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An Application of the Krein-Milman Theorem to Bernstein and Markov Inequalities

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Given a trinomial of the form $p(x) = ax^m + bx^n + c$ with $a, b, c \in \mathbb{R}$, we obtain, explicitly, the best possible constant $\mathcal{M}_{m,n}(x)$ in the inequality

$$|p'(x)| \leq \mathcal{M}_{m,n}(x) \cdot \|p\|,$$

where $x \in [-1, 1]$ is fixed and $\|p\|$ is the sup norm of p over $[-1, 1]$. This answers a question to an old problem, first studied by Markov, for a large family of trinomials. We obtain the mappings $\mathcal{M}_{m,n}(x)$ by means of classical convex analysis techniques, in particular, using the Krein-Milman approach.

1. Introduction and background

The aim of this paper is to obtain sharp estimates on the derivative of certain types of trinomials. In a more general setting this problem has been studied since the end of the 19^{th} century. The first reference to deal with this problem is due to the popular chemist D. Mendeleev (the author of the Periodic Table of the Elements). In particular, he was interested in the following problem:

> If $p(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ and we define $\|p\|_{[\alpha,\beta]} := \max\{|p(x)| : x \in [\alpha, \beta]\}$ then, what is the smallest possible constant $M_2(\alpha, \beta) > 0$ so that $|p'(x)| \leq M_2(\alpha, \beta)\|p\|_{[\alpha,\beta]}$ for every $x \in [\alpha, \beta]$?

Using the change of variable $x \to [\alpha+\beta-(\alpha-\beta)x]/2$ it can be seen that $M_2(\alpha, \beta) = 2/(\beta-\alpha)M_2(-1, 1)$, and hence, we can restrict ourselves to polynomials on the standard interval $[-1, 1]$. Mendeleev gave his own solution to the problem proving that $M_2(-1, 1) = 4$. Mendeleev's result was generalized by A. A. Markov for polynomials of arbitrary degree ([5]). In order to understand this generalization we need some notation. If $n \in \mathbb{N}$, $\alpha < \beta$ and p is a polynomial of degree at most n then $\|p\|_{[\alpha,\beta]}$ will denote the sup norm over the

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interval $[\alpha, \beta]$. When working over the interval $[-1, 1]$ we will just write $\|p\|$. $M_n(\alpha, \beta)$ will denote the best constant in the inequality $|p'(x)| \leq M_n(\alpha, \beta)\|p\|_{[\alpha,\beta]}$ for every $x \in [\alpha, \beta]$. When working over the interval $[-1, 1]$ we will just write M_n.

Theorem 1.1 (A. A. Markov, 1889). *If p is a polynomial of degree at most $n \in \mathbb{N}$, then $|p'(x)| \leq n^2\|p\|$. Equality is attained at the end points of $[-1, 1]$ for the n-th Chebyshev polynomial of the first kind, defined by $T_n(x) = \cos(n \arccos x)$ for $x \in [-1, 1]$, in other words, $M_n = n^2$.*

Markov's original paper ([5]) is written in old Russian and it is not readily accessible. Nevertheless an English translation can be found in [6]. For a modern proof we refer to [4]. Markov's estimate for the first derivative is, in fact, an estimate on the norm of the first derivative. His estimate, $M_n = n^2$, can be substantially improved if we fix a point in the interval $[-1, 1]$. If $x \in [-1, 1]$ is fixed, then we define $\mathcal{M}_n(x)$ as the best constant in the following inequality:

$$|p'(x)| \leq \mathcal{M}_n(x) \cdot \|p\|, \tag{1}$$

for any polynomial p of degree less than or equal to n. An estimate on $\mathcal{M}_n(x)$ can be easily derived from the complex version of Markov's Theorem due to S. Bernstein ([2, 3]):

Theorem 1.2 (S. Bernstein, 1912). *If p is a polynomial of degree less than or equal than $n \in \mathbb{N}$, then $|p'(x)| \leq \frac{n}{\sqrt{1-x^2}} \cdot \|p\|$ for every $x \in (-1, 1)$. In other words, $\mathcal{M}_n(x) \leq \frac{n}{\sqrt{1-x^2}}$.*

Bernstein's estimate, $\mathcal{M}_n(x) \leq \frac{n}{\sqrt{1-x^2}}$, coincides with $\mathcal{M}_n(x)$ in n points in $[-1, 1]$, but it is far from being optimal in most of the interval $[-1, 1]$. The importance of Bernstein's estimates rests on the fact that it has been used in modern proofs of Markov's theorem in order to simplify the original proof.

Several attempts to find $\mathcal{M}_n(x)$ for every $n \in \mathbb{N}$ have been performed. The most successful one is due to E. V. Voronovskaja who, in [8], studied the polynomials p with $\|p\| = 1$ and $|p'(x)| = \mathcal{M}_n(x)$, as well as properties of the function $\mathcal{M}_n(x)$. She also produced a method ([9]) to obtain $\mathcal{M}_n(x)$ for every n. It turns out that the so called Zolotarev's polynomials, suitable normalized, are the class of extremal polynomials in Bernstein's inequality (1). However the construction of these polynomials is not explicit in the sense that, for higher values of n, it involves solving elliptic integrals and requires numerical calculus. For a complete account on Bernstein and Markov type inequalities we refer to [4].

