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# On Generalized Differentials, Viability and Invariance of Differential Inclusions*

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Forward viability and invariance of time-dependent differential inclusions are studied with the aid of generalized differentials. Contingent derivative is compared with a newer concept of generalized differential quotient. It is shown that the latter is more suitable for expressing criteria of viability and invariance, as it better describes the directions tangent to invariant trajectories of differential inclusions. The concept of generalized differential quotient is related to Cellina continuously approximable set-valued functions whose properties are used.

*Keywords:* Differential inclusion, viability, invariance, Cellina continuously approximable multifunction, contingent derivative, generalized differential quotient

## 1. Introduction

In 1942 M. Nagumo [15] formulated necessary and sufficient conditions under which all trajectories of a vector field starting at points of a closed set $K_0$ (constraint set) stay in this set. If we replace the vector field by an orientor field, uniqueness of trajectories is lost and one may be interested in two different possibilities: either all trajectories starting from all points of $K_0$ stay in $K_0$ (forward invariance) or for each point of $K_0$ at least one trajectory starting at this point stays in $K_0$ (viability). Another extension of the problem leads to a differential inclusion with the right-hand side depending on time and the closed set changing in time as well.

In the original result of Nagumo and in its later generalizations [1, 3, 9], the tangent cone to the set $K_0$ or to the graph of the time-dependent multifuction $K$ is used to express the conditions for viability or invariance. In the latter case the tangent cone is equal to the contingent derivative of the multifuction $K$. This concept is one of the generalizations of the ordinary derivative of a (single-valued) function. Contingent derivative may be applied to functions that are not differentiable in the ordinary sense or to multifunctions. It seems however that the contingent derivative may contain directions that are not essential in the problems of viability or invariance. Such directions appear as limits of sequences of points of the graph, while in these problems important directions come from (absolutely continuous) trajectories of differential equations or differential inclusions. In [13], for the

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first time, we applied Sussmann's generalized differential quotients to express sufficient conditions for viability of time-dependent differential inclusions. In the very concept of generalized differential quotient, approximation of a multifunction by continuous functions plays a fundamental role thus bringing this tool closer to trajectories and their properties.

Set-valued maps that are called here *Cellina continuously approximable* (CCA) were introduced and studied by Arrigo Cellina in the 1960s [4, 5, 6] and later rediscovered by Hector Sussmann in his papers on generalized differentials [16, 17, 18] (they were initially called *regular*).

We recall here the main properties of CCA set-valued maps and generalized differential quotients (GDQs). Contrary to the contingent derivative, a generalized differential quotient of a set valued map (at some point of its graph) is not unique. Though in some cases this lack of uniqueness may be convenient, we see this rather as a flaw. As in [13] we use the related concept of SGDQ and show its connections with contingent derivative.

The main results of this paper give necessary conditions for viability and invariance of a differential inclusion $\dot{x}(t) \in F(t, x(t))$ with respect to a constraint multifunction $K$. Under some basic assumptions on $F$, we show that both viability and invariance imply GDQ-differentiability of $K$ at almost every $t$ and every $x \in K(t)$, in the direction of $\mathbb{R}_+$. This means that the corresponding SGDQ of $K$ is not empty. Moreover, viability implies that $F(t, x)$ has nonempty intersection with SGDQ of $F$ at $(t, x)$ in the direction of $\mathbb{R}_+$, for almost every $t$ and every $x \in K(t)$. Similarly, invariance implies that $F(t, x)$ is contained in SGDQ of $F$ at $(t, x)$ in the direction of $\mathbb{R}_+$, for almost every $t$ and every $x \in K(t)$. The key point of the proof is the fact that the orientor field evaluated at some $(t, x)$ is actually a generalized differential quotient of $K$ at $(t, x)$.

Similar theorems were formulated earlier with SGDQ replaced by the contingent derivative evaluated at 1. As SGDQ is in general smaller than this set, we get a sharper estimate of the admissible directions. We give an example in which the contingent derivative of $K$ produces many directions that, because of the form of $K$, cannot be tangent to invariant trajectories of the orientor field. On the other hand, SGDQ contains only good directions.

We also show that the conditions shown as necessary for viability and invariance become sufficient if we add extra assumptions on $F$ and $K$. To achieve viability we assume a stronger measurability property of $F$ and to get invariance we impose a Lipschitz property. These results, however, are simple consequences of earlier works.

## 2. Preliminaries

By a set-valued map (multifunction) we mean a triple $F = (X, Y, G)$ such that $X$ and $Y$ are sets and $G$ is a subset of $X \times Y$. The sets $X,\ Y,\ G$ are called, respectively, the *source*, *target* and *graph* of $F$, and we write $X = So(F)$, $Y = Ta(F)$, $G = Gr(F)$. For $x \in So(F)$ we write $F(x) = \{y : (x, y) \in Gr(F)\}$ (it can happen that $F(x) = \emptyset$ for $x \in So(F)$). The sets $Do(F) = \{x \in So(F) : F(x) \neq \emptyset\}$, $Im(F) = \bigcup_{x \in So(F)} F(x)$, are called, respectively, the *domain* and *image* of $F$. If $F = (X, Y, G)$ is a set-valued map, we say that $F$ is a set-valued map from $X$ to $Y$, and write $F : X \twoheadrightarrow Y$. We use $SVM(X, Y)$ to denote the set of all set-valued maps from $X$ to $Y$. We reserve capital letters for set-valued maps and small ones for ordinary (single-valued and everywhere defined) maps.

