Brøndsted-Rockafellar Property and Maximality of Monotone Operators Representable by Convex Functions in Non-Reflexive Banach Spaces

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In this work we are concerned with maximality of monotone operators representable by certain convex functions in non-reflexive Banach spaces. We also prove that these maximal monotone operators satisfy a Brøndsted-Rockafellar type property.

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1. Introduction

Let X be a real Banach space. We use the notation X^* for the topological dual of X and $\langle \cdot, \cdot \rangle$ stands for both duality products in $X \times X^*$ and $X^* \times X^{**}$,

$$\langle x, x^* \rangle = x^*(x), \quad \langle x^*, x^{**} \rangle = x^{**}(x^*), \quad x \in X, \ x^* \in X^*, \ x^{**} \in X^{**}.$$

A point to set operator $T : X \rightrightarrows X^*$ is a relation on X to X^*:

$$T \subset X \times X^*$$

and $x^* \in T(x)$ means $(x, x^*) \in T$. An operator $T : X \rightrightarrows X^*$ is *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in T.$$

The operator T is *maximal monotone* if it is monotone and maximal in the family of monotone operators of X into X^* (with respect to order of inclusion).

In [11] Fitzpatrick has put in light the possibility to represent maximal monotone operators by convex functions on $X \times X^*$. Before that, Krauss [12] managed to represent maximal

monotone operators by subdifferentials of saddle functions. Fitzpatrick's approach was constructive: Given a maximal monotone operator $T : X \rightrightarrows X^*$, he has defined the lower semicontinuous convex function $\varphi_T : X \times X^* \to \bar{\mathbb{R}}$ as

$$\varphi_T(x, x^*) = \sup_{(y,y^*)\in T} \langle x - y, y^* - x^* \rangle + \langle x, x^* \rangle. \tag{1}$$

Follows directly from maximal monotonicity of T, that φ_T majorizes the duality product on $X \times X^*$. On the other hand, φ_T is equal to the duality product in the graph of T. In this sense, it is said that φ_T is a convex representation of T or the *Fitzpatrick function* of T. It was also proved [11] that φ_T is the smallest function in the family of lower semicontinuous convex functions on $X \times X^*$ which have the above proprieties:

Theorem 1.1 ([11, Theorem 3.10]). *If T is a maximal monotone operator on a real Banach space X, then* (1) *is the smallest element of the Fitzpatrick family $\mathcal{F}_T$,*

$$\mathcal{F}_T = \left\{ h \in \bar{\mathbb{R}}^{X \times X^*} \left| \begin{array}{l} h \text{ is convex and lower semicontinuous} \\ \langle x, x^* \rangle \le h(x, x^*), \ \forall (x, x^*) \in X \times X^* \\ (x, x^*) \in T \Rightarrow h(x, x^*) = \langle x, x^* \rangle \end{array} \right. \right\} \tag{2}$$

Moreover, for any $h \in \mathcal{F}_T$,

$$(x, x^*) \in T \iff h(x, x^*) = \langle x, x^* \rangle.$$

Note that *any $h \in \mathcal{F}_T$ fully characterizes T.* Fitzpatrick family of convex representations of a maximal monotone operator was recently rediscovered by Burachik and Svaiter [9] and Martinez-Legaz and Théra [14]. Since then, this subject has been object of intense research [9, 21, 10, 13, 1, 3, 18, 15].

In [9], Burachik and Svaiter also proved that this family has a biggest element:

Proposition 1.2. *Let T be a maximal monotone operator on a real Banach space X. There exists a (unique) maximum element $\sigma_T \in \mathcal{F}_T$,*

$$\sigma_T = \sup_{h \in \mathcal{F}_T} \{h\},$$

which satisfies, in a reflexive and in a generic space, respectively

$$\varphi_T^*(x^*, x) = \sigma_T(x, x^*), \quad \sigma_T^*(x^*, x) = \varphi_T(x, x^*).$$

Moreover, σ_T can be characterized as

$$\sigma_T(x, x^*) = \operatorname{clconv}(\pi + \delta_T)(x, x^*),$$

where π denotes the duality product on $X \times X^*$ and δ_T is the indicator function of T.