In this paper we consider spaces of trinomials with constant term,

$$\mathcal{P}_{m,n}(\mathbb{R}) := \{ax^m + bx^n + c : \ a, b, c \in \mathbb{R}\},$$

and provide explicit formulas for the smallest constant $\mathcal{M}_{m,n}(x)$ in the inequality

$$|p'(x)| \leq \mathcal{M}_{m,n}(x)\|p\|, \tag{2}$$

for any $p \in \mathcal{P}_{m,n}(\mathbb{R})$ and $x \in [-1, 1]$ fixed. The mapping that assigns to each polynomial of the form $ax^m + bx^n + c$ its coordinates (a, b, c) in the basis $\{x^m, x^n, 1\}$ of $\mathcal{P}_{m,n}(\mathbb{R})$ is a linear isomorphism that lets us identify $\mathcal{P}_{m,n}(\mathbb{R})$ with $\mathbb{R}^3$. We will, then, denote the norm

$\|\cdot\|_{m,n}$, defined on $\mathbb{R}^3$, by

$$\|(a,b,c)\|_{m,n} := \max_{x\in[-1,1]} |ax^m + bx^n + c|.$$

The technique we will use in order to obtain these sharp functions $\mathcal{M}_{m,n}(x)$ will rely on the following well known consequence of Krein-Milman's Theorem:

Remark 1.3. If C is a convex body in a Banach space and $f : C \to \mathbb{R}$ is a convex function that attains its maximum, then there is an extreme point $e \in C$ so that $f(e) = \max\{f(x) : x \in C\}$.

Notice that dividing (2) by $\|p\|$ we can restrict attention to polynomials in the unit ball of the space $(\mathbb{R}^3, \|\cdot\|_{m,n})$, that we denote by $\mathsf{B}_{m,n}$. Thus, to obtain the optimal $\mathcal{M}_{m,n}(x)$, it suffices to work with the convex function $\mathsf{B}_{m,n} \ni p \mapsto |p'(x)| \in \mathbb{R}$ and the extreme points of $\mathsf{B}_{m,n}$, characterized in [7]. This is exactly what we will do. Also, $\mathsf{S}_{m,n}$ will denote the unit sphere of $(\mathbb{R}^3, \|\cdot\|_{m,n})$. The rest of the notation will be rather usual and, from now on and unless mentioned otherwise, we will only work over the interval $[-1,1]$.

2. $\mathcal{M}_{m,n}(x)$ for $m, n \in \mathbb{N}$ odd

This section will provide an explicit formula for the smallest constant $\mathcal{M}_{m,n}(x)$ in the inequality

$$|p'(x)| \le \mathcal{M}_{m,n}(x)\|p\|,$$

for $m > n$ odd natural numbers and every $p \in \mathcal{P}_{m,n}(\mathbb{R})$. In order to achieve this goal, we will need a series of technical lemmas, related to extreme points of the unit ball of $(\mathbb{R}^3, \|\cdot\|_{m,n})$, and certain inequalities.

Lemma 2.1 (Muñoz-Fernández and Seoane-Sepúlveda, [7]). *If $m, n \in \mathbb{N}$ are odd numbers with $m > n$ then the equation*

$$|n + mx| = (m-n)|x|^{\frac{m}{m-n}} \tag{3}$$

has only three roots, one at $x = -1$, another one at a point $\lambda_0 \in (-\frac{n}{m}, 0)$ and a third one at a point $\lambda_1 > 0$.

Remark 2.2. The key in the previous lemma is the value of λ_0 for every $m, n \in \mathbb{N}$ odd. Some values for λ_0 can be obtained using symbolic calculus, for instance it can be proved that $\lambda_0 = -\frac{1}{4}$ when $m = 3$ and $n = 1$,

$$\lambda_0 = \frac{-\sqrt[6]{3}\sqrt[3]{25}(4\sqrt{2}+9\sqrt{3})\sqrt[3]{9+4\sqrt{6}} - 113\sqrt[3]{(9+4\sqrt{6})^2} + \sqrt[3]{15}(49+24\sqrt{6})}{768\sqrt[3]{(9+4\sqrt{6})^2}},$$

when $m = 5$ and $n = 1$, and $\lambda_0 = \frac{4+\sqrt[3]{10}-2\sqrt[3]{100}}{6}$ for $m = 5$ and $n = 3$. More values for λ_0 can be obtained numerically. The reader can find below a table with 21 values for λ_0 with an accuracy of 5 decimal digits.

λ_0	$m=3$	$m=5$	$m=7$	$m=9$	$m=11$	$m=13$
$n=1$	-0.25	-0.13471	-0.09072	-0.06795	-0.05414	-0.04491
$n=3$	—	-0.52145	-0.34142	-0.25000	-0.19558	-0.15983
$n=5$	—	—	-0.65076	-0.47306	-0.36750	-0.29838
$n=7$	—	—	—	-0.72537	-0.56186	-0.45475
$n=9$	—	—	—	—	-0.77380	-0.62538
$n=11$	—	—	—	—	—	-0.80774

The following lemma characterizes the extreme points of $\mathsf{B}_{m,n}$ for $m>n$ odd numbers. This lemma will be crucial when trying to find the sharp constant for Bernstein's and Markov's inequalities.

Lemma 2.3 (Muñoz-Fernández and Seoane-Sepúlveda, [7]). *Let $m>n$ be odd natural numbers, then the extreme points of the unit ball of $(\mathbb{R}^3, \|\cdot\|_{m,n})$ are*

$$\left\{\pm\left(t, -\frac{m}{(m-n)^{\frac{m-n}{m}} n^{\frac{n}{m}}} t^{\frac{n}{m}}, 0\right) : \frac{n}{m-n} \le t \le \frac{n}{n+m\lambda_0}\right\} \cup \{\pm(0,0,1)\},$$

where λ_0 is given in Lemma 2.1.

Now we state the main theorem in this section.