If $X$ is a metric space supplied with a metric $d$ and $K \subseteq X$, then $\operatorname{dist}(x, K) := \inf_{y \in K} d(x, y)$

denotes the *distance from $x$ to $K$*, where we set $\text{dist}(x, \emptyset) := +\infty$. For $\varepsilon > 0$, the *ball of radius $\varepsilon$ around the set $K$* (or *$\varepsilon$-neighborhood of $K$*) is $B(K, \varepsilon) := K^\varepsilon := \{x \in X : \text{dist}(x, K) < \varepsilon\}$. The balls $B(K, \varepsilon)$ are neighborhoods of $K$. When $K$ is compact, each neighborhood of $K$ contains such a ball around $K$. If $X$ is a normed space, $B$ will denote the unit ball centered at the origin.

Let $X$ and $Y$ be metric spaces. We say that a sequence of multifunctions $F_n : X \twoheadrightarrow Y$ *graph converges* to a multifunction $F : X \twoheadrightarrow Y$, and write $F_n \xrightarrow{gr} F$, if

$$\lim_{n\to\infty} \triangle(Gr(F_n), Gr(F)) = 0$$

where

$$\triangle(A, B) = \sup\{\text{dist}(q, B) : q \in A\}.$$

Note that $\triangle(A, B)$ in not the usual symmetric distance between two sets. Indeed, if $A \subset B$ then $\triangle(A, B) = 0$ while $\triangle(B, A) \neq 0$.

We say that a set-valued map $F : X \twoheadrightarrow Y$ is upper semicontinuous (abbr. u.s.c.) at $\bar{x} \in Do(F)$ if and only if for any neighborhood $U$ of $F(x)$ there exists $\delta > 0$ such that for every $x \in B(\bar{x}, \delta)$, $F(x) \subset U$.

**Definition 2.1 ([17]).** Let $X$ and $Y$ be metric spaces. A set-valued map $F : X \twoheadrightarrow Y$ is *Cellina continuously approximable (abbreviated 'CCA')* if for every compact subset $K$ of $X$

(1) $Gr(F\,|_K)$ is compact;
(2) there exists a sequence $\{f_j\}_{j=1}^\infty$ of single-valued continuous maps $f_j : K \to Y$ such that $f_j \xrightarrow{gr} F\,|_K$.

We use $CCA(X, Y)$ to denote the set of all CCA set-valued maps from $X$ to $Y$.

When $f : X \to Y$ is a single-valued map, then $f$ belongs to $CCA(X, Y)$ if and only if $f$ is continuous.

The following results characterize CCA set-valued maps. Convexity plays here an important role.

**Theorem 2.2 ([17]).** *Assume that $K$ is a compact metric space, $Y$ is a normed space, and $C$ is a convex subset of $Y$. Let $F \in SVM(K, C)$ be a set-valued map such that the graph of $F$ is compact and the value $F(x)$ is a nonempty convex set for every $x \in K$. Then $F$ is CCA as a set-valued map from $K$ to $C$.*

**Theorem 2.3 ([4]).** *Let $X$ be a metric space, $Y$ a metric locally convex space and let $F \in SVM(X, Y)$ with convex closed values be u.s.c. such that the range $F(X) = \bigcup_{x\in X} F(x)$ is totally bounded. Then for every $\varepsilon > 0$ there exists a continuous single-valued mapping $f : X \to coF(X) \cap B(F(X), \varepsilon)$ such that $\triangle(Gr(f), Gr(F)) < \varepsilon$.*

The last theorem implies the following.

**Theorem 2.4 ([17]).** *Assume that $X$ is a metric space, $Y$ is a normed space, and $C$ is a convex subset of $Y$. Let $F \in SVM(X, C)$ be an upper semicontinuous set-valued map with nonempty compact convex values. Then $F \in CCA(X, C)$.*

**Theorem 2.5 ([17]).** *Assume that $X$, $Y$, $Z$ are metric spaces. Let $F \in CCA(X,Y)$, $G \in CCA(Y,Z)$. Then the composite map $G \circ F$ belongs to $CCA(X;Z)$.*

**Example 2.6.** Consider $F : \mathbb{R} \twoheadrightarrow \mathbb{R}$ such that

$$F(x) = \begin{cases} -1 & \text{if } x < 0 \\ [-1,1] & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

To show that $F$ is CCA it is enough to approximate $F$, in the graph sense, by a piecewise linear single-valued function

$$f(x) = \begin{cases} -1 & \text{if } x < -\varepsilon \\ \frac{x}{\varepsilon} & \text{if } -\varepsilon \le x \le \varepsilon \\ 1 & \text{if } x > \varepsilon \end{cases}$$

where $\varepsilon$ is sufficiently small.

