Journal of Convex Analysis
Volume 15 (2008), No. 2, 395–409

Maximal Monotone Operators and Maximal Monotone Functions for Equilibrium Problems

Koji Aoyama
*Department of Economics, Chiba University,
Yayoi-cho, Inage-ku, Chiba-shi, Chiba 263-8522, Japan
aoyama@le.chiba-u.ac.jp*

Yasunori Kimura
*Department of Mathematical and Computing Sciences,
Tokyo Institute of Technology, Ookayama, Meguro-ku, Tokyo 152-8552, Japan
yasunori@is.titech.ac.jp*

Wataru Takahashi
*Department of Mathematical and Computing Sciences,
Tokyo Institute of Technology, Ookayama, Meguro-ku, Tokyo 152-8552, Japan
wataru@is.titech.ac.jp*

Received: June 12, 2007

In this paper, we deal with multi-valued monotone operators and two variable monotone functions for equilibrium problems in Banach spaces. We first study close relationships concerning resolvents and maximal monotonicity for monotone operators and monotone functions. We also consider a one-to-one correspondence between a class of maximal monotone operators and that of maximal monotone functions with certain conditions.

Keywords: equilibrium problem, maximal monotone operator, maximal monotone function, resolvent

2000 Mathematics Subject Classification: Primary 47H05, 74G10

1. Introduction

Let E be a real Banach space, C a nonempty subset of E, and f a function of $E \times E$ into $\mathbb{R} \cup \{-\infty, \infty\}$ such that $f(x,x) = 0$ for all $x \in C$. Then the equilibrium problem (with respect to C) is formulated as follows: find $\bar{x} \in C$ such that

$$f(\bar{x}, y) \geq 0 \quad \text{for all } y \in C.$$

In this case, such a point $\bar{x} \in C$ is called a solution to the problem. The equilibrium problem has been widely discussed in the literature. For example, existence results of solutions were obtained in [6], [19], [4] and [7], while some methods for finding a solution were studied in [8], [13], [5] and [18]; see also [3] and references therein. On the other hand, let A be a maximal monotone operator on E. If $0 \in Ax$, then x is called a zero of A. It is known that the problem of finding such a point is one of the most important problems in convex analysis and mathematical optimization; see, for instance, [12], [17], [14], [9], [10] and [11].

ISSN 0944-6532 / \$ 2.50

The main purpose of this paper is to investigate a relationship between the problem of finding a solution of the equilibrium problem and the problem of finding a zero of a maximal monotone operator. Roughly speaking, in this paper, we assert that these two problems are equivalent under some conditions.

In the next section, we treat some definitions and notions of geometrical properties of a Banach space. The convexity, monotonicity, and semicontinuity for extended real-valued functions are also introduced.

In §3, using existence results due to Blum and Oettli [4], we deal with the resolvent for a maximal monotone function f, where f is the objective function of a given equilibrium problem. Then we construct a maximal monotone operator for f, and moreover, we see that the resolvent of the maximal monotone operator coincides with that of f. We also examine the reverse: for a given maximal monotone operator, we provide a method of constructing a maximal monotone function and prove that their resolvents are identical.

The last section is devoted to studying a one-to-one correspondence between a maximal monotone operator and a maximal monotone function for an equilibrium problem. We construct a bijective mapping from the family of all maximal monotone operators having a fixed nonempty closed convex effective domain to the family of maximal monotone functions satisfying certain conditions.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers, $\mathbb{R}$ the set of real numbers, E a real Banach space with norm $\|\cdot\|$, E^* the dual of E, $\langle x, \xi^* \rangle$ the value of $\xi^* \in E^*$ at $x \in E$. The set of extended real numbers is denoted by $[-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$. Let us define $a + \infty = \infty + a = \infty$ and $a + (-\infty) = -\infty + a = -\infty$ if $a \in \mathbb{R}$, and

$$-(\infty) = -\infty \quad \text{and} \quad -(-\infty) = \infty.$$

The (normalized) duality mapping J of E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|_{E^*}^2\}$$

for $x \in E$. It is known that

$$\|x\|^2 - \|y\|^2 \leq 2 \langle x - y, x^* \rangle \tag{1}$$

holds for all $x, y \in E$ and $x^* \in Jx$. It is also known that the duality mapping J of E is surjective if E is reflexive.

The norm $\|\cdot\|$ of E is said to be Gâteaux differentiable if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in E$ with $\|x\| = \|y\| = 1$. In this case a Banach space E is said to be smooth. We know that the duality mapping J is single-valued and norm-to-weak* continuous if E is smooth. A Banach space E is said to be strictly convex if $\|x + y\| < 2$ for all $x, y \in E$ whenever $\|x\| = \|y\| = 1$ and $x \neq y$. It is known that the duality mapping J of E is injective, that is, $Jx \cap Jy = \emptyset$ for all $x, y \in E$ with $x \neq y$ if E is strictly convex; see [20] for more details.

Let ϕ be a function of E into $[-\infty, \infty]$. The effective domain $\{x \in E : \phi(x) < \infty\}$ of ϕ is denoted by $D(\phi)$. A function ϕ is said to be convex if

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y) \tag{2}$$

for all $x, y \in E$ and $\lambda \in (0,1)$ such that the right hand side is well defined. A convex function ϕ is said to be proper if $\phi(x) > -\infty$ for all $x \in E$ and there is $y \in E$ such that $\phi(y) < \infty$. A function ϕ is said to be lower semicontinuous if the set $\{x \in E : \phi(x) \leq l\}$ is closed in E for all $l \in \mathbb{R}$. We know that if a function ϕ is lower semicontinuous, then

$$x_n \to x \text{ implies } \phi(x) \leq \liminf_{n\to\infty} \phi(x_n);$$

see, for example, [2, 20].