Theorem 2.4. *Let $m>n$ be odd natural numbers and*

$$I_{m,n} = \left[\left(\frac{|\lambda_0| n}{m}\right)^{\frac{1}{m-n}}, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right],$$

then

$$\mathcal{M}_{m,n}(x) = \begin{cases} \dfrac{mn}{n+m\lambda_0} \cdot x^{n-1} \cdot \left|x^{m-n} + \lambda_0\right| & \text{if } |x| \in [0,1] \setminus I_{m,n}, \\ n\left(\dfrac{n}{m}\right)^{\frac{n}{m-n}} \cdot \dfrac{1}{|x|} & \text{if } |x| \in I_{m,n}, \end{cases} \tag{4}$$

where λ_0 is given in Lemma 2.1.

In order to prove Theorem 2.4 we will need a series of technical lemmas, which we proceed to state and prove in what follows.

Lemma 2.5. *Let $m>n$ be odd natural numbers and let λ_0 be the real number from Lemma 2.1. Then, we have*

$$|\lambda_0| \cdot \frac{n}{m} < |\lambda_0| \cdot \frac{1-|\lambda_0|^{\frac{n}{m-n}}}{1-|\lambda_0|^{\frac{m}{m-n}}} < \frac{n}{m}. \tag{5}$$

Proof. To see that $|\lambda_0| \cdot \frac{n}{m} < |\lambda_0| \cdot \frac{1-|\lambda_0|^{\frac{n}{m-n}}}{1-|\lambda_0|^{\frac{m}{m-n}}}$, let us recall that $|\lambda_0| < \frac{n}{m} < 1$, and consider the following inequality:

$$\frac{n}{m} < \frac{1-x^n}{1-x^m}. \tag{6}$$

We will show that (6) holds for every $x \in \left(0, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right)$. Indeed, if $0 < x < 1$, we have that (6) is equivalent to

$$m - n > mx^n - nx^m \tag{7}$$

Now, since $mx^n - nx^m$ is strictly increasing on $(0,1)$ (it is enough to check its derivative), then $mx^n - nx^m$ meet the line $y = m - n$ at, at most, one point, and this happens when $x = 1$. Therefore $mx^n - nx^m$ is either smaller that $m - n$ over $(0,1)$ or it is bigger than $m - n$ over $(0,1)$. It is straightforward to check that the first option is the right one. Thus, (7) holds for every $x \in \left(0, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right)$, and the first inequality is verified. Next, in order to show that $|\lambda_0| \cdot \frac{1-|\lambda_0|^{\frac{n}{m-n}}}{1-|\lambda_0|^{\frac{m}{m-n}}} < \frac{n}{m}$, it suffices to perform some simple calculations and use that $n + m\lambda_0 = (m-n)|\lambda_0|^{\frac{m}{m-n}}$ (equation (3)).

□

Lemma 2.6. *Let $m > n$ be odd natural numbers and let λ_0 be the real number from Lemma 2.1. Then, if*

$$f(x) = \frac{mn}{m-n} \cdot x^{n-1} \cdot |x^{m-n} - 1| \quad \textit{and}$$
$$g(x) = \frac{mn}{n + m\lambda_0} \cdot x^{n-1} \cdot \left|x^{m-n} + \lambda_0\right|,$$

we have $g(x) \geq f(x)$, whenever

$$0 \leq |x| \leq \left(\frac{|\lambda_0| n}{m}\right)^{\frac{1}{m-n}} \quad \textit{or} \quad \left(\frac{n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq 1. \tag{8}$$

Proof. By symmetry we can assume that $x > 0$. It is a simple exercise to check that $f(x)$ and $g(x)$ meet at the points

$$x_1 = \left(\frac{n}{m}\right)^{\frac{1}{m-n}} \quad \text{and} \quad x_2 = \left(|\lambda_0| \cdot \frac{1 - |\lambda_0|^{\frac{n}{m-n}}}{1 - |\lambda_0|^{\frac{m}{m-n}}}\right)^{\frac{1}{m-n}}.$$

Now, using Lemma 2.5 one can see that neither x_1 nor x_2 are in the interior of the intervals $\left[0, \left(\frac{|\lambda_0| n}{m}\right)^{\frac{1}{m-n}}\right]$ or $\left[\left(\frac{n}{m}\right)^{\frac{1}{m-n}}, 1\right]$. Thus, one of the functions f or g must be bigger (or equal) than the other one on each of the previous intervals. Now it is clear that $f(1) < g(1)$. On the other hand $f\left(\left(\frac{|\lambda_0| n}{m}\right)^{\frac{1}{m-n}}\right) < g\left(\left(\frac{|\lambda_0| n}{m}\right)^{\frac{1}{m-n}}\right)$. Indeed, the latter inequality is equivalent to

$$\left|\frac{\lambda_0 n}{m} + 1\right| < \frac{1}{|\lambda_0|^{\frac{m}{m-n}}} \cdot \left|\frac{\lambda_0 n}{m} - \lambda_0\right|.$$

Now, by means of (3) we, equivalently, obtain

$$|\lambda_0|^{\frac{m}{m-n}} < |\lambda_0|,$$

which is obviously true since (Lemma 2.1) $-1 < -\frac{n}{m} < \lambda_0 < 0$. This finishes the proof. □

Lemma 2.7. *Let $m > n$ be odd natural numbers and let λ_0 be the real number from Lemma 2.1. Then, if*

$$f(x) = \frac{mn}{m-n} \cdot x^{n-1} \cdot |x^{m-n} - 1|,$$
$$g(x) = \frac{mn}{n+m\lambda_0} \cdot x^{n-1} \cdot \left|x^{m-n} + \lambda_0\right| \quad \text{and}$$
$$h(x) = n \left(\frac{n}{m}\right)^{\frac{n}{m-n}} \cdot \frac{1}{|x|},$$

we have $h(x) \geq \max\{f(x), g(x)\}$, whenever

$$\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq \left(\frac{n}{m}\right)^{\frac{1}{m-n}}. \tag{9}$$

Proof. We will show that, if

$$\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq \left(\frac{n}{m}\right)^{\frac{1}{m-n}}, \tag{10}$$

then $h(x) \geq f(x)$ and $h(x) \geq g(x)$. Suppose that (10) holds and that (by symmetry) $x > 0$.

(i) That $f(x) \leq h(x)$ follows from the fact that the function $x^n - x^m$ is strictly increasing over the interval $\left(0, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right)$. Indeed, the derivative of $x^n - x^m$ is positive on $\left(0, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right)$, thus, the maximum of $x^n - x^m$ over $\left(0, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right)$ is attained at $x = \left(\frac{n}{m}\right)^{\frac{1}{m-n}}$, and its value is

$$\left(\frac{n}{m}\right)^{\frac{n}{m-n}} - \left(\frac{n}{m}\right)^{\frac{m}{m-n}} = \frac{(m-n)\cdot n^{\frac{n}{m-n}}}{m^{\frac{m}{m-n}}}.$$

Thus,

$$x^n - x^m \leq \frac{(m-n)\cdot n^{\frac{n}{m-n}}}{m^{\frac{m}{m-n}}}, \tag{11}$$

for $\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \leq x \leq \left(\frac{n}{m}\right)^{\frac{1}{m-n}}$. Rearranging terms in inequality (11), one obtains (after some simple calculations) that $f(x) \leq h(x)$.

(ii) In order to see that $g(x) \leq h(x)$ for $\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \leq x \leq \left(\frac{n}{m}\right)^{\frac{1}{m-n}}$, notice that the inequality

$$\frac{mn}{n+m\lambda_0} \cdot x^{n-1} \cdot \left|x^{m-n} + \lambda_0\right| \leq n \left(\frac{n}{m}\right)^{\frac{n}{m-n}} \frac{1}{|x|}$$

is equivalent to

$$\frac{m}{n+m\lambda_0} \cdot |x^m + \lambda_0 x^n| \leq \left(\frac{n}{m}\right)^{\frac{n}{m-n}}. \tag{12}$$

Since the derivative of $x^m + \lambda_0 x^n$ vanishes only at 0 and $\pm\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}}$, then $x^m + \lambda_0 x^n$ is monotone over the interval $\left[\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}}, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right]$, thus we just need to evaluate at the endpoints of this interval. A straightforward calculation shows that (12) holds. □

Now, we are ready to proof the main theorem of this section.

Proof of Theorem 2.4. By definition we have that

$$\mathcal{M}_{m,n}(x) = \sup_{p \in \mathsf{S}_{m,n}} |p'(x)|.$$

Using Remark 1.3 in the previous supremum, we can restrict ourselves to the extreme polynomials of $\mathsf{B}_{m,n}$, which are given in Lemma 2.3. Notice that the contribution of ± 1 to $\mathcal{M}_{m,n}(x)$ is irrelevant. Hence it suffices to consider the polynomials

$$p_t(x) = \pm\left(tx^m - \frac{m}{(m-n)^{\frac{m-n}{m}} n^{\frac{n}{m}}} t^{\frac{n}{m}} x^n\right),$$

where $\frac{n}{m-n} \leq t \leq \frac{n}{n+m\lambda_0}$. Therefore

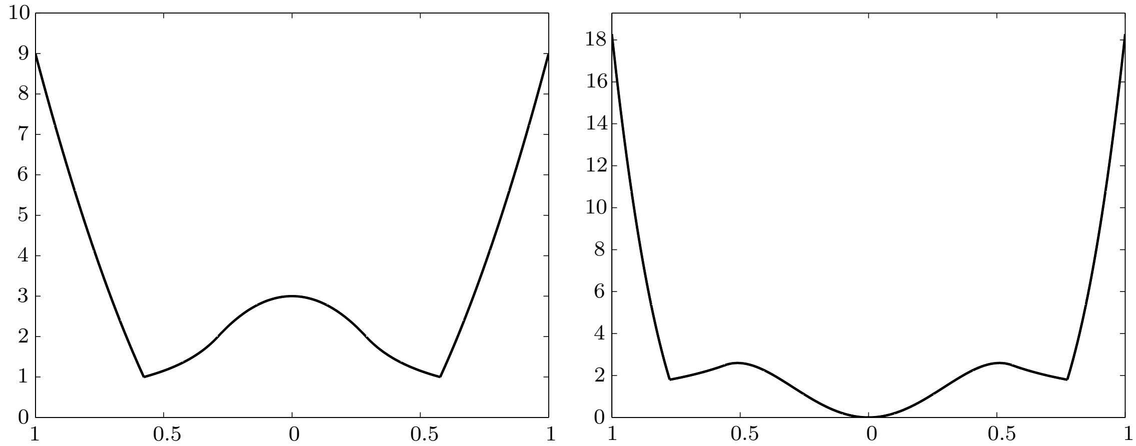
$$\begin{aligned}
\mathcal{M}_{m,n}(x) &= \sup_{\frac{n}{m-n} \leq t \leq \frac{n}{n+m\lambda_0}} \left|\frac{dp_t}{dx}(x)\right| \\
&= \sup_{\frac{n}{m-n} \leq t \leq \frac{n}{n+m\lambda_0}} \left| mtx^{m-1} - \frac{mnt^{\frac{n}{m}}}{(m-n)^{\frac{m-n}{m}} n^{\frac{n}{m}}} x^{n-1} \right| \\
&= \sup_{\frac{n}{m-n} \leq t \leq \frac{n}{n+m\lambda_0}} \left| m \cdot x^{n-1} \cdot \left[tx^{m-n} - \left(\frac{n}{m-n}\right)^{\frac{m-n}{m}} t^{\frac{n}{m}} \right] \right|.
\end{aligned}$$