## 3. Generalized differentials

Let us start with the definition of directional generalized differential quotients.

**Definition 3.1 ([17]).** Let $m, n \in \mathbb{N}$, $F : \mathbb{R}^m \twoheadrightarrow \mathbb{R}^n$ be a set-valued map, $\overline{x} \in \mathbb{R}^m$, $\overline{y} \in F(\overline{x})$ and let $\Lambda$ be a nonempty compact subset of $\mathbb{R}^{n\times m}$ (then an element of $\Lambda$ is an $n \times m$ matrix). Let $S$ be a subset of $\mathbb{R}^m$. We say that $\Lambda$ is a *generalized differential quotient (gdq) of $F$ at $(\overline{x}, \overline{y})$ in the direction $S$*, and write $\Lambda \in GDQ(F; \overline{x}, \overline{y}; S)$, if for every positive real number $\delta$ there exist $U, G$ such that

(1) $U$ is a compact neighborhood of 0 in $\mathbb{R}^m$ and $U \cap S$ is compact;
(2) $G$ is a CCA set-valued map from $\overline{x} + U \cap S$ to the $\delta$-neighborhood $\Lambda^\delta$ of $\Lambda$ in $\mathbb{R}^{n\times m}$;
(3) $G(x) \cdot (x - \overline{x}) \subseteq F(x) - \overline{y}$ for every $x - \overline{x} \in U \cap S$.

For $S = \mathbb{R}^m$ we write $\Lambda \in GDQ(F; \overline{x}, \overline{y})$ and say that $\Lambda$ is a generalized differential quotient of $F$ at $(\overline{x}, \overline{y})$.

Observe that gdqs are not unique. If $\Lambda \in GDQ(F; \overline{x}, \overline{y}; S)$, then for any compact overset $\Lambda'$ of $\Lambda$ also $\Lambda' \in GDQ(F; \overline{x}, \overline{y}; S)$.

We say that $F : \mathbb{R}^m \twoheadrightarrow \mathbb{R}^n$ is *GDQ-differentiable at $(\overline{x}, \overline{y})$ in the direction $S$* if $GDQ(F; \overline{x}, \overline{y}; S)$ is not empty.

As elements of $GDQ(F; \overline{x}, \overline{y}; S)$ are sets, $GDQ(F; \overline{x}, \overline{y}; S)$ is a partially ordered set with respect to the inclusion relation.

**Definition 3.2.** Let $F$ be GDQ-differentiable at $(\overline{x}, \overline{y})$ in the direction $S$. A *minimal gdq* of $F$ at $(\overline{x}, \overline{y})$ in the direction $S$ is a minimal element of $GDQ(F; \overline{x}, \overline{y}; S)$ with respect to inclusion.

**Theorem 3.3 (Minimality Theorem, [14]).** *If $GDQ(F; \overline{x}, \overline{y}; S)$ is not empty, then there exists in this set at least one minimal gdq. Every element $\Lambda$ of $GDQ(F; \overline{x}, \overline{y}; S)$ contains a minimal element of $GDQ(F; \overline{x}, \overline{y}; S)$.*

We use $minGDQ(F;x,y;S)$ to denote the collection of all minimal gdqs of $F$ at $(x,y)$ in the direction $S$.

**Example 3.4.** Consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=|x|$. Then one can show that $[-1,1]\in GDQ(f;0,0)$ and that this is the minimal gdq. This interval is also the Clarke generalized gradient of $f$ at 0 (see e.g. [7]). However for $f(x)=x^2\sin\frac{1}{x}$ when $x\neq 0$ and $f(0)=0$ the same interval is again the Clarke generalized gradient of $f$ at 0, while the minimal gdq is just the ordinary derivative at 0, equal 0.

**Example 3.5.** Let $F:\mathbb{R}\twoheadrightarrow\mathbb{R}$ be the set-valued map defined by

$$F(x)=\begin{cases}[-|x|,|x|] & \text{if } x\neq 0\\ \{0\} & \text{if } x=0.\end{cases}$$

Then any singleton $\{a\}$ for $a\in[-1,1]$ is a minimal gdq of $F$ at $(0,0)$.

Let $X=\mathbb{R}^m$ and $Y=\mathbb{R}^n$.

**Definition 3.6.** We say that a set-valued map $F:X\twoheadrightarrow Y$ is *Lipschitz at the point* $x_0$ if there exists $L\geq 0$ and a neighborhood $N(x_0)$ of $x_0$ such that $Do(F)=N(x_0)$ and

$$\forall x\in N(x_0),\quad F(x)\subseteq F(x_0)+L||x-x_0||B$$

where $B$ is a unit ball in $Y$.