Remark 2.1. Let $\phi\colon E \to [-\infty, \infty]$ be a lower semicontinuous and convex function. If there is $y \in E$ such that $\phi(y) \in \mathbb{R}$, then $\phi(x) > -\infty$ for all $x \in E$. In fact, let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that $\lambda_n \to 0$. If $\phi(z) = -\infty$, then

$$\phi(\lambda_n z + (1-\lambda_n)y) \leq \lambda_n\phi(z) + (1-\lambda_n)\phi(y) = -\infty$$

for every $n \in \mathbb{N}$. Since ϕ is lower semicontinuous, we have that

$$\phi(y) \leq \liminf_{n\to\infty} \phi(\lambda_n z + (1-\lambda_n)y) = -\infty.$$

This is a contradiction. So, we have that $\phi(x) > -\infty$ for all $x \in E$.

Let ϕ be a function of E into $[-\infty, \infty]$. The function $\phi^*\colon E^* \to [-\infty, \infty]$ defined by

$$\phi^*(x^*) = \sup_{x\in E}(\langle x, x^*\rangle - \phi(x))$$

for $x^* \in E^*$ is called the conjugate function of ϕ. The biconjugate function $\phi^{**}\colon E \to [-\infty, \infty]$ of ϕ is defined by

$$\phi^{**}(x) = \sup_{x^*\in E^*}(\langle x, x^*\rangle - \phi^*(x^*))$$

for $x \in E$. We know that if ϕ is proper, lower semicontinuous, and convex, then $\phi^{**} = \phi$; see [2, 21].

For a nonempty subset C of a Banach space E, the indicator function $i_C\colon E \to [-\infty, \infty]$ for C is defined by

$$i_C(x) = \begin{cases} 0 & \text{if } x \in C; \\ \infty & \text{if } x \notin C. \end{cases}$$

Let f be a function of $E \times E$ into $[-\infty, \infty]$ and C a nonempty subset of E. We say that f is monotone with respect to C if

$$f(x, y) \leq -f(y, x)$$

for all $x, y \in C$. In this case, if $f(u, v) > -\infty$ for all $u, v \in C$, then $f(x, y) \in \mathbb{R}$ for all $x, y \in C$. Indeed, suppose that $f(y, x) = \infty$ for some $x, y \in C$. Then, from the monotonicity of f, we have

$$-\infty < f(x, y) \leq -f(y, x) = -(\infty) = -\infty,$$

which is a contradiction. Therefore we see that $f(y, x) \in \mathbb{R}$.

Remark 2.2. Let C be a nonempty subset of a Banach space E and f a function of $E \times E$ into $[-\infty, \infty]$. Suppose that $f(x,x) = 0$ and $f(x,\cdot)\colon E \to [-\infty,\infty]$ is lower semicontinuous and convex for all $x \in C$. In this case it follows from Remark 2.1 that $f(x,y) > -\infty$ for all $x \in C$ and $y \in E$. In addition, assuming that f is monotone with respect to C, we conclude that $f(x,y) \in \mathbb{R}$ for all $x, y \in C$.

Let C be a nonempty subset of E. A function $f\colon E \times E \to [-\infty,\infty]$ is said to be maximal monotone with respect to C if, for every $x \in C$ and $x^* \in E^*$,

$$f(x,y) + \langle y - x, x^* \rangle \geq 0 \quad \text{for all } y \in C$$

whenever $\langle z - x, x^* \rangle \geq f(z,x)$ for all $z \in C$; see [4].

Let A be a multi-valued mapping of E into E^*. For convenience, we regard a multi-valued mapping A as its graph $\{(x,x^*) \in E \times E^* : x^* \in Ax\}$. That is, $(x,x^*) \in A$ if and only if $x^* \in Ax$. The effective domain of A is denoted by $D(A)$, that is, $D(A) = \{x \in E : Ax \neq \emptyset\}$. A multi-valued mapping A is said to be monotone if $\langle x - y, x^* - y^* \rangle \geq 0$ for all $(x,x^*) \in A$ and $(y,y^*) \in A$. A multi-valued mapping A is said to be a maximal monotone operator if A is monotone and its graph is not properly contained in the graph of any other monotone operator of $E \times E^*$. It is known that a monotone operator $A \subset E \times E^*$ is maximal monotone if and only if

$$(x,x^*) \in E \times E^* \quad \text{and} \quad \langle x - y, x^* - y^* \rangle \geq 0 \quad \text{for all } (y,y^*) \in A$$

imply $(x,x^*) \in A$; see, for instance, [2, 20]. We also know the following.