Beside that, a complete study of the epigraphical structure of the function σ_T is also presented in [9] and it is proved that $\mathcal{F}_T$ is invariant under a suitable generalized conjugation operator.

Such invariance can be expressed as: If $T : X \rightrightarrows X^*$ is maximal monotone and $h \in \mathcal{F}_T$, then

$$\begin{aligned} h(x, x^*) &\ge \langle x, x^* \rangle, \\ h^*(x^*, x) &\ge \langle x, x^* \rangle, \end{aligned} \tag{3}$$

for all $(x, x^*) \in X \times X^*$.

Condition (3) was proved [10] to be not only a necessary condition but also a sufficient condition for maximal monotonicity in a reflexive Banach space.

Theorem 1.3 ([10, Theorem 3.1]). *Let X be a reflexive Banach space. If $h : X \times X^* \to \bar{\mathbb{R}}$ is proper, convex, lower semicontinuous and*

$$h(x, x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*$$
$$h^*(x^*, x) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*$$

then the operator $T : X \rightrightarrows X^*$ defined as

$$T = \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle\}$$

is maximal monotone and $T = \{(x, x^*) \in X \times X^* \mid h^*(x^*, x) = \langle x, x^* \rangle\}$.

Theorem 1.3 has been used for characterizing maximal monotonicity [19, 2] in reflexive Banach spaces. It is an open question whether (3) is also a sufficient condition for maximal monotonicity in a non-reflexive Banach space. A natural generalization of (3) in a generic Banach space is

$$h(x, x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^* \tag{4}$$
$$h^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

In this paper, we prove that (4) is a sufficient condition for a lower semicontinuous convex function h to represent a maximal monotone operator in a generic Banach space.

The theory of convex representations of maximal monotone operators is closely related to the study of a family of enlargements of such operators [9] introduced in [20]. In particular, an important question concerning the study of ε-enlargements [6, 7, 8], T^ε, of a maximal monotone operator T is whether an element in the graph of T^ε can be approximated by an element in the graph of T. This question has been successfully solved for the extension $\partial_\varepsilon f$, of ∂f, by Brøndsted and Rockafellar in [5]: Given $\varepsilon > 0$ and $x^* \in \partial_\varepsilon f(x)$, for all $\lambda > 0$ there exists $\bar{x}^*_\lambda \in \partial f(\bar{x}_\lambda)$, such that

$$\|\bar{x}_\lambda - x\| \leq \lambda, \quad \|\bar{x}^*_\lambda - x^*\| \leq \frac{\varepsilon}{\lambda}. \tag{5}$$

It does make sense to ask if the same property is valid for maximal monotone operators, that are not subdifferentials, with respect to its ε-enlargements: Let X is a real Banach space, $T : X \rightrightarrows X^*$ a maximal monotone operator and $x^* \in T^\varepsilon(x)$ for some $\varepsilon > 0$. Given $\lambda > 0$, does there exists $\bar{x}^*_\lambda \in T^\varepsilon(\bar{x}_\lambda)$ such that (5) is valid?

The answer is affirmative in a reflexive Banach space setting [8] but is negative in a non-reflexive Banach space [17]. From now on, we will refer to this fact as *Brøndsted-Rockafellar property*.

The major goal of this paper, is to show that (4) is a sufficient condition for a lower semicontinuous convex function h to represent a maximal monotone operator in a generic Banach space and that such operators satisfy a *strict Brøndsted-Rockafellar property* (see Theorem 4.2, item *4.*).

The manuscript is organized as follows: In Section 2 we establish some well known results and the notation to be used in the article. In Section 3 we are concerned with preliminary technical results and in Section 4 we prove our main results.

2.   Basic Results and Notation

The norms on X, X^* and X^{**} will be denoted by $\|\cdot\|$. We use the notation $\bar{\mathbb{R}}$ for the extended real numbers:

$$\bar{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}.$$

A convex function $f : X \to \bar{\mathbb{R}}$ is said to be proper if $f > -\infty$ and there exists a point $\hat{x} \in X$ for which $f(\hat{x}) < \infty$. The *subdifferential* of f is the point to set operator $\partial f : X \rightrightarrows X^*$ defined at $x \in X$ by

$$\partial f(x) = \{x^* \in X^* \mid f(y) \geq f(x) + \langle y - x, x^* \rangle, \text{ for all } y \in X\}.$$

For each $x \in X$, the elements $x^* \in \partial f(x)$ are called *subgradients* of f. Rockafellar proved that if f is proper, convex and lower semicontinuous, then ∂f is maximal monotone on X [16].

Fenchel-Legendre conjugate of $f : X \to \bar{\mathbb{R}}$ is $f^* : X^* \to \bar{\mathbb{R}}$ defined by

$$f^*(x^*) = \sup\{\langle x, x^* \rangle - f(x) \mid x \in X\}.$$

Note that f^* is always convex and lower semicontinuous. If f is proper convex and lower semicontinuous, then f^* is proper and from its definition, follows directly *Fenchel-Young inequality*: for all $x \in X$, $x^* \in X^*$,

$$f(x) + f^*(x^*) \geq \langle x, x^* \rangle \quad \text{and} \quad f(x) + f^*(x^*) = \langle x, x^* \rangle \quad \text{iff } x^* \in \partial f(x). \tag{6}$$

Note that $h(x, x^*) := f(x) + f^*(x^*)$ fully characterizes ∂f.

The concept of *ε-subdifferential* of a convex function f was introduced by Brøndsted and Rockafellar [5]. It is a point to set operator $\partial_\varepsilon f : X \rightrightarrows X^*$ defined at each $x \in X$ as

$$\partial_\varepsilon f(x) = \{x^* \in X^* \mid f(y) \geq f(x) + \langle y - x, x^* \rangle - \varepsilon, \text{ for all } y \in X\},$$

where $\varepsilon \geq 0$. Note that $\partial f = \partial_0 f$ and $\partial f(x) \subset \partial_\varepsilon f(x)$, for all $\varepsilon \geq 0$. Using the conjugate function f^* of f it is easy to see that

$$x^* \in \partial_\varepsilon f(x) \quad \Leftrightarrow \quad f(x) + f^*(x^*) \leq \langle x, x^* \rangle + \varepsilon. \tag{7}$$

An important tool to be used in the next sections is the classical Fenchel duality formula, which we present now.

Theorem 2.1 ([4, pp. 11]). *Let us consider two proper and convex functions f and g such that f (or g) is continuous at a point $\hat{x} \in X$ for which $f(\hat{x}) < \infty$ and $g(\hat{x}) < \infty$. Then,*

$$\inf_{x \in X} \{f(x) + g(x)\} = \max_{x^* \in X^*} \{-f^*(-x^*) - g^*(x^*)\}. \tag{8}$$

3.   Preliminary Results

In this section we present some preliminary technical results which will be used in the next sections.

Theorem 3.1. *Suppose that $h : X \times X^* \to \bar{\mathbb{R}}$ is proper, convex, lower semicontinuous and*

$$h(x, x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*$$
$$h^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

Then, for any $\varepsilon > 0$ there exists $(\tilde{x}, \tilde{x}^*) \in X \times X^*$ such that

$$h(\tilde{x}, \tilde{x}^*) + \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 < \varepsilon, \qquad \|\tilde{x}\|^2 \leq h(0,0), \quad \|\tilde{x}^*\|^2 \leq h(0,0),$$

where the two last inequalities are strict in the case $h(0,0) > 0$.

Proof. If $h(0,0) < \varepsilon$ then $(\tilde{x}, \tilde{x}^*) = (0,0)$ has the desired properties. The non-trivial case is

$$\varepsilon \leq h(0,0), \tag{9}$$

which we consider now. Using the first assumption on h, we conclude that for any $(x, x^*) \in X \times X^*$,

$$\begin{aligned}
h(x, x^*) + \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2 &\geq \langle x, x^* \rangle + \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2 \\
&\geq -\|x\|\,\|x^*\| + \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2 \\
&= \tfrac{1}{2}\left(\|x\| - \|x^*\|\right)^2 \geq 0.
\end{aligned} \tag{10}$$

The second assumption on h also gives, for all $(z^*, z^{**}) \in X^* \times X^{**}$,

$$\begin{aligned}
h^*(z^*, z^{**}) + \tfrac{1}{2}\|z^*\|^2 + \tfrac{1}{2}\|z^{**}\|^2 &\geq \langle z^*, z^{**} \rangle + \tfrac{1}{2}\|z^*\|^2 + \tfrac{1}{2}\|z^{**}\|^2 \\
&\geq -\|z^*\|\,\|z^{**}\| + \tfrac{1}{2}\|z^*\|^2 + \tfrac{1}{2}\|z^{**}\|^2 \\
&= \tfrac{1}{2}\left(\|z^*\| - \|z^{**}\|\right)^2 \geq 0.
\end{aligned} \tag{11}$$

Now using Theorem 2.1 with $f, g : X \times X^* \to \bar{\mathbb{R}}$,

$$f(x, x^*) = h(x, x^*), \qquad g(x, x^*) = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2$$

we conclude that there exists $(\hat{z}^*, \hat{z}^{**}) \in X^* \times X^{**}$ such that

$$\inf \; h(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = -h^*(\hat{z}^*, \hat{z}^{**}) - \frac{1}{2}\|\hat{z}^*\|^2 - \frac{1}{2}\|\hat{z}^{**}\|^2.$$

As the right hand side of the above equation is non positive and the left hand side is non negative, these two terms are zero. Therefore,

$$\inf \; h(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0, \tag{12}$$

and

$$h^*(\hat{z}^*, \hat{z}^{**}) + \frac{1}{2}\|\hat{z}^*\|^2 + \frac{1}{2}\|\hat{z}^{**}\|^2 = 0. \tag{13}$$

For $(z^*, z^{**}) = (\hat{z}^*, \hat{z}^{**})$, all inequalities on (11) must hold as equalities. Therefore,

$$\|\hat{z}^*\|^2 = \|\hat{z}^{**}\|^2 = -h^*(\hat{z}^*, \hat{z}^{**}) \leq h(0, 0), \tag{14}$$

where the last inequality follows from the definition of conjugate.

Using (12) we conclude that for any $\eta > 0$, there exists $(x_\eta, x_\eta{}^*) \in X \times X^*$ such that

$$h(x_\eta, x_\eta{}^*) + \frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|x_\eta{}^*\|^2 < \eta. \tag{15}$$

If $h(0, 0) = \infty$, then, taking $\eta = \varepsilon$ and $(\tilde{x}, \tilde{x}^*) = (x_\eta, x_\eta{}^*)$ we conclude that the theorem holds. Now, we discuss the case $h(0, 0) < \infty$. In this case, using (14) we have

$$\|\hat{z}^*\| = \|\hat{z}^{**}\| \leq \sqrt{h(0, 0)}. \tag{16}$$

Note that from (9) we are considering

$$\varepsilon \leq h(0, 0) < \infty. \tag{17}$$

Combining (15) with (13) and using Fenchel-Young inequality (6) we obtain

$$\begin{aligned}
\eta &> h(x_\eta, x_\eta{}^*) + \tfrac{1}{2}\|x_\eta\|^2 + \tfrac{1}{2}\|x_\eta{}^*\|^2 + h^*(\hat{z}^*, \hat{z}^{**}) + \tfrac{1}{2}\|\hat{z}^*\|^2 + \tfrac{1}{2}\|\hat{z}^{**}\|^2 \\
&\geq \langle x_\eta, \hat{z}^* \rangle + \langle x_\eta{}^*, \hat{z}^{**} \rangle + \tfrac{1}{2}\|x_\eta\|^2 + \tfrac{1}{2}\|x_\eta{}^*\|^2 + \tfrac{1}{2}\|\hat{z}^*\|^2 + \tfrac{1}{2}\|\hat{z}^{**}\|^2 \\
&\geq \tfrac{1}{2}\|x_\eta\|^2 - \|x_\eta\|\|\hat{z}^*\| + \tfrac{1}{2}\|\hat{z}^{**}\|^2 + \tfrac{1}{2}\|x_\eta{}^*\|^2 - \|x_\eta{}^*\|\|\hat{z}^{**}\| + \tfrac{1}{2}\|\hat{z}^{**}\|^2 \\
&= \tfrac{1}{2}\left(\|x_\eta\| - \|\hat{z}^*\|\right)^2 + \tfrac{1}{2}\left(\|x_\eta{}^*\| - \|\hat{z}^{**}\|\right)^2.
\end{aligned}$$

As the two terms in the last inequality are non negative,

$$\|x_\eta\| < \|\hat{z}^*\| + \sqrt{2\eta}, \qquad \|x_\eta{}^*\| < \|\hat{z}^{**}\| + \sqrt{2\eta}.$$

Therefore, using (16) we obtain

$$\|x_\eta\| < \sqrt{h(0, 0)} + \sqrt{2\eta}, \qquad \|x_\eta{}^*\| < \sqrt{h(0, 0)} + \sqrt{2\eta}.$$

To end the proof, take in (15)

$$0 < \eta < \frac{\varepsilon^2}{2h(0, 0)} \tag{18}$$

and let

$$\tau = \frac{\sqrt{h(0, 0)}}{\sqrt{h(0, 0)} + \sqrt{2\eta}}, \qquad \tilde{x} = \tau\, x_\eta, \qquad \tilde{x}^* = \tau\, x_\eta{}^*. \tag{19}$$

Then,

$$\|\tilde{x}\| < \sqrt{h(0, 0)}, \qquad \|\tilde{x}^*\| < \sqrt{h(0, 0)}.$$

Now, using the convexity of h and of the square of the norms and (15), we have

$$\begin{aligned}
&h(\tilde{x}, \tilde{x}^*) + \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 \\
&\leq (1 - \tau)\, h(0, 0) + \tau\left(h(x_\eta, x_\eta{}^*) + \frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|x_\eta{}^*\|^2\right) \\
&< (1 - \tau)\, h(0, 0) + \tau\, \eta \\
&= h(0, 0) - \tau(h(0, 0) - \eta).
\end{aligned}$$

Therefore, using also (18)

$$
\begin{aligned}
\varepsilon - \left(h(\tilde{x}, \tilde{x}^*) + \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 \right) &\geq \varepsilon - h(0,0) + \tau(h(0,0) - \eta) \\
&> \varepsilon - h(0,0) + \tau(h(0,0) - 2\eta) \\
&= \varepsilon - h(0,0) + \sqrt{h(0,0)}\left(\sqrt{h(0,0)} - \sqrt{2\eta}\right) \\
&= \varepsilon - \sqrt{2h(0,0)\eta} > 0.
\end{aligned}
$$

which completes the proof. $\qquad\square$

In Theorem 3.1 the origin has a special role. In order to use this theorem with an arbitrary point, define, for $h : X \times X^* \to \bar{\mathbb{R}}$ and $(z, z^*) \in X \times X^*$,

$$
\begin{aligned}
h_{(z,z^*)} &: X \times X^* \to \bar{\mathbb{R}}, \\
h_{(z,z^*)}(x, x^*) &= h(x + z, x^* + z^*) - \left[\langle x, z^* \rangle + \langle z, x^* \rangle + \langle z, z^* \rangle \right].
\end{aligned}
\tag{20}
$$

The next proposition follows directly from algebraic manipulations and from (20).

Proposition 3.2. *Take $h : X \times X^* \to \bar{\mathbb{R}}$ and $(z, z^*) \in X \times X^*$.*

1. *If h is proper, convex and lower semicontinuous, then $h_{(z,z^*)}$ is also proper, convex and lower semicontinuous.*
2. *$(h_{(z,z^*)})^* = (h^*)_{(z^*,z)}$, where in the right hand side z is identified with its image by the canonical injection of X into X^{**}:*

$$
(h^*)_{(z^*,z)}(x^*, x^{**}) = h^*(x^* + z^*, x^{**} + z) - \left[\langle x^*, z \rangle + \langle z^*, x^{**} \rangle + \langle z^*, z \rangle \right].
$$

3. *For any $(x, x^*) \in X \times X^*$,*

$$
h_{(z,z^*)}(x, x^*) - \langle x, x^* \rangle = h(x + z, x^* + z^*) - \langle x + z, x^* + z^* \rangle.
$$

4. *If h majorizes the duality product in $X \times X^*$ then $h_{(z,z^*)}$ also majorizes the duality product in $X \times X^*$.*
5. *If h^* majorizes the duality product in $X^* \times X^{**}$ then $(h_{(z,z^*)})^*$ also majorizes the duality product in $X^* \times X^{**}$.*

Corollary 3.3. *Suppose that $h : X \times X^* \to \bar{\mathbb{R}}$ is proper, convex, lower semicontinuous and*

$$
\begin{aligned}
h(x, x^*) &\geq \langle x, x^* \rangle, &\forall (x, x^*) \in X \times X^* \\
h^*(x^*, x^{**}) &\geq \langle x^*, x^{**} \rangle, &\forall (x^*, x^{**}) \in X^* \times X^{**}.
\end{aligned}
$$

Then, for any $(z, z^*) \in X \times X^*$ and $\varepsilon > 0$ there exist $(\tilde{x}, \tilde{x}^*) \in X \times X^*$ such that

$$
\begin{aligned}
h(\tilde{x}, \tilde{x}^*) &< \langle \tilde{x}, \tilde{x}^* \rangle + \varepsilon, \\
\|\tilde{x} - z\|^2 &\leq h(z, z^*) - \langle z, z^* \rangle, \\
\|\tilde{x}^* - z^*\|^2 &\leq h(z, z^*) - \langle z, z^* \rangle.
\end{aligned}
$$

where the two last inequalities are strict in the case $\langle z, z^* \rangle < h(z, z^*)$.

Proof. If $h(z, z^*) = \langle z, z^* \rangle$ then $(\tilde{x}, \tilde{x}^*) = (z, z^*)$ satisfy the desired conditions. Assume that

$$0 < h(z, z^*) - \langle z, z^* \rangle. \tag{21}$$

Using Proposition 3.2 and applying Theorem 3.1 for the function $h_{(z,z^*)}$ we conclude that there exists $(\tilde{z}, \tilde{z}^*) \in X \times X^*$ such that

$$h_{(z,z^*)}(\tilde{z}, \tilde{z}^*) + \frac{1}{2}\|\tilde{z}\|^2 + \frac{1}{2}\|\tilde{z}^*\|^2 < \varepsilon, \quad \|\tilde{z}\|^2 < h_{(z,z^*)}(0,0), \quad \|\tilde{z}^*\|^2 < h_{(z,z^*)}(0,0). \tag{22}$$

By (20), note that $h_{(z,z^*)}(0,0) = h(z, z^*) - \langle z, z^* \rangle$. Let

$$\tilde{x} = \tilde{z} + z, \qquad \tilde{x}^* = \tilde{z}^* + z^*.$$

Therefore, using (22) and (21), we have

$$\|\tilde{x} - z\|^2 < h(z, z^*) - \langle z, z^* \rangle, \qquad \|\tilde{x}^* - z^*\|^2 < h(z, z^*) - \langle z, z^* \rangle.$$

To end the proof of the first part of the corollary, use Proposition 3.2 and (22) to obtain

$$
\begin{aligned}
h(\tilde{x}, \tilde{x}^*) - \langle \tilde{x}, \tilde{x}^* \rangle &= h_{(z,z^*)}(\tilde{z}, \tilde{z}^*) - \langle \tilde{z}, \tilde{z}^* \rangle \\
&\leq h_{(z,z^*)}(\tilde{z}, \tilde{z}^*) + \tfrac{1}{2}\|\tilde{z}\|^2 + \tfrac{1}{2}\|\tilde{z}^*\|^2 < \varepsilon.
\end{aligned}
$$

$$\square$$

Theorem 3.4. *Suppose that $h : X \times X^* \to \bar{\mathbb{R}}$ is proper, convex, lower semicontinuous and*

$$
\begin{aligned}
h(x, x^*) &\geq \langle x, x^* \rangle, & \forall (x, x^*) \in X \times X^* \\
h^*(x^*, x^{**}) &\geq \langle x^*, x^{**} \rangle, & \forall (x^*, x^{**}) \in X^* \times X^{**}.
\end{aligned}
$$

If $(x, x^*) \in X \times X^*$, $\varepsilon > 0$ and

$$h(x, x^*) < \langle x, x^* \rangle + \varepsilon,$$

then, there exists $(\bar{x}, \bar{x}^*) \in X \times X^*$ such that

$$h(\bar{x}, \bar{x}^*) = \langle \bar{x}, \bar{x}^* \rangle, \qquad \|x - \bar{x}\| < \sqrt{\varepsilon}, \qquad \|x^* - \bar{x}^*\| < \sqrt{\varepsilon}.$$

Moreover, for any $\lambda > 0$ there exists $(\bar{x}_\lambda, \bar{x}_\lambda^*) \in X \times X^*$ such that

$$h(\bar{x}_\lambda, \bar{x}_\lambda^*) = \langle \bar{x}, \bar{x}_\lambda^* \rangle, \qquad \|\bar{x}_\lambda - x\| < \lambda, \qquad \|\bar{x}_\lambda^* - x^*\| < \frac{\varepsilon}{\lambda}.$$

Proof. Let

$$h(x, x^*) - \langle x, x^* \rangle < \varepsilon_0 < \varepsilon. \tag{23}$$

For an arbitrary $\theta \in (0, 1)$, define inductively a sequence $\{(x_k, x_k^*)\}$ as follows: For $k = 0$, let

$$(x_0, x_0^*) = (x, x^*). \tag{24}$$

Given k and (x_k, x_k^*), use Corollary 3.3 to conclude that there exists some (x_{k+1}, x_{k+1}^*) such that

$$h(x_{k+1}, x_{k+1}^*) - \langle x_{k+1}, x_{k+1}^* \rangle < \theta^{k+1}\varepsilon_0 \tag{25}$$

and

$$
\begin{aligned}
\|x_{k+1} - x_k\|^2 &\leq h(x_k, x_k^*) - \langle x_k, x_k^* \rangle, \\
\|x_{k+1}^* - x_k^*\|^2 &\leq h(x_k, x_k^*) - \langle x_k, x_k^* \rangle.
\end{aligned}
\tag{26}
$$

Using (23) and (25) we conclude that for all k,

$$
0 \leq h(x_k, x_k^*) - \langle x_k, x_k^* \rangle < \theta^k \varepsilon_0.
\tag{27}
$$

which, combined with (26) yields

$$
\sum_{k=0}^{\infty} \|x_{k+1} - x_k\| < \sqrt{\varepsilon_0} \sum_{k=0}^{\infty} \sqrt{\theta^k}, \qquad \sum_{k=0}^{\infty} \|x_{k+1}^* - x_k^*\| < \sqrt{\varepsilon_0} \sum_{k=0}^{\infty} \sqrt{\theta^k}.
$$

In particular, the sequences $\{x_k\}$ and $\{x_k^*\}$ are convergent. Let

$$
\bar{x} = \lim_{k \to \infty} x_k, \qquad \bar{x}^* = \lim_{k \to \infty} x_k^*.
$$

Then, using the previous equation we have

$$
\|\bar{x} - x\| < \frac{\sqrt{\varepsilon_0}}{1 - \sqrt{\theta}}, \qquad \|\bar{x}^* - x^*\| < \frac{\sqrt{\varepsilon_0}}{1 - \sqrt{\theta}}.
$$

Since, by (23), $\varepsilon_0 < \varepsilon$, for $\theta \in (0,1)$ sufficiently small,

$$
\|\bar{x} - x\| < \sqrt{\varepsilon}, \qquad \|\bar{x}^* - x^*\| < \sqrt{\varepsilon}.
$$

Using (27) we have

$$
\lim_{k \to \infty} h(x_k, x_k^*) - \langle x_k, x_k^* \rangle = 0.
$$

As h is lower semicontinuous and the duality product is continuous,

$$
h(\bar{x}, \bar{x}^*) - \langle \bar{x}, \bar{x}^* \rangle \leq 0.
$$

Therefore, $h(\bar{x}, \bar{x}^*) - \langle \bar{x}, \bar{x}^* \rangle = 0$, which ends the proof of the first part of the theorem.

To prove the second part of the theorem, use in X the norm

$$
|||x||| = \frac{\sqrt{\varepsilon}}{\lambda} \|x\|,
$$

and apply the first part of the theorem in this re-normed space. $\qquad \square$

4.   Main Result

In this section we present our main result, Theorem 4.2. Before that, we recall a well known result of theory of convex functions.

Lemma 4.1. *Let E be a real topological linear space and $f : E \to \bar{\mathbb{R}}$ be a convex function. If $g : E \to \mathbb{R}$ is Gateaux differentiable at x_0, $f(x_0) = g(x_0)$ and $f \geq g$ in a neighborhood of x_0, then $g'(x_0) \in \partