in S$,

$$|y_1 - y_2|\, |x_1 - x_2| \geq (y_1 - y_2) \cdot (x_1 - x_2) \geq \sigma |y_1 - y_2|^2.$$

$(b) \Longrightarrow (a)$: Essential differentiability of f follows from ∂f being single-valued, and entails continuity, and thus local boundedness, of ∇f on $\operatorname{int} \operatorname{dom} f$. Local boundedness, and local Lipschitz continuity of ∇f relative to $\operatorname{gph} \nabla f$, implies local Lipschitz continuity of ∇f on $\operatorname{int} \operatorname{dom} f$. □

Essentially, condition (d) above says that inequality (2) holds for some $\sigma > 0$ for each compact – but not necessarily convex – subset of $\operatorname{dom} \partial f^*$ and for each bounded selection from ∂f^* on that subset.[1] It turns out that when such a compact subset is in fact convex, satisfaction of (2) for one bounded selection from ∂f^* (provided it exists) is enough to guarantee that f^* is "strongly convex on that subset". More generally, we have:

Proposition 4.2. *Let $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper, lower semicontinuous, and convex function, let K be a compact and convex subset of* $\operatorname{dom} \partial g$*, and let $\sigma > 0$. Consider the inequality*

$$g(x') \geq g(x) + y \cdot (x' - x) + \frac{1}{2}\sigma |x' - x|^2. \tag{3}$$

The following are equivalent:

- (a) *for all $x \in K$ there exists $y \in \partial g(x)$ such that* (3) *holds for all $x' \in K$;*
- (b) *g is strongly convex with constant σ on K (in the sense that* (1) *is satisfied by g for all $x_1, x_2 \in K$ and all $\lambda \in (0, 1)$);*
- (c) *for all $x \in K$ and all $y \in \partial g(x)$,* (3) *holds for all $x' \in K$.*

Proof. $(a) \Longrightarrow (b)$ Pick any $x_1, x_2 \in K$ and $\lambda \in (0, 1)$. Let $x = (1 - \lambda)x_1 + \lambda x_2$ and let $y \in \partial g(x)$ be such that (3) holds for all $x' \in K$. Inequalities obtained from (3) by substituting x_1 and x_2 for x', multiplied, respectively, by $1 - \lambda$ and λ, and added to each other, yield

$$(1 - \lambda)g(x_1) + \lambda g(x_2) \geq g(x) + \frac{1}{2}\sigma \left((1 - \lambda)|x_1 - x|^2 + \lambda |x_2 - x|^2\right).$$

Now, algebra yields $(1 - \lambda)|x_1 - x|^2 + \lambda |x_2 - x|^2 = \lambda(1 - \lambda)|x_1 - x_2|^2$, and so g is strongly convex on K.

[1]Monotone operators for which there exist locally bounded selections were studied in [3] under the name "locally efficient".

(*b*) $\Longrightarrow$ (*c*) Let $\phi = g - \frac{1}{2}\sigma|\cdot|^2$. From (*b*) we have that $\phi_K = \phi + \delta_K$ is convex. (Here δ_K is the indicator of K: $\delta_K(x) = 0$ if $x \in K$ and $\delta_K(x) = \infty$ otherwise.) Thus for any $x, x' \in K$, any $z \in \partial\phi_K(x)$, we have

$$\phi(x') \geq \phi(x) + z \cdot (x' - x).$$

But $\partial\phi + \partial\delta_K \subset \partial\phi_K$ and $0 \in \partial\delta_K(x)$, so the inequality above holds for all $x, x' \in K$, all $z \in \partial\phi(x)$. As the function $\frac{1}{2}\sigma|\cdot|^2$ is finite everywhere, $\partial g(x) = \partial\phi + \sigma x$. Thus, the inequality displayed above translates to

$$g(x') - \frac{1}{2}\sigma|x'|^2 \geq g(x) - \frac{1}{2}\sigma|x|^2 + (y - \sigma x) \cdot (x' - x)$$

and this inequality holds for all $x, x' \in K$, all $y \in \partial g(x)$. Algebra shows that this inequality is exactly (3).

(*c*) $\Longrightarrow$ (*a*) Obvious. □

Corollary 4.3. *Let $f : I\!R^n \to \overline{I\!R}$ be a proper, lower semicontinuous, and convex function. If*

($\diamond$) *for every compact and convex set $K \subset \operatorname{dom}\partial f^*$ there exists a compact set $K' \subset I\!R^n$ such that $\partial f^*(y) \cap K' \neq \emptyset$ for all $x \in K$,*

then the condition that

(*a*) *f is essentially differentiable and ∇f is locally Lipschitz continuous on* $\operatorname{int}\operatorname{dom} f$

implies that

(*b*) *f^* is essentially locally strongly convex.*

On the other hand, if $\operatorname{dom}\partial f^*$ *is convex then* (*b*) *implies* (*a*). *Thus, if* ($\diamond$) *holds and* $\operatorname{dom}\partial f^*$ *is convex, then* (*a*) *is equivalent to* (*b*).

Corollary 4.3 in particular says that for a coercive (and proper and lsc) convex function f, essential differentiability of f and local Lipschitz continuity of ∇f is equivalent to essential local strong convexity of f^*.

There are some natural examples of classes of convex functions g for which the two assumptions used in Corollary 4.3 are met, that is, for which $\operatorname{dom}\partial g$ is convex and there exist bounded selections from $\partial g(x)$ with x ranging over compact subsets $K \subset \operatorname{dom}\partial g$. One example is given by functions $g = g_0 + \delta_C$ where $g_0 : I\!R^n \to I\!R$ is a convex function (finite-valued!) and δ_C is the indicator of a nonempty closed convex set C. Indeed, then ∂g_0 is locally bounded and $\partial g_0(x) \subset \partial g(x)$ for each $x \in \operatorname{dom}\partial g = \operatorname{dom} g = C$. Another class is given by functions g for which $\operatorname{dom}\partial g$ is open, as then ∂g is bounded on each compact subset of $\operatorname{dom}\partial g$. More generally, it is enough that $\operatorname{dom}\partial g$ be *relatively open*, that is, open in the relative topology of the affine hull of $\operatorname{dom}\partial g$, and equivalently, of $\operatorname{dom} g$. Then $\operatorname{dom}\partial g$ equals to the relative interior of $\operatorname{dom} g$, and so is relatively open. Furthermore, if P denotes the orthogonal projection onto the affine hull of $\operatorname{dom} g$, the function $G(x) := g(Px)$ is proper, lsc, and convex, $\operatorname{dom}\partial G$ is open, and given any compact $K \subset \operatorname{dom}\partial g \subset \operatorname{dom}\partial G$, ∂G is bounded on K and $\partial G(x) \subset \partial g(x)$ for all $x \in K$.

To conclude, we make the following observation, which follows from the fact that for any essentially differentiable f, $\operatorname{dom}\partial f$ is open and convex.

Corollary 4.4. *Let $\mathcal{C}$ be the class of all proper, lower semicontinuous, and convex functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that f is essentially differentiable, ∇f is locally Lipschitz continuous, and f is essentially locally strongly convex. Then*

$$f \in \mathcal{C} \quad \Longleftrightarrow \quad f^* \in \mathcal{C}.$$

References

[1] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton (1970).

[2] R. T. Rockafellar, R. J.-B. Wets: Variational Analysis, Springer, Berlin (1998).

[3] A. Verona, M. E. Verona: Locally efficient monotone operators, Proc. Amer. Math. Soc. 109(1) (1990) 195–204.