Lemma 2.3 (Rockafellar [15]). *Let E be a strictly convex, smooth, and reflexive Banach space and $A \subset E \times E^*$ a monotone operator. Then A is maximal monotone if and only if $R(J + rA) = E^*$ for all $r > 0$, where $R(J + rA)$ is the range of $J + rA$.*

Let E be a strictly convex, smooth, and reflexive Banach space and $A \subset E \times E^*$ a maximal monotone operator. Let $r > 0$ and $x \in E$ be given. Using Lemma 2.3, we know that there exists a unique $x_r \in D(A)$ such that $Jx \in Jx_r + rAx_r$. Thus we may define a single-valued mapping $J_r\colon E \to D(A)$ by $J_r x = x_r$, that is, $J_r = (J + rA)^{-1}J$. Such a mapping J_r is said to be the resolvent of A for r. The set of all zeros of A is denoted by $A^{-1}0$, that is, $A^{-1}0 = \{x \in E : Ax \ni 0\}$. It is known that $A^{-1}0 = F(J_r)$ for all $r > 0$, where $F(J_r)$ is the set of fixed points of J_r. It is also known that

$$\frac{1}{r}(J - JJ_r)x \in AJ_r x \tag{3}$$

for all $r > 0$ and $x \in E$; see, for example, [9, 10, 11].

Let E be a Banach space, C a nonempty subset of E, and f a function of $E \times E$ into $[-\infty,\infty]$ such that $f(x,x) = 0$ for all $x \in C$. The solution set of an equilibrium problem (with respect to C) is denoted by $\mathrm{EP}(f)$. Namely,

$$\mathrm{EP}(f) = \{z \in C : f(z,y) \geq 0 \text{ for all } y \in C\}.$$

3. Resolvents and Maximal Monotonicity

We first discuss the existence of resolvents of a maximal monotone function. The following theorem is essentially due to Blum and Oettli [4, Corollary 1].

Theorem 3.1. *Let E be a smooth and reflexive Banach space and C a nonempty closed convex subset of E. Let f be a function of $E \times E$ into $[-\infty, \infty]$ that satisfies the following conditions:*

(*F1*) $f(x, x) = 0$ *for all* $x \in C$;
(*F2*) f *is monotone with respect to* C;
(*F3*) $f(x, \cdot)$ *is lower semicontinuous and convex for all* $x \in C$;
(*F4*) f *is maximal monotone with respect to* C.

Then for every $x^* \in E^*$, there exists $z \in C$ such that

$$0 \leq f(z, y) + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|z\|^2 - \langle y - z, x^* \rangle$$

for all $y \in C$.

Remark 3.2. The assumption for f of [4, Corollary 1] is stronger than that of Theorem 3.1. However, it is not hard to prove this theorem by using the techniques developed in [4].

If, in addition to the hypotheses of Theorem 3.1, E is strictly convex, then it is easy to see that such z is unique for each $x^* \in E^*$. Thus we have the following theorem, which is also essentially obtained in [4]. We give the proof for the sake of completeness.

Theorem 3.3. *Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty closed convex subset of E. Let f be a function of $E \times E$ into $[-\infty, \infty]$ that satisfies (F1), (F2), (F3) and (F4) in Theorem 3.1. Then for every $x^* \in E^*$, there exists a unique point $z \in C$ such that*

$$0 \leq f(z, y) + \langle y - z, Jz - x^* \rangle$$

for all $y \in C$.

Proof. Let $x^* \in E^*$ be fixed. From the consequence of Theorem 3.1, it follows that there exists a point $z \in C$ such that

$$0 \leq f(z, y) + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|z\|^2 - \langle y - z, x^* \rangle$$

for all $y \in C$. Note that $f(z, y) \in \mathbb{R}$ for all $y \in C$ by Remark 2.2. Thus the monotonicity of f implies that

$$f(y, z) \leq \frac{1}{2}\|y\|^2 - \frac{1}{2}\|z\|^2 - \langle y - z, x^* \rangle \tag{4}$$

for all $y \in C$. Let $z_t = (1 - t)z + ty \in C$ for $y \in C$ and $t \in (0, 1)$. Then it is clear that $z_t \in C$ for all $t \in (0, 1)$. By (4) and (1), we have

$$\begin{aligned} f(z_t, z) &\leq \frac{1}{2}\|z_t\|^2 - \frac{1}{2}\|z\|^2 - \langle z_t - z, x^* \rangle \\ &\leq \langle z_t - z, Jz_t \rangle - \langle z_t - z, x^* \rangle \\ &= \langle z_t - z, Jz_t - x^* \rangle = t \langle y - z, Jz_t - x^* \rangle . \end{aligned} \tag{5}$$

Note that $f(z_t, z)$, $f(z_t, y)$ and $f(y, z_t) \in \mathbb{R}$ because of Remark 2.2 and the fact that $y, z, z_t \in C$. Using (*F1*), (*F3*), (5) and (*F2*), we obtain

$$\begin{aligned} 0 &= f(z_t, z_t) \\ &\leq (1-t)f(z_t, z) + tf(z_t, y) \\ &\leq (1-t)t \langle y - z, Jz_t - x^* \rangle + tf(z_t, y) \\ &\leq (1-t)t \langle y - z, Jz_t - x^* \rangle + t(-f(y, z_t)). \end{aligned}$$