Now, call $R(t) = m \cdot x^{n-1} \cdot \left[tx^{m-n} - \left(\frac{n}{m-n}\right)^{\frac{m-n}{m}} t^{\frac{n}{m}}\right]$. Clearly, the above supremum will be attained at either $t = \frac{n}{m-n}$, or $t = \frac{n}{n+m\lambda_0}$, or at a critical point of $R(t)$ inside the interval $\left(\frac{n}{m-n}, \frac{n}{n+m\lambda_0}\right)$. A simple calculation shows that the only critical point of $R(t)$ corresponds to $t_0 = \frac{n}{m-n} \cdot \left(\frac{n}{m}\right)^{\frac{m}{m-n}} \cdot \frac{1}{|x|^m}$, which leads to

$$R(t_0) = n \left(\frac{n}{m}\right)^{\frac{n}{m-n}} \frac{1}{|x|}.$$

On the other hand, the condition $\frac{n}{m-n} \leq t_0 \leq \frac{n}{n+m\lambda_0}$ is equivalent to

$$\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq \left(\frac{n}{m}\right)^{\frac{1}{m-n}}.$$



$\mathcal{M}_{3,1}(x)$. In this case $M_{3,1} = 9$. $\mathcal{M}_{5,3}(x)$. Here, $M_{5,3} \simeq 18.278286$.

Figure 2.1: $\mathcal{M}_{m,1}(x)$ with $m \in \mathbb{N}$ odd is similar to $\mathcal{M}_{3,1}(x)$ whereas $\mathcal{M}_{m,n}(x)$ with $n \in \mathbb{N}$ odd and $n > 1$ is similar to $\mathcal{M}_{5,3}(x)$.

Thus, an easy computation gives us the following:

$$\mathcal{M}_{m,n}(x) = \sup_{\frac{n}{m-n} \le t \le \frac{n}{n+m\lambda_0}} |R(t)|$$

$$= \begin{cases} \max\left\{\left|R\left(\frac{n}{m-n}\right)\right|, \left|R\left(\frac{n}{n+m\lambda_0}\right)\right|, n\left(\frac{n}{m}\right)^{\frac{n}{m-n}} \frac{1}{|x|}\right\} & \text{if } \left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \le |x| \le \left(\frac{n}{m}\right)^{\frac{1}{m-n}}, \\ \max\left\{\left|R\left(\frac{n}{m-n}\right)\right|, \left|R\left(\frac{n}{n+m\lambda_0}\right)\right|\right\} & \text{if } |x| \le \left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \text{ or } \left(\frac{n}{m}\right)^{\frac{1}{m-n}} \le |x| \le 1, \end{cases}$$

where

$$R\left(\frac{n}{m-n}\right) = \frac{mnx^{n-1}}{m-n} \cdot |x^{m-n} - 1|$$

and

$$R\left(\frac{n}{n+m\lambda_0}\right) = \frac{mnx^{n-1}}{n+m\lambda_0} \cdot \left|x^{m-n} + \lambda_0\right|.$$

Next, it suffices to use Lemmas 2.6 and 2.7 to conclude the desired result. □

Corollary 2.8. *Let $m > n$ be odd natural numbers, then*

$$M_{m,n} = \mathcal{M}_{m,n}(\pm 1) = \frac{mn(1+\lambda_0)}{n+m\lambda_0}.$$

and equality is attained for the polynomials

$$p(x) = \frac{\pm(nx^m + \lambda_0 mx^n)}{n+m\lambda_0}.$$

3. $\mathcal{M}_{m,n}(x)$ for m odd and n even

Here we will obtain an explicit formula for the best constant $\mathcal{M}_{m,n}(x)$ in the inequality

$$|p'(x)| \le \mathcal{M}_{m,n}(x)\|p\|,$$

for $m \in \mathbb{N}$ even and $n \in \mathbb{N}$ odd. Again, we will make use of the extreme points of $\mathsf{B}_{m,n}$, described in the following lemma.

Lemma 3.1 (Muñoz-Fernández and Seoane-Sepúlveda, [7]). *If $m, n \in \mathbb{N}$ are such that m is odd, n is even and $m > n$, the extreme points of the unit ball of $(\mathbb{R}^3, \|\cdot\|_{m,n})$ are*

$$\{\pm(0,2,-1), \pm(1,1,-1), \pm(1,-1,1), \pm(0,0,1)\}.$$

Now we are ready to give the expression of the function $\mathcal{M}_{m,n}(x)$ in the case treated in this section.

Theorem 3.2. *Let $m, n \in \mathbb{N}$ be such that m is odd, n is even and $m > n$. Then*

$$\mathcal{M}_{m,n}(x) = \begin{cases} 2n|x|^{n-1} & \text{if } |x| \leq \left(\frac{n}{m}\right)^{\frac{1}{m-n}}, \\ mx^{m-1} + n|x|^{n-1} & \text{if } \left(\frac{n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq 1. \end{cases} \tag{13}$$

Proof. If $x \in [-1,1]$, by definition we have that

$$\mathcal{M}_{m,n}(x) = \sup_{p \in \mathsf{S}_{m,n}} |p'(x)|.$$

Applying, again, Remark 1.3 to the previous supremum, it suffices to work just with the extreme polynomials of $\mathsf{B}_{m,n}$, which are given in Lemma 3.1. Notice that the contribution of ± 1 to $\mathcal{M}_{m,n}(x)$ is irrelevant. Hence it suffices to consider the polynomials

$$p_1(x) = \pm(2x^n - 1), \quad p_2(x) = \pm(x^m + x^n - 1) \quad \text{and} \quad p_3(x) = \pm(x^m - x^n + 1).$$

Therefore

$$\begin{aligned} \mathcal{M}_{m,n}(x) &= \max\{|p_1'(x)|, |p_2'(x)|, |p_3'(x)|\} \\ &= \max\{2n|x|^{n-1}, |mx^{m-1} + nx^{n-1}|, |mx^{m-1} - nx^{n-1}|\} \\ &= \max\{2n|x|^{n-1}, mx^{m-1} + n|x|^{n-1}\} \\ &= |x|^{n-1} \max\{2n, m|x|^{m-n} + n\}, \end{aligned}$$

and since $2n \leq m|x|^{m-n} + n$ provided $\left(\frac{n}{m}\right)^{\frac{1}{m-n}} \leq |x|$, the result follows immediately. $\square$

Corollary 3.3. *If $m, n \in \mathbb{N}$ are such that m is odd, n is even and $m > n$, then*

$$M_{m,n} = \mathcal{M}_{m,n}(\pm 1) = m + n,$$

and equality is attained for the polynomials $p(x) = \pm(x^m \pm x^n - 1)$.