**Definition 3.7.** We say that a set-valued map $\tilde{F}:X\twoheadrightarrow Y$ is a *multiselection* of a set-valued map $F:X\twoheadrightarrow Y$ if for every $x\in X$, $\tilde{F}(x)\subseteq F(x)$.

**Theorem 3.8 ([13]).** *Let $F:\mathbb{R}\twoheadrightarrow\mathbb{R}^n$ be a set-valued map. Then $F$ is GDQ-differentiable at $(x_0,y_0)$ if and only if there is a compact neighborhood $U$ of $x_0$ such that $F\,|_U$ has a CCA multiselection $\tilde{F}$ that is Lipschitz at the point $x_0$ and $\tilde{F}(x_0)=y_0$.*

**Corollary 3.9.** *$F:\mathbb{R}\twoheadrightarrow\mathbb{R}^n$ is GDQ-differentiable at $(0,0)$ in the direction $S=\mathbb{R}_+$ iff there is $U=[0,c]$ such that $F\,|_U$ has a CCA multiselection, Lipschitz at $0$ and equal $0$ at $0$.*

**Corollary 3.10.** *Let $F:\mathbb{R}\twoheadrightarrow\mathbb{R}^n$ and $0\in F(0)$. If there is a continuous map $\gamma:[0,c]\to\mathbb{R}^n$ such that $\gamma(0)=0$, $\gamma(t)\in F(t)$ and $\gamma$ has the right-side derivative at $0$, then $\gamma'(0)$ is a minimal gdq of $F$ at $(0,0)$ in the direction of $\mathbb{R}_+$.*

Let $SGDQ(K;t,x;\mathbb{R}_+)$ denote the closure of the union of all minimal gdqs of $K$ at $(t,x)\in GrK$ in the direction $\mathbb{R}_+$. We will apply this concept to characterize viability and invariance of differential inclusions in Section 4.

Let $X$ be a normed space and $C\subset X$. Let us recall that *the contingent cone* (the 'Bouligand cone') to $C$ at $x$ is the set defined by

$$T_C(x)=\left\{w\in X:\liminf_{t\downarrow 0}\frac{\operatorname{dist}(x+tw,C)}{t}=0\right\}.$$

**Definition 3.11 ([2]).** Let $X$ and $Y$ be normed spaces and $F:K\twoheadrightarrow Y$, where $K\subset X$ and $Do(F)=K$. The *contingent derivative* of $F$ at $x_0\in K$ and $y_0\in F(x_0)$, denoted by $DF(x_0, y_0)$, is a set-valued map from $X$ to $Y$ whose graph is the *contingent cone* $T_{Gr(F)}(x_0, y_0)$ to the graph of $F$ at $(x_0, y_0)$.

From the definition of the contingent cone we have

$$v_0 \in DF(x_0, y_0)(u_0) \ \Leftrightarrow\ (u_0, v_0) \in T_{Gr(F)}(x_0, y_0)$$

or equivalently

$$v_0 \in DF(x_0, y_0)(u_0) \ \Leftrightarrow\ \liminf_{h \to 0^+, u \to u_0} \ \mathrm{dist}\left(v_0, \frac{F(x_0 + hu) - y_0}{h}\right) = 0.$$

When $F$ is a locally Lipschitz set-valued map, the definition of the contingent derivative reduces to the following (see e.g. [1])

$$v_0 \in DF(x_0, y_0)(u_0) \ \Leftrightarrow\ \liminf_{h \to 0^+} \ \mathrm{dist}\left(v_0, \frac{F(x_0 + hu_0) - y_0}{h}\right) = 0.$$

**Theorem 3.12 ([13]).** *Let $F : \mathbb{R} \twoheadrightarrow \mathbb{R}^n$, $Do(F) = T \subseteq \mathbb{R}$, be a set-valued map and $y \in F(t)$. Then*

$$\Lambda \in minGDQ(F; t, y; \mathbb{R}_+) \ \Rightarrow\ \Lambda \subseteq DF(t, y)(1).$$

**Remark 3.13.** Similarly, one can show that

$$\Lambda \in minGDQ(F; t, y; \mathbb{R}_-) \ \Rightarrow\ \Lambda \subseteq DF(t, y)(-1).$$

**Corollary 3.14.** *Under assumptions of Theorem* 3.12 *we have the following inclusion*

$$SGDQ(F; t, y; \mathbb{R}_+) \subseteq DF(t, y)(1).$$

**Corollary 3.15.** *Consider $F : \mathbb{R} \twoheadrightarrow \mathbb{R}^n$. If $F$ is GDQ-differentiable at the point $(x, y)$ in the direction of $\mathbb{R}_+$ ($\mathbb{R}_-$), then $DF(t, y)(1) \neq \emptyset$ ($DF(t, y)(-1) \neq \emptyset$).*

The next example shows that the contingent derivative in general is larger than the union of minimal generalized differential quotients.