Therefore we get

$$f(y, z_t) \leq (1-t) \langle y - z, Jz_t - x^* \rangle$$

for all $y \in C$ and $t \in (0,1)$. Since J is norm-to-weak* continuous and since $\lim_{t\to 0} z_t = z$ and $f(y, \cdot)$ is lower semicontinuous, we conclude that

$$f(y, z) \leq \liminf_{t\to 0} f(y, z_t) \leq \liminf_{t\to 0}(1-t) \langle y - z, Jz_t - x^* \rangle = \langle y - z, Jz - x^* \rangle$$

for all $y \in C$. Consequently, (*F4*) implies that

$$f(z, y) + \langle y - z, Jz - x^* \rangle \geq 0$$

for all $y \in C$. On the other hand, it follows from (1) that if $z \in C$ and $f(z, y) + \langle y - z, Jz - x^* \rangle \geq 0$ for all $y \in C$, then

$$0 \leq f(z, y) + \frac{1}{2} \|y\|^2 - \frac{1}{2} \|z\|^2 - \langle y - z, x^* \rangle$$

for all $y \in C$. Since E is strictly convex, such z is unique. So, we conclude the desired result. □

Using Theorem 3.3, we immediately obtain the following:

Corollary 3.4. *Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty closed convex subset of E. Let f be a function of $E \times E$ into $[-\infty, \infty]$ that satisfies (F1), (F2), (F3) and (F4) in Theorem* 3.1. *Then for every $x \in E$ and $r > 0$, there exists a unique point $z_r \in C$ such that*

$$0 \leq f(z_r, y) + \frac{1}{r} \langle y - z_r, Jz_r - Jx \rangle \tag{6}$$

for all $y \in C$.

Proof. Let $x \in E$ and $r > 0$ be given. Note that a function rf also satisfies the conditions from (*F1*) to (*F4*). Therefore, for $Jx \in E^*$, there exists a unique point $z_r \in C$ such that

$$0 \leq rf(z_r, y) + \langle y - z_r, Jz_r - Jx \rangle$$

for all $y \in C$. This completes the proof. □

Under the same assumptions in Corollary 3.4, for every $r > 0$, we may define a single-valued mapping $F_r \colon E \to C$ by

$$F_r x = \left\{ z \in C : 0 \leq f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \text{ for all } y \in C \right\} \tag{7}$$

for $x \in E$, which is called the resolvent of f for r.

Next, we study a relationship between a maximal monotone operator and a maximal monotone function. For a given maximal monotone function f, we introduce a method of constructing the maximal monotone operator A_f whose resolvent coincides with that of f.

Theorem 3.5. *Let E be a real Banach space and C a nonempty subset of E. Let f be a function of $E \times E$ into $[-\infty, \infty]$ that satisfies the following conditions:*

(F1) $f(x, x) = 0$ for all $x \in C$;
(F2) f is monotone with respect to C;
(F3) $f(x, \cdot)$ is lower semicontinuous and convex for all $x \in C$.

Let A_f be a multi-valued mapping of E into E^* defined by

$$A_f x = \begin{cases} \{x^* \in E^* : f(x, y) \geq \langle y - x, x^* \rangle \text{ for all } y \in C\} & \text{if } x \in C; \\ \emptyset & \text{if } x \notin C. \end{cases} \tag{8}$$

Then $\mathrm{EP}(f) = A_f^{-1}0$ and A_f is monotone. Moreover, suppose that E is smooth, strictly convex, and reflexive and C is closed and convex. If f is maximal monotone with respect to C, then A_f is a maximal monotone operator and the resolvent F_r of f coincides with the resolvent $(J + rA_f)^{-1}J$ of A_f for each $r > 0$.

Proof. The equality $\mathrm{EP}(f) = A_f^{-1}0$ is obvious from the definition of A_f. We first prove that A_f is monotone. Let $(x, x^*), (y, y^*) \in A_f$. By the definition of A_f, we have

$$f(x, z) \geq \langle z - x, x^* \rangle \quad \text{and} \quad f(y, z) \geq \langle z - y, y^* \rangle$$

for all $z \in C$. In particular we obtain

$$f(x, y) \geq \langle y - x, x^* \rangle \quad \text{and} \quad f(y, x) \geq \langle x - y, y^* \rangle . \tag{9}$$

Note that, from Remark 2.2, $f(x, y)$ and $f(y, x) \in \mathbb{R}$. It follows from (*F2*) and (9) that

$$0 \geq f(x, y) + f(y, x) \geq \langle y - x, x^* \rangle + \langle x - y, y^* \rangle = - \langle x - y, x^* - y^* \rangle .$$

Therefore we conclude that $\langle x - y, x^* - y^* \rangle \geq 0$ for all $(x, x^*), (y, y^*) \in A_f$. This means that A_f is monotone.

We next prove that A_f is maximal monotone. Suppose that C is a closed convex subset of a smooth, strictly convex, and reflexive Banach space E. Suppose that f is maximal monotone with respect to C. Let $r > 0$ be given. From Lemma 2.3, it is enough to show $R(J + rA_f) \supset E^*$, where $R(J + rA_f)$ is the range of $J + rA_f$. Let $x^* \in E^*$ be fixed. Since E is reflexive and smooth, J is surjective and single-valued, that is, there exists $x \in E$ such that $Jx = x^*$. Let F_r be the resolvent of f for $r > 0$. Corollary 3.4 and (7) imply that

$$f(F_r x, y) + \frac{1}{r} \langle y - F_r x, JF_r x - Jx \rangle \geq 0$$

for all $y \in C$, that is,

$$f(F_r x, y) \geq \left\langle y - F_r x, \frac{1}{r}(Jx - JF_r x) \right\rangle$$

for all $y \in C$. This shows $(Jx - JF_r x)/r \in A_f F_r x$ and hence

$$x^* = Jx \in (J + rA_f)F_r x. \tag{10}$$

This means that $x^* \in R(J + rA_f)$. Therefore we conclude that $R(J + rA_f) \supset E^*$ and A_f is a maximal monotone operator on E. According to (10), we have

$$(J + rA_f)^{-1}Jx \ni F_r x$$

for all $x \in E$ and $r > 0$. In this case, we know that the resolvent $(J + rA_f)^{-1}J$ of A_f for r is single-valued. Consequently,

$$(J + rA_f)^{-1}Jx = F_r x$$

for all $x \in E$ and $r > 0$. It follows that the resolvent of f coincides with the resolvent of A_f. This completes the proof. □