4. $\mathcal{M}_{m,n}(x)$ for $m, n \in \mathbb{N}$ with m even

The application of Remark 1.3 in order to estimate $\mathcal{M}_{m,n}(x)$ in the case treated in this section is not as simple as in the previous sections. The reason for this has to be found in the complexity of the extreme points of $\mathsf{B}_{m,n}$ whenever m is even. Indeed, if n is odd then ([7, Theorem 4.6]) the extreme polynomials of $\mathsf{B}_{m,n}$ are

$$\left\{\pm\big(t, \pm\Gamma(t), 1 - t - \Gamma(t)\big) : \frac{n}{m} \leq t \leq 2\right\} \cup \{\pm(0,0,1)\},$$

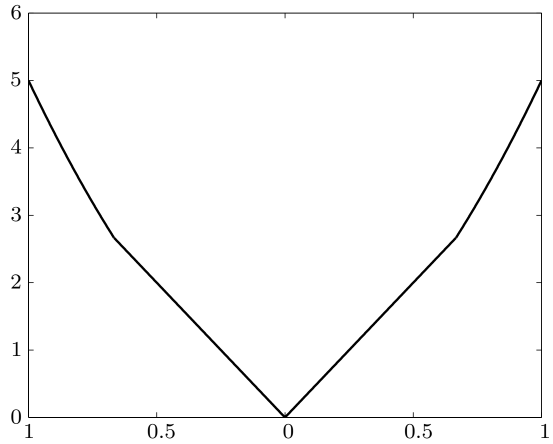


Figure 3.1: $\mathcal{M}_{3,2}(x)$. In the general case $\mathcal{M}_{m,n}(x)$, being m odd and n even, is of similar shape. Here, $M_{3,2} = 5$.

and $\Gamma(t)$ is defined as an implicit function of t in the equation

$$\frac{(m-n)t}{n}\left(\frac{n\Gamma(t)}{mt}\right)^{\frac{m}{m-n}} = 2 - t - \Gamma(t). \tag{14}$$

On the other hand if n is even then according to [7, Theorem 5.5] the extreme polynomials of $\mathsf{B}_{m,n}$ are

$$\{\pm(s, \Lambda(s), -1-s-\Lambda(s))\} \cup \{\pm(t, -\Upsilon(t), 1)\} \cup \{\pm(0,0,1)\},$$

where $s \in [\gamma_0, -2]$, $t \in [-\gamma_1, -\gamma_0]$, $\gamma_0 = -\frac{2}{m-n} \cdot \left(\frac{m^m}{n^n}\right)^{\frac{1}{m-n}}$, $\gamma_1 = \frac{-2n}{m-n}$ and Λ and Υ are defined by

$$\Lambda(s) = -\Gamma(|s|) \quad \text{and} \quad \Upsilon(t) = \left(\frac{2m}{m-n}\right)^{\frac{m-n}{m}} \cdot \left(\frac{mt}{n}\right)^{\frac{n}{m}}.$$

The fact that equation (14) has no explicit solution in general accounts for the difficulty in tackling the problem whenever m is even. However we can give explicit formulas for $\mathcal{M}_{m,n}(x)$ for the cases $m = 2n$ with n being either odd or even. Interestingly, we will obtain very different results.

4.1. $\mathcal{M}_{2n,n}(x)$ for $n \in \mathbb{N}$ odd

We will make use of the extreme points of $(\mathbb{R}^3, \|\cdot\|_{2n,n})$ with $n \in \mathbb{N}$ odd, which are characterize in the following result:

Lemma 4.1 (Aron and Klimek, [1]). *If $n \in \mathbb{N}$ is odd, then the extreme points of the unit ball of $(\mathbb{R}^3, \|\cdot\|_{2n,n})$ are*

$$\pm(0,0,1) \quad \textit{and} \quad \pm(t, \pm 2(\sqrt{2t} - t), 1 + t - 2\sqrt{2t}),$$

where $t \in [1/2, 2]$.

Theorem 4.2. *Let $n \in \mathbb{N}$ be odd. Then*

$$\mathcal{M}_{2n,n}(x) = \begin{cases} \dfrac{n|x|^{n-1}}{1-x^n} & \text{if } |x| \le \dfrac{1}{\sqrt[n]{2}}, \\ 4n|x|^{2n-1} & \text{if } \dfrac{1}{\sqrt[n]{2}} \le |x| \le 1. \end{cases}$$

Proof. If $x \in [-1,1]$ we have that

$$\mathcal{M}_{2n,n}(x) = \sup_{p \in \mathsf{S}_{2n,n}} |p'(x)|.$$

Again, Remark 1.3 helps us since we just need to consider the extreme polynomials of $\mathsf{B}_{2n,n}$, which are given in Lemma 4.1 (also, the contribution of ± 1 to $\mathcal{M}_{2n,n}(x)$ is irrelevant). Thus, it is enough to consider the polynomials

$$p_t(x) = \pm(tx^{2n} \pm 2(\sqrt{2t} - t)x^n + 1 + t - 2\sqrt{2t}),$$

where $t \in [1/2, 2]$. Therefore

$$\begin{aligned} \mathcal{M}_{2n,n}(x) &= \max\left\{\left|\frac{dp_t}{dx}(x)\right| : t \in [1/2, 2]\right\} \\ &= 2nx^{n-1} \max\{|tx^n \pm (\sqrt{2t} - t)| : t \in [1/2, 2]\} \\ &= 2nx^{n-1} \max\{t|x|^n + \sqrt{2t} - t : t \in [1/2, 2]\}. \end{aligned}$$

Now, call $R(t) = t|x|^n + \sqrt{2t} - t$. Clearly, the above supremum will be attained at either $t = \frac{1}{2}$, or $t = 2$, or at a critical point of $R(t)$ inside the interval $\left(\frac{1}{2}, 2\right)$. An easy calculation shows that the only critical point of $R(t)$ corresponds to $t_0 = \frac{1}{2(1-x^n)^2}$, which leads to

$$2nx^{n-1}R(t_0) = \frac{nx^{n-1}}{1-|x|^n}.$$

On the other hand, the condition $\frac{1}{2} \le t_0 \le 2$ is equivalent to $|x| \le \frac{1}{\sqrt[n]{2}}$. Thus, an easy calculation gives us the following:

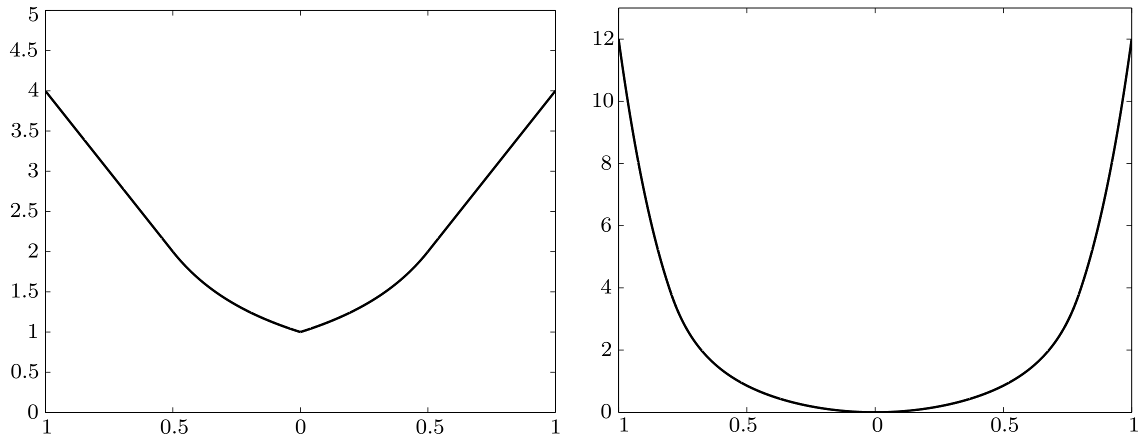
$$\begin{aligned} \mathcal{M}_{2n,n}(x) &= 2nx^{n-1} \sup_{\frac{1}{2} \le t \le 2} |R(t)| \\ &= 2nx^{n-1} \cdot \begin{cases} \max\{|R(1/2)|, |R(2)|, |R(t_0)|\} & \text{if } |x| \le \frac{1}{\sqrt[n]{2}}, \\ \max\{|R(1/2)|, |R(2)|\} & \text{if } \frac{1}{\sqrt[n]{2}} \le |x| \le 1 \end{cases} \\ &= \begin{cases} \max\left\{n|x|^{n-1}(1-|x|^n), 4n|x|^{2n-1}, \frac{n|x|^{n-1}}{1-|x|^n}\right\} & \text{if } |x| \le \frac{1}{\sqrt[n]{2}}, \\ \max\{n|x|^{n-1}(1-|x|^n), 4n|x|^{2n-1}\} & \text{if } \frac{1}{\sqrt[n]{2}} \le |x| \le 1. \end{cases} \end{aligned}$$

Now, some simple calculations lead to the result. $\square$

Corollary 4.3. *If $n \in \mathbb{N}$ is odd, then*

$$M_{2n,n} = \mathcal{M}_{2n,n}(\pm 1) = 4n,$$

and equality is attained for the polynomials $p(x) = \pm(2x^{2n} - 1)$.



$\mathcal{M}_{2,1}(x)$. In this case, $M_{2,1} = 4$. $\mathcal{M}_{6,3}(x)$. Here, $M_{6,3} = 12$.

Figure 4.1: $\mathcal{M}_{2n,n}(x)$ with $n \in \mathbb{N}$ odd and $n > 3$ is similar to $\mathcal{M}_{6,3}(x)$.

4.2. $\mathcal{M}_{2n,n}(x)$ for $n \in \mathbb{N}$ even

Analogously as we did before, we need the extreme points of the unit ball of $(\mathbb{R}^3, \|\cdot\|_{2n,2})$ with $n \in \mathbb{N}$ even, characterized in the following result:

Lemma 4.4 (Muñoz-Fernández and Seoane-Sepúlveda, [7]). *If $n \in \mathbb{N}$ is even, then the extreme points of the unit ball of $(\mathbb{R}^3, \|\cdot\|_{2n,n})$ are*

$$\pm(0,0,1), \quad \pm(t, 2(\sqrt{2t}-t), 1+t-2\sqrt{2t}) \quad \textit{and} \quad \pm(t, -2\sqrt{2t}, 1),$$

where $t \in [2,8]$.

Theorem 4.5. *Let $n \in \mathbb{N}$ be even, then*

$$\mathcal{M}_{2n,n}(x) = \begin{cases} 8n(-2|x|^{2n-1} + |x|^{n-1}) & \textit{if } 0 \le |x| \le \left(\frac{1}{4}\right)^{\frac{1}{n}}, \\ \frac{n}{|x|} & \textit{if } \left(\frac{1}{4}\right)^{\frac{1}{n}} \le |x| \le \left(\frac{1}{2}\right)^{\frac{1}{n}}, \\ \frac{n|x|^{n-1}}{1-x^n} & \textit{if } \left(\frac{1}{2}\right)^{\frac{1}{n}} \le |x| \le \left(\frac{3}{4}\right)^{\frac{1}{n}}, \\ 8n(2|x|^{2n-1} - |x|^{n-1}) & \textit{if } \left(\frac{3}{4}\right)^{\frac{1}{n}} \le |x| \le 1. \end{cases} \tag{15}$$

Proof. We will proceed in the same spirit as we did in the proof of Theorem 4.2. If $x \in [-1,1]$, by definition we have that

$$\mathcal{M}_{2n,n}(x) = \sup_{p \in \mathsf{S}_{2n,n}} |p'(x)|.$$

Now, by means of Remark 1.3, it suffices to consider the polynomials

$$p_t(x) = \pm(tx^{2n} \pm 2(\sqrt{2t}-t)x^n + 1 + t - 2\sqrt{2t}) \quad \text{and} \quad q_t(x) = \pm(tx^{2n} - 2\sqrt{2t}x^n + 1),$$

where $t \in [2,8]$. Performing similar calculations to those from Theorem 4.2 one can arrive

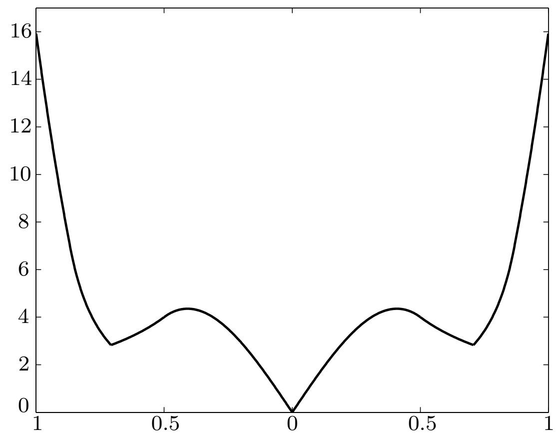


Figure 4.2: $\mathcal{M}_{4,2}(x)$. In the general case $\mathcal{M}_{2n,n}(x)$, being $n \in \mathbb{N}$ even, is of similar shape. In this example, $M_{4,2} = 16$.

to the following conclusions:

$$\max_{t\in[2,8]}\left|\frac{dp_t}{dx}(x)\right| = 2nx^{n-1}\max\{|tx^n + (\sqrt{2t}-t)| : t \in [2,8]\}$$

$$= \begin{cases} \max\left\{|8nx^{n-1}(1-2x^n)|, 4n|x|^{2n-1}, \frac{n|x|^{n-1}}{1-x^n}\right\} & \text{if } \sqrt[n]{\frac{1}{2}} \le |x| \le \sqrt[n]{\frac{3}{4}}, \\ \max\left\{|8nx^{n-1}(1-2x^n)|, 4n|x|^{2n-1}\right\} & \text{if } |x| \le \sqrt[n]{\frac{1}{2}} \text{ or } \sqrt[n]{\frac{3}{4}} \le |x| \le 1, \end{cases}$$

and, also,

$$\max_{t\in[2,8]}\left|\frac{dq_t}{dx}(x)\right| = 2nx^{n-1}\max\{|tx^n - \sqrt{2t}| : t \in [2,8]\}$$

$$= \begin{cases} \max\left\{|8nx^{n-1}(1-2x^n)|, 4n|x|^{n-1}|x^n-1|, \frac{n}{|x|}\right\} & \text{if } \sqrt[n]{\frac{1}{4}} \le |x| \le \sqrt[n]{\frac{1+\sqrt{2}}{4}}, \\ \max\left\{|8nx^{n-1}(1-2x^n)|, 4n|x|^{n-1}|x^n-1|\right\} & \text{if } |x| \le \sqrt[n]{\frac{1}{4}} \text{ or } \sqrt[n]{\frac{1+\sqrt{2}}{4}} \le |x| \le 1. \end{cases}$$

Clearly, for $n \in \mathbb{N}$ even,

$$\mathcal{M}_{2n,n}(x) = \max\left\{\max_{t\in[2,8]}\left|\frac{dp_t}{dx}(x)\right|, \max_{t\in[2,8]}\left|\frac{dq_t}{dx}(x)\right|\right\}.$$

Now, it suffices to consider $x > 0$, and compare the above functions over the intervals $\left[0, \sqrt[n]{\frac{1}{4}}\right]$, $\left[\sqrt[n]{\frac{1}{4}}, \sqrt[n]{\frac{2}{5}}\right]$, $\left[\sqrt[n]{\frac{2}{5}}, \sqrt[n]{\frac{1}{2}}\right]$, $\left[\sqrt[n]{\frac{1}{2}}, \sqrt[n]{\frac{1+\sqrt{2}}{4}}\right]$, $\left[\sqrt[n]{\frac{1+\sqrt{2}}{4}}, \sqrt[n]{\frac{3}{4}}\right]$ and $\left[\sqrt[n]{\frac{3}{4}}, 1\right]$. Some technical (but simple) calculations, similar to those from Theorem 4.2, lead us to (15). $\square$

Corollary 4.6. *If $n \in \mathbb{N}$ is even, then*

$$M_{2n,n} = \mathcal{M}_{2n,n}(\pm 1) = 8n,$$

and equality is attained for the polynomials $p(x) = \pm(8x^{2n} - 8x^n + 1)$.

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