**Example 3.16.** Consider a set-valued map $F : \mathbb{R} \twoheadrightarrow \mathbb{R}$ defined as follows

$$K(t) = \begin{cases} \{t - \frac{1}{n} \mid n \in \mathbb{N},\ n > \frac{1}{t}\} \cup \{t\} & \text{if } t \neq 0 \\ \{0\} & \text{if } t = 0. \end{cases}$$

Then $DK(0, 0)(1) = [0, 1]$, while $SGDQ(K; 0, 0; \mathbb{R}_+) = \{1\}$.

## 4. Viability and invariance of differential inclusions

Let $K : T \twoheadrightarrow \mathbb{R}^n$, where $Do(K) = T = [0, 1] \subseteq \mathbb{R}$, be a constraint multifunction and $F : [0, 1] \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ be an orientor field (i.e. multivalued vector field). Consider the multivalued Cauchy problem:

$$\begin{cases} \dot{x}(t) \in F(t, x(t)), \\ x(t_0) = x_0 \in K(t_0), t_0 \in [0, 1). \end{cases} \tag{1}$$

By a (forward) *solution* to this problem we mean an absolutely continuous function $x : [t_0, 1] \to \mathbb{R}^n$ that satisfies the inclusion almost everywhere and satisfies the initial condition. The set of all such solutions is denoted by $Sol(F, t_0, x_0)$. The solution is (forward) *invariant* (with respect to $K$) if $x(t) \in K(t)$ for every $t \in [t_0, 1]$.

The differential inclusion $\dot{x}(t) \in F(t, x(t))$ is called (forward) *invariant* (with respect to $K$) if for every $t_0 \in [0, 1)$ and every $x_0 \in K(t_0)$ all solutions of (1) are invariant.

The differential inclusion $\dot{x}(t) \in F(t, x(t))$ is called (forward) *viable* (with respect to $K$) if for every $t_0 \in [0, 1)$ and every $x_0 \in K(t_0)$ there exists an invariant solution to (1).

Observe that if $Sol(F, t_0, x_0)$ is not empty for every $t_0 \in [0, 1)$ and every $x_0 \in K(t_0)$ and the inclusion is invariant then it is also viable. Moreover, if for every $t_0 \in [0, 1)$ and every $x_0 \in K(t_0)$, $Sol(F, t_0, x_0)$ consists of one element, then both properties coincide.

Let $K : T \twoheadrightarrow \mathbb{R}^n$, where $Do(K) = T$, be closed-valued. We say that $K$ is *left absolutely continuous* on $[0, 1]$ if the following holds:

$$\begin{aligned} &\forall \varepsilon > 0,\ \forall \text{ compact } P \subset \mathbb{R}^n,\ \exists \delta > 0,\ \forall I \subset \mathbb{N}, \\ &\forall\ \{t_i, \tau_i : t_i < \tau_i, i \in I\} \text{ with } (t_i, \tau_i) \cap (t_j, \tau_j) = \emptyset \text{ for } i \neq j, \\ &\sum (\tau_i - t_i) \leq \delta \Rightarrow \sum \triangle(K(t_i) \cap P, K(\tau_i)) \leq \varepsilon \end{aligned} \tag{2}$$

where $\mathbb{N}$ is the set of natural numbers.

We get the definition of right absolute continuity and absolute continuity by replacing $\triangle(K(t_i) \cap P, K(\tau_i))$ in (2), respectively, by $\triangle(K(\tau_i) \cap P, K(t_i))$ and $\max\{\triangle(K(t_i) \cap P, K(\tau_i)), \triangle(K(\tau_i) \cap P, K(t_i))\}$.

We gather some assumptions on the multifunction $F : T \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ that will be useful in the sequel.

(A0) For every measurable $\gamma : [0, 1] \to \mathbb{R}^n$ the multifunction $t \twoheadrightarrow F(t, \gamma(t))$ is measurable.
(A1) $t \twoheadrightarrow F(t, x)$ is measurable for every $x \in \mathbb{R}^n$.
(A2) $||F(t, x)|| \leq \mu(t)(1 + ||x||)$ for almost all $t \in [0, 1]$ and all $x \in \mathbb{R}^n$, where $\mu$ is integrable.
(A3) The graphs $Gr(F(t, \cdot))$ are closed for almost all $t \in [0, 1]$.
(A4) The multifunction $x \twoheadrightarrow F(t, x)$ is continuous for almost every $t \in [0, 1]$.
(A5) $\forall\ k > 0\ \exists\ c_k \in L^1(0, 1)$ such that for almost all $t \in [0, 1]$, $F(t, \cdot)$ is $c_k(t)$ Lipschitz on $kB$.

Now we are going to show that viability and invariance are naturally related to generalized differential quotients. For that we need the following lemma.