Remark 3.6. In Theorem 3.5, we can not guarantee $D(A_f) = C$. For the case of $D(A_f) = C$, see §4.

We know from [1] that every reflexive Banach space has an equivalent strictly convex and smooth norm. So, we have the following result.

Corollary 3.7. *Let E be a reflexive Banach space and C a nonempty closed convex subset of E. Let f be a function of $E \times E$ into $[-\infty, \infty]$ that satisfies the conditions (F1), (F2) and (F3) in Theorem* 3.5. *Then, if f is maximal monotone with respect to C, then an operator A_f defined by* (8) *is maximal monotone.*

Next, we construct a maximal monotone function f_A from a given maximal monotone operator $A \subset E \times E^*$.

Theorem 3.8. *Let E be a real Banach space and $A \subset E \times E^*$ a monotone operator such that the effective domain $D(A) = \{x \in E : Ax \neq \emptyset\}$ is nonempty. Let f_A be a function of $E \times E$ into $[-\infty, \infty]$ defined by*

$$f_A(x, y) = \begin{cases} \sup\{\langle y - x, x^* \rangle : x^* \in Ax\} & \text{if } x \in D(A); \\ -\infty & \text{if } x \notin D(A) \end{cases} \tag{11}$$

for $x, y \in E$. Let $\mathrm{EP}(f_A)$ be the solution set of an equilibrium problem of f_A with respect to $D(A)$. Then $A^{-1}0 \subset \mathrm{EP}(f_A)$ and the following are satisfied:

(F1) $f_A(x, x) = 0$ for all $x \in D(A)$;
(F2) f_A is monotone with respect to $D(A)$;
(F3) $f_A(x, \cdot)$ is lower semicontinuous and convex for all $x \in D(A)$.

Moreover, if A is maximal monotone, then f_A is maximal monotone with respect to $D(A)$. Furthermore, suppose that E is smooth, strictly convex, and reflexive. If $D(A)$ is closed and convex, then the resolvent of A coincides with the resolvent of f_A.

Proof. The inclusion $A^{-1}0 \subset \mathrm{EP}(f_A)$ and (F1) is trivial from the definition of f_A.

Let $x \in D(A)$. Then $Ax \neq \emptyset$. Since a function $\phi(y) = \langle y - x, x^* \rangle$ is affine and continuous for every $x^* \in Ax$, it follows that $f_A(x, \cdot)$ is convex and lower semicontinuous. This implies *(F3)*.

We next show *(F2)*: Let $x, y \in D(A)$ be given. Since A is monotone, we have

$$-\langle x - y, y^* \rangle \geq \langle y - x, x^* \rangle$$

for all $x^* \in Ax$ and $y^* \in Ay$. This implies that

$$\inf\{-\langle x - y, y^* \rangle : y^* \in Ay\} \geq \sup\{\langle y - x, x^* \rangle : x^* \in Ax\} = f_A(x, y)$$

for all $x, y \in D(A)$. Therefore $f_A(x, y) \in \mathbb{R}$ for all $x, y \in D(A)$. It is clear that

$$\inf\{-\langle x - y, y^* \rangle : y^* \in Ay\} = -\sup\{\langle x - y, y^* \rangle : y^* \in Ay\} = -f_A(y, x)$$

for all $x, y \in D(A)$. From these facts, we obtain *(F2)*.

Assuming that A is maximal monotone, we show that f_A is maximal monotone with respect to $D(A)$. Let $(x, \xi^*) \in D(A) \times E^*$ be fixed. Suppose that $\langle z - x, \xi^* \rangle \geq f_A(z, x)$ for all $z \in D(A)$. Then we have $\langle z - x, \xi^* \rangle \geq \langle x - z, z^* \rangle$ for all $(z, z^*) \in A$, and hence

$$\langle z - x, z^* - (-\xi^*) \rangle \geq 0$$

for all $(z, z^*) \in A$. Since A is maximal monotone, we conclude that $(x, -\xi^*) \in A$. Therefore we obtain that

$$f_A(x, y) = \sup\{\langle y - x, x^* \rangle : x^* \in Ax\} \geq \langle y - x, -\xi^* \rangle$$

for all $y \in E$. Consequently, it follows that

$$f_A(x, y) + \langle y - x, \xi^* \rangle \geq 0$$

for all $y \in E$. Hence, we deduce that f_A is a maximal monotone function.

Suppose that E is smooth, strictly convex, and reflexive and $D(A)$ is closed and convex. From all observation above, using Corollary 3.4, we can define the resolvent F_r of f_A for each $r > 0$, that is,

$$F_r x = \left\{ z \in D(A) : f_A(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0 \text{ for all } y \in D(A) \right\}$$

is a singleton for all $x \in E$ and $r > 0$. On the other hand, let J_r be the resolvent of A for $r > 0$. From (3) and the definition of f_A, we obtain

$$\begin{aligned} f_A(J_r x, y) = & \sup\{\langle y - J_r x, z^* \rangle : z^* \in AJ_r x\} \\ \geq & \left\langle y - J_r x, \frac{Jx - JJ_r x}{r} \right\rangle. \end{aligned}$$

Consequently, we have

$$f_A(J_r x, y) + \frac{1}{r} \langle y - J_r x, JJ_r x - Jx \rangle \geq 0$$

for all $x, y \in E$. This means that $J_r x \in F_r x$. Since each $F_r x$ is a singleton, we deduce that $J_r x = F_r x$ for all $x \in E$. This completes the proof. □

Remark 3.9. Let E be a reflexive Banach space and $A \subset E \times E^*$ a maximal monotone operator. Rockafellar [16] showed that the strong closure of $D(A)$ is convex.

In §4, we also show that $A^{-1}0 = \mathrm{EP}(f_A)$ under the same setting as Theorem 3.8.

4. Maximal monotone operators and maximal monotone functions

In this section, we investigate a correspondence between a class of maximal monotone operators and a class of maximal monotone functions.

Theorem 4.1. *Let E be a reflexive Banach space, $A \subset E \times E^*$ a monotone operator which has a nonempty closed convex effective domain $C = D(A)$, f_A a function of $E \times E$ into $[-\infty, \infty]$ induced by A with* (11)*, and A_{f_A} a monotone operator induced by f_A with*

$$A_{f_A}x = \begin{cases} \{x^* \in E^* : f_A(x,y) \geq \langle y - x, x^* \rangle \text{ for all } y \in C\} & \text{if } x \in C; \\ \emptyset & \text{if } x \notin C. \end{cases} \tag{12}$$

If A is maximal monotone, then $A_{f_A} = A$.

Proof. Suppose that A is maximal monotone. Let $x \in E$. If $A_{f_A}x = \emptyset$, then clearly $A_{f_A}x \subset Ax$. Otherwise, there exists $u^* \in A_{f_A}x$. Since f_A is monotone by Theorem 3.8, we have that for all $(y, w^*) \in A$,

$$\langle y - x, u^* \rangle \leq f_A(x,y) \leq -f_A(y,x) \leq -\langle x - y, w^* \rangle = \langle y - x, w^* \rangle .$$

Therefore we have $\langle y - x, u^* \rangle \leq \langle y - x, w^* \rangle$, or equivalently

$$\langle y - x, w^* - u^* \rangle \geq 0$$

for all $(y, w^*) \in A$. Since A is maximal monotone, we obtain that $u^* \in Ax$ and thus $A_{f_A}x \subset Ax$ for every $x \in E$. On the other hand, by Corollary 3.7 and Theorem 3.8, we have that A_{f_A} is also maximal monotone if A is maximal monotone. Hence we have $A_{f_A} = A$. □

We also have the following corollary.

Corollary 4.2. *Let E be a reflexive Banach space, $A \subset E \times E^*$ a maximal monotone operator which has a nonempty closed convex effective domain $C = D(A)$, and f_A a function of $E \times E$ into $[-\infty, \infty]$ induced by A with* (11)*. Then, $A^{-1}0 = \mathrm{EP}(f_A)$.*

Proof. It follows from Theorem 4.1 that $A^{-1}0 = A_{f_A}^{-1}0$. Therefore, using Theorem 3.5, we have that

$$A^{-1}0 = A_{f_A}^{-1}0 = \mathrm{EP}(f_A),$$

which completes the proof. □

Next, we consider the conditions which imply that $f_{A_f} = f$ for a function f satisfying (*F1*), (*F2*) and (*F3*) in Theorem 3.5. We first prove the following lemmas.

Lemma 4.3. *For a monotone operator $A \subset E \times E^*$ with a nonempty domain $C = D(A)$, let f_A be a function induced by A with* (11)*. Then,*

(*F5*) *$f_A(x, ty + (1-t)x) = tf_A(x,y)$ for $x \in C$, $y \in E$, $t > 0$, and $f_A(v,y) = -\infty$ for $v \notin C$ and $y \in E$.*

Proof. Let $x \in C$, $y \in E$, and $t > 0$. Then we have that

$$\begin{aligned} f_A(x, ty + (1-t)x) &= \sup_{z^* \in Ax} \langle ty + (1-t)x - x, z^* \rangle \\ &= \sup_{z^* \in Ax} t \langle y - x, z^* \rangle \\ &= t \sup_{z^* \in Ax} \langle y - x, z^* \rangle \\ &= t f_A(x, y). \end{aligned}$$

It is obvious from the definition of f_A that $f_A(v, y) = -\infty$ for $v \notin C$ and $y \in E$. $\square$

Lemma 4.4. *Let f be a function of $E \times E$ into $[-\infty, \infty]$ and C a nonempty subset of E satisfying (F1), (F2) and (F3) in Theorem 3.5. Let A_f be a monotone operator on E induced by f with (8). Suppose the following:*

(F6) $f(y, v) \geq \langle v - y, w^* \rangle$ for every $v \in E$ whenever $y \in C$, $w^* \in E^*$, and $f(y, v) \geq \langle v - y, w^* \rangle$ for every $v \in C$.