**Lemma 4.1 ([9, 10]).** *Let* $F : [0, 1] \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ *fulfil the assumptions* (A1), (A2), (A3) *and have nonempty closed convex values. Then there exists a set* $A \subset [0, 1]$ *of full (Lebesgues) measure such that*

$$\forall\ (\tau, x_\tau) \in A \times \mathbb{R}^n,\ \forall\ \epsilon > 0,\ \exists\ \delta > 0,\ \forall\ x \in Sol(F, \tau, x_\tau),\ \forall\ 0 < |h| < \delta$$
$$\frac{1}{h}(x(\tau + h) - x_\tau) \in F(\tau, x_\tau) + \varepsilon B.$$

Similarly as in [9], Lemma 4.1 implies the following:

**Corollary 4.2.** *Let $V$ be a separable metric space and let $f : [0,1] \times \mathbb{R}^n \times V \to \mathbb{R}^n$ satisfy the following conditions:*

1. *for almost all $t \in [0,1]$ the map $(x, v) \mapsto f(t, x, v)$ is continuous;*
2. *for all $(x, v) \in \mathbb{R}^n \times V$ the map $t \mapsto f(t, x, v)$ is measurable;*
3. *there is an integrable function $\mu : [0,1] \to \mathbb{R}$ such that for almost all $t \in [0,1]$ and all $(x, v) \in \mathbb{R}^n \times V$, $||f(t, x, v)|| \leq \mu(t)(1 + ||x||)$.*

*Then there exists a set $A \subset [0,1]$ of full measure such that for every $\tau \in A$, every $v \in V$ and every solution $x(\cdot)$ of $\dot{x}(t) = f(t, x(t), v)$ the derivative $\dot{x}(\tau)$ exists and is equal to $f(\tau, x(\tau), v)$.*

**Theorem 4.3.** *Let $K : [0,1] \twoheadrightarrow \mathbb{R}^n$ and let $F : [0,1] \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ fulfil assumptions* (A1), (A2), (A3) *and have nonempty compact convex values. If the inclusion $\dot{x}(t) \in F(t, x(t))$ is viable with respect to $K$, then for almost every $\tau \in [0,1]$ and every $x_\tau \in K(\tau)$, $K$ is GDQ-differentiable at $(\tau, x_\tau)$ in the direction $\mathbb{R}_+$ and*

$$F(\tau, x_\tau) \in GDQ(K; \tau, x_\tau; \mathbb{R}_+).$$

**Proof.** Consider a set $A \subset [0,1]$ of full measure from Lemma 4.1. Let us assume that $0 \in A$ and $(0,0) \in Gr(K)$. Without loss of generality, we can put $(\tau, x_\tau) = (0,0)$. We have to show that for every $\varepsilon > 0$ there exists $\delta > 0$ and a CCA set-valued map $G : [0,\delta] \twoheadrightarrow F(0,0)^\varepsilon$ such that $G(h)h \subseteq K(h)$ for $h \in [0,\delta]$. Let us choose any $\varepsilon > 0$ and let $\delta > 0$ be a number, dependent on $\varepsilon$, that is guaranteed by Lemma 4.1. Choose an invariant solution to (1) with $t_0 = 0$ and $x(0) = 0$ and define the set-valued map $G$ on $[0,\delta]$ as follows

$$G(h) = \begin{cases} F(0,0), & h = 0 \\ \frac{x(h)}{h}, & h \neq 0. \end{cases}$$

From Lemma 4.1

$$\frac{x(h)}{h} \in F(0,0)^\varepsilon \tag{3}$$

for $h \in [0,\delta]$, which means that $G$ is u.s.c. at $h = 0$. It is also u.s.c. at all $h > 0$ as $h \mapsto \frac{x(h)}{h}$ is a single-valued continuous function. Moreover, (3) implies that $G(h) \subset F(0,0)^\varepsilon$ for $h > 0$, and obviously $G(0) \subset F(0,0)^\varepsilon$.

Since $G$ is u.s.c. with nonempty compact convex values, then, by Theorem 2.4, $G$ is a CCA set-valued map. To finish the proof let us notice that $G(h)h \subseteq K(h)$ for every $h \in [0,\delta]$. Indeed, for $h \neq 0$ we have $x(h) \subseteq K(h)$ by viability and for $h = 0$, $G(0) \cdot 0 \subseteq K(0)$ by assumption that $(0,0) \in Gr(K)$. □

**Remark 4.4.** Invariant solutions are global, i.e. they are defined on the entire interval $[t_0, 1]$. To get the conclusion of Theorem 4.3 we need only local solutions defined on $[t_0, t_0 + \gamma]$, for some $\gamma > 0$, that satisfy $x(t) \in K(t)$.

**Corollary 4.5.** *Let all the assumptions of Theorem* 4.3 *hold. If the inclusion $\dot{x}(t) \in F(t, x(t))$ is viable with respect to $K$, then for almost every $\tau \in [0,1]$ and every $x_\tau \in K(\tau)$ there exists $\Lambda \in minGDQ(K; \tau, x_\tau; \mathbb{R}_+)$ (so $\Lambda \subseteq SGDQ(K; \tau, x_\tau; \mathbb{R}_+)$) such that $\Lambda \subseteq F(\tau, x_\tau)$.*

**Proof.** From Theorem 4.3, $F(\tau, x_\tau)$ is a gdq of $K$ at $(\tau, x_\tau)$. By Theorem 3.3 it contains some minimal gdq $\Lambda$. $\square$

**Corollary 4.6.** *Let all the assumptions of Theorem* 4.3 *hold. If the inclusion* $\dot{x}(t) \in F(t, x(t))$ *is viable with respect to* $K$*, then there exists a set* $A \subseteq [0,1]$ *of full measure such that for every* $t \in A$ *and every* $x \in K(t)$ *we have*

$$F(t,x) \cap SGDQ(K;t,x;\mathbb{R}_+) \neq \emptyset. \tag{4}$$

Corollary 4.6 gives a necessary condition for viability of the differential inclusion. However, to reverse the implication we need slightly stronger assumptions on $K$ and $F$.