Then, for $x \in C$ and $z^* \in E^*$, $z^* \in A_f x$ if and only if $\langle x, z^* \rangle = \sup_{v \in E}(\langle v, z^* \rangle - f(x, v))$.

Proof. Suppose $z^* \in A_f x$. Then we have that $f(x, v) \geq \langle v - x, z^* \rangle$ for all $v \in C$. By assumption, it follows that $f(x, v) \geq \langle v - x, z^* \rangle$ for all $v \in E$, or equivalently that

$$\langle x, z^* \rangle \geq \sup_{v \in E}(\langle v, z^* \rangle - f(x, v)).$$

On the other hand, since $f(x, x) = 0$, it always holds that

$$\sup_{v \in E}(\langle v, z^* \rangle - f(x, v)) \geq \langle x, z^* \rangle - f(x, x) = \langle x, z^* \rangle.$$

Hence we have that $\langle x, z^* \rangle = \sup_{v \in E}(\langle v, z^* \rangle - f(x, v))$. Next, we suppose that $\langle x, z^* \rangle = \sup_{v \in E}(\langle v, z^* \rangle - f(x, v))$. Then it follows that

$$\langle x, z^* \rangle = \sup_{v \in E}(\langle v, z^* \rangle - f(x, v)) \geq \sup_{v \in C}(\langle v, z^* \rangle - f(x, v))$$

and thus $\langle x, z^* \rangle \geq \langle v, z^* \rangle - f(x, v)$ for all $v \in C$. Therefore we obtain that $f(x, v) \geq \langle v - x, z^* \rangle$ for all $v \in C$, which implies that $z^* \in A_f x$. This completes the proof. $\square$

Theorem 4.5. *Let f be a function of $E \times E$ into $[-\infty, \infty]$, C a nonempty convex subset of E, and A_f an operator on E induced by f with (8). Suppose the conditions (F1), (F2) and (F3) in Theorem 3.5 and $C = D(A_f)$. Then $f_{A_f} = f$ if and only if the following conditions hold:*

(F5) $f(x, ty + (1-t)x) = t f(x, y)$ for $x \in C$, $y \in E$, $t > 0$, and $f(v, y) = -\infty$ for $v \notin C$ and $y \in E$;

(F6) $f(y, v) \geq \langle v - y, w^* \rangle$ for every $v \in E$ whenever $y \in C$, $w^* \in E^*$, and $f(y, v) \geq \langle v - y, w^* \rangle$ for every $v \in C$,

where f_{A_f} is a function of $E \times E$ into $[-\infty, \infty]$ defined by

$$f_{A_f}(x, y) = \begin{cases} \sup\{\langle y - x, x^* \rangle : x^* \in A_f x\} & \text{if } x \in C; \\ -\infty & \text{if } x \notin C. \end{cases}$$

Proof. Suppose that $f_{A_f} = f$. Lemma 4.3 shows that (*F5*) holds. For (*F6*), suppose $x \in C$ and $z^* \in E^*$ satisfy $f(x,y) \geq \langle y-x, z^* \rangle$ for every $y \in C$. It follows from the definition of A_f that $z^* \in A_f x$. Thus we have that

$$f(x,y) = f_{A_f}(x,y) = \sup_{w^* \in A_f x} \langle y-x, w^* \rangle \geq \langle y-x, z^* \rangle$$

for every $y \in E$. Hence (*F6*) holds.

Next, let us suppose that (*F5*) and (*F6*) hold. Let $x \in C$ and define a function $g_x \colon E \to (-\infty, \infty]$ by $g_x(y) = f(x,y)$ for $y \in E$. Using (*F5*), we have that

$$\begin{aligned}
\inf_{t>0} \frac{g_x(x+t(y-x))}{t} &= \inf_{t>0} \frac{f(x, ty+(1-t)x)}{t} \\
&= \inf_{t>0} \frac{tf(x,y)}{t} \\
&= f(x,y)
\end{aligned}$$

for every $y \in E$. Define $h_x \colon E \to (-\infty, \infty]$ by $h_x(v) = \inf_{t>0} g_x(x+tv)/t$ for $v \in E$. Then, since $h_x(v) = f(x, v+x)$ and f satisfies the conditions (*F1*) and (*F3*) in Theorem 3.5, one has that h_x is proper, lower semicontinuous and convex. For $z^* \in E^*$, it follows that

$$\begin{aligned}
h_x^*(z^*) &= \sup_{v \in E} (\langle v, z^* \rangle - h_x(v)) \\
&= \sup_{v \in E} \left(\langle v, z^* \rangle - \inf_{t>0} \frac{g_x(x+tv)}{t} \right) \\
&= \sup_{v \in E} \sup_{t>0} \left(\langle v, z^* \rangle - \frac{g_x(x+tv)}{t} \right) \\
&= \sup_{t>0} \sup_{v \in E} \frac{1}{t} (t \langle v, z^* \rangle - g_x(x+tv)) \\
&= \sup_{t>0} \frac{1}{t} \sup_{v \in E} (\langle x+tv, z^* \rangle - g_x(x+tv) - \langle x, z^* \rangle) \\
&= \sup_{t>0} \frac{1}{t} \left(\sup_{u \in E} (\langle u, z^* \rangle - g_x(u)) - \langle x, z^* \rangle \right).
\end{aligned}$$

Since $\sup_{u \in E}(\langle u, z^* \rangle - g_x(u)) - \langle x, z^* \rangle \geq \langle x, z^* \rangle - g_x(x) - \langle x, z^* \rangle = 0$, it follows that $h_x^*(z^*) \geq 0$. By Lemma 4.4 with the condition (*F6*), we get that

$$h_x^*(z^*) = i_{A_f x}(z^*) = \begin{cases} 0 & \text{if } z^* \in A_f x; \\ \infty & \text{if } z^* \notin A_f x. \end{cases}$$

Hence we have that

$$\begin{aligned} f(x,y) &= h_x(y-x) \\ &= h_x^{**}(y-x) \\ &= \sup_{z^*\in E^*} (\langle y-x, z^*\rangle - h_x^*(z^*)) \\ &= \sup_{z^*\in E^*} (\langle y-x, z^*\rangle - i_{A_f x}(z^*)) \\ &= \sup_{z^*\in A_f x} \langle y-x, z^*\rangle \\ &= f_{A_f}(x,y), \end{aligned}$$

for every $y \in E$. On the other hand, suppose that $x \notin C$. Then we get from the definition of f_{A_f} and *(F5)* that

$$f_{A_f}(x,y) = -\infty = f(x,y),$$

for every $y \in E$. Hence we get that $f_{A_f} = f$, which completes the proof. □

Lemma 4.6. *Let E be a reflexive Banach space. For a maximal monotone operator $A \subset E \times E^*$ with a closed convex effective domain C, let f_A be a function induced by A with* (11). *Then, the following holds:*

(F6) *$f_A(y,v) \geq \langle v-y, w^*\rangle$ for every $v \in E$ whenever $y \in C$, $w^* \in E^*$, and $f_A(y,v) \geq \langle v-y, w^*\rangle$ for every $v \in C$.*

Proof. Let $y \in C$ and $w^* \in E^*$ and suppose that $f_A(y,v) \geq \langle v-y, w^*\rangle$ for every $v \in C$. Then we have that $w^* \in A_{f_A}y$ by the definition of A_{f_A}. Using Theorem 4.1, we obtain that $A = A_{f_A}$ and hence $w^* \in Ay$. Therefore we have that

$$f_A(y,v) = \sup_{z^*\in Ay} \langle v-y, z^*\rangle \geq \langle v-y, w^*\rangle$$

for every $v \in E$. □

Now we summarize our results.

Theorem 4.7. *Let C be a nonempty closed convex subset of a reflexive Banach space E. Let $\mathcal{M}$ be the family of all maximal monotone operators with an effective domain C, and let $\mathcal{F}$ be the family of all functions of $E \times E$ into $[-\infty, \infty]$ satisfying the following conditions:*

(F1) *$f(x,x) = 0$ for all $x \in C$;*
(F2) *f is monotone with respect to C;*
(F3) *$f(x,\cdot)$ is lower semicontinuous and convex for all $x \in C$;*
(F4) *f is maximal monotone with respect to C;*
(F5) *$f(x, ty+(1-t)x) = tf(x,y)$ for $x \in C$, $y \in E$, $t > 0$, and $f(v,y) = -\infty$ for $v \notin C$ and $y \in E$;*
(F6) *$f(y,v) \geq \langle v-y, w^*\rangle$ for every $v \in E$ whenever $y \in C$, $w^* \in E^*$, and $f(y,v) \geq \langle v-y, w^*\rangle$ for every $v \in C$.*

Define a function $\Phi\colon \mathcal{M} \to \mathcal{F}$ by

$$\Phi(A) = f_A$$

for $A \in \mathcal{M}$, where f_A is a function induced by A with (11). *Then Φ is an injection. Moreover, let*

$$\mathcal{F}_0 = \{f \in \mathcal{F} : D(A_f) = C\}.$$

Then, Φ is a bijection of $\mathcal{M}$ onto $\mathcal{F}_0$.

Proof. Theorem 3.8, Lemma 4.3, and Lemma 4.6 guarantee that $f_A \in \mathcal{F}$ and hence Φ is well defined. Let $A_1, A_2 \in \mathcal{M}$ and suppose that $\Phi(A_1) = \Phi(A_2)$. Then, by Theorem 4.1, we have that

$$A_1 = A_{f_{A_1}} = A_{\Phi(A_1)} = A_{\Phi(A_2)} = A_{f_{A_2}} = A_2.$$

Hence Φ is a one-to-one mapping. We also have that $\Phi(A) \in \mathcal{F}_0$ for every $A \in \mathcal{M}$ since $D(A_{\Phi(A)}) = D(A) = C$.

On the other hand, let $f \in \mathcal{F}_0$. Then, it follows from Theorem 3.5 that A_f is monotone. Moreover, since f is maximal monotone, A_f is also maximal monotone by Corollary 3.7. By assumption, $D(A_f) = C$. Therefore one has that $A_f \in \mathcal{M}$. Theorem 4.5 implies that

$$f = f_{A_f} = \Phi(A_f)$$

and hence Φ maps $\mathcal{M}$ onto $\mathcal{F}_0$. □

In addition to the theorem above, suppose that E is smooth and strictly convex. Then, it follows from Theorem 3.8 that Φ preserves the resolvents. Namely, the resolvent of A coincides with the resolvent of $\Phi(A)$ for each $A \in \mathcal{M}$.

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