**Theorem 4.7 ([13]).** *Assume that* $K : T \twoheadrightarrow \mathbb{R}^n$*, where* $T = [0,1]$*, is an u.s.c. multifunction with nonempty compact values such that for all* $(t,x) \in Gr(K)$*,* $K$ *is GDQ differentiable at* $(t,x)$ *in the direction of* $\mathbb{R}_+$ *and for every* $\varepsilon > 0$ *there exists* $T_\varepsilon \subseteq T$ *such that* $\lambda(T \backslash T_\varepsilon) < \varepsilon$ *and the map* $(t,x) \mapsto SGDQ(K;t,x;\mathbb{R}_+)$ *is u.s.c. on* $(T_\varepsilon \times \mathbb{R}^n) \cap GrK$*. Let* $F : GrK \twoheadrightarrow \mathbb{R}^n$ *with nonempty compact convex values satisfy conditions* (A0), (A2) *and* (A3).
*If* $F(t,x) \cap SGDQ(K;t,x;\mathbb{R}_+) \neq \emptyset$ *for almost every* $t$ *and every* $x \in K(t)$*, then the inclusion* $\dot{x}(t) \in F(t,x(t))$ *is viable with respect to* $K$*.*

To study invariance we represent the multifunction $F$ by a single-valued map depending on an additional variable. To do this we need stronger assumptions on $F$.

**Definition 4.8 ([2]).** Let $U$ be a metric space. We say that a single-valued map

$$f : [0,1] \times \mathbb{R}^n \times U \to \mathbb{R}^n$$

is a Caratheodory parametrization of $F : [0,1] \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ if

(i) $\forall (t,x) \in [0,1] \times \mathbb{R}^n$, $F(t,x) = f(t,x,U)$,
(ii) $\forall (x,u) \in \mathbb{R}^n \times U$, $f(\cdot,x,u)$ is measurable,
(iii) $\forall (t,u) \in [0,1] \times U$, $f(t,\cdot,u)$ is continuous,
(iv) $\forall (t,x) \in [0,1] \times \mathbb{R}^n$, $f(t,x,\cdot)$ is continuous.

Existence of a Caratheodory parametrization is assured by the following:

**Lemma 4.9 ([2]).** *Consider a set-valued map* $F : [0,1] \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ *with nonempty compact convex images. Assume that* $F$ *satisfies* (A1), (A2) *and* (A4).
*Then there exists a Caratheodory parametrization* $f$ *of* $F$ *with* $U = B$ *– the unit ball in* $\mathbb{R}^n$ *– such that:*

$$\exists c > 0, \forall t \in [0,1], \forall x \in \mathbb{R}^n, \forall u,v \in B, ||f(t,x,u) - f(t,x,v)|| \leq c\mu(t)(1 + ||x||)||u - v||$$

Now we can prove a necessary condition of invariance.

**Theorem 4.10.** *Let* $K : [0,1] \twoheadrightarrow \mathbb{R}^n$*,* $Do(K) = [0,1]$ *and let* $F : [0,1] \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ *with nonempty compact convex values fulfil assumption* (A1), (A2) *and* (A4).
*If the inclusion* $\dot{x}(t) \in F(t,x(t))$ *is invariant with respect to* $K$*, then there exists a set* $A \subset [0,1]$ *of full measure such that for every* $t \in A$ *and for every* $x \in K(t)$*,* $K$ *is GDQ-differentiable at* $(t,x)$ *in the direction of* $\mathbb{R}_+$ *and*

$$F(t,x) \subseteq SGDQ(K;t,x;\mathbb{R}_+). \tag{5}$$

**Proof.** Let $f$ be a Caratheodory parametrization of $F$ that exists by Lemma 4.9. Let $A$ be the set that is guaranteed by Corollary 4.2 for the function $f$. We can assume that $1 \notin A$. Let us choose $\tau \in A$, $x_\tau \in K(\tau)$ and $w \in F(\tau, x_\tau)$. We are going to show that $w \in SGDQ(K; \tau, x_\tau; \mathbb{R}_+)$. There exists $v \in B$ such that $w = f(\tau, x_\tau, v)$. Let $x(\cdot)$ satisfy the equation $\dot{x}(t) = f(t, x(t), v)$ for a.e. $t \in [\tau, 1]$ and the initial condition $x(\tau) = x_\tau$. Then $x(\cdot)$ also satisfies the inclusion $\dot{x}(t) \in F(t, x(t))$ a.e. on $[\tau, 1]$ and $\dot{x}(\tau) = w$. Since the inclusion is invariant with respect to $K$, $x(t) \in K(t)$ for all $t \in [\tau, 1]$. Observe that, by Corollary 3.10, $w$ is a minimal gdq of $K$ at the point $(\tau, x_\tau)$ in the direction $\mathbb{R}_+$, so eventually $w \in SGDQ(K; \tau, x_\tau; \mathbb{R}_+)$. □

**Remark 4.11.** In [9] it is shown that a necessary condition for invariance is that

$$F(t, x) \subseteq DK(t, x)(1) \tag{6}$$

holds for all $t$ from a full measure set and all $x \in K(t)$. Since, in general $DK(t, x)(1)$ is larger than $SGDQ(K; t, x; \mathbb{R}_+)$, condition (5) is stronger than condition (6). This means that Theorem 4.10 is stronger than its counterparts using the contingent derivative.

**Remark 4.12.** If in Theorem 4.10 we replace assumption (A4) by (A3) (as was in Theorem 4.3 and Corollary 4.6 concerning viability), its conclusion does not hold in general. Let, for example, $F(t, x) = 0$ for $x \neq 0$ and let $F(t, 0) = [0, 1] \subset \mathbb{R}$. Let $K(t) = \{0\}$. Then for every $t \in [0, 1]$ the graph of $F(t, \cdot)$ is closed, but $F(t, \cdot)$ is not continuous. The inclusion $\dot{x}(t) \in F(t, x(t))$ is invariant with respect to $K$. However, condition (5) does not hold for $x = 0$ and any $t$.

To reverse Theorem 4.10 we need more assumptions on the orientor field $F$ and the constraint multifunction $K$. In particular, we need uniqueness of solutions for the Caratheodory parametrization and for that we require that $F$ satisfies the Lipschitz condition (A5).

**Example 4.13.** Let $K(t) = \{0\} \subset \mathbb{R}$ and $F(t, x) = \sqrt{|x|}$ for $t \in [0, 1]$ and $x \in \mathbb{R}$. Then for any $t \in [0, 1]$, $F(t, 0) = \{0\} = K'(t) = SGDQ(K; t, 0, \mathbb{R}_+)$. However the differential inclusion (in fact, the equation) is not invariant with respect to $K$. This is due to the fact that the equation $\dot{x}(t) = F(t, x(t))$ has nonzero solutions corresponding to the initial condition $x(\tau) = 0$.

**Theorem 4.14.** *Let $K : [0, 1] \twoheadrightarrow \mathbb{R}^n$, $Do(K) = [0, 1]$, be GDQ-differentiable at every point $(t, x) \in Gr(K)$ and be absolutely continuous and with compact values. Let $F : [0, 1] \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ with nonempty compact convex values fulfil assumption* (A1), (A2) *and* (A5). *Then the following conditions are equivalent*

$(i)$ *there exists a set $A \subset [0, 1]$ of full measure such that for every $t \in A$ and for every $x \in K(t)$, $F(t, x) \subseteq SGDQ(K; t, x; \mathbb{R}_+)$*

$(ii)$ *the inclusion $\dot{x}(t) \in F(t, x(t))$ is invariant with respect to $K$.*

**Proof.** Let us assume that $(i)$ is satisfied. By Theorem 3.12 we know that $SGDQ(K; t, x; \mathbb{R}_+) \subseteq DK(t, x)(1)$. Thus, by Theorem 4.10 from [9] we have the thesis.
The implication $(ii) \Rightarrow (i)$ follows from Theorem 4.10. □

The following example illustrates the last theorem and shows that, in some cases, $SGDQ$ is a more adequate generalized differential than the classical contingent derivative.

**Example 4.15.** Let us consider $K : [0,1] \twoheadrightarrow \mathbb{R}$ defined as in Example 3.16. We consider the following Cauchy problem

$$\begin{cases} \dot{x}(t) \in F(t, x(t)), & \text{a.e. on } [0,1] \\ x(0) = 0 \in K(0) \end{cases} \tag{7}$$

where $F : [0,1] \times \mathbb{R} \twoheadrightarrow \mathbb{R}$ and $F(t,x) = 1$. Observe that $SGDQ(K; t, x; \mathbb{R}_+) = 1$. Then, of course, the tangential condition from Theorem 4.14 is fulfilled. Notice that $(t,x) \twoheadrightarrow SGDQ(K; t, x; \mathbb{R}_+)$ is the 'optimal' set-valued differential of $K$ in the sense that the tangent vectors to all invariant solutions to (7) are included in it and there are no superfluous parts. On the other hand, one can observe that for $t > 0$, $DK(t,t)(1) = [-\infty, 1]$ and although the tangential condition $F(t,x) \subset DK(t,x)(1)$ from [9] is fulfilled, the contingent derivative is very big and contains a superfluous part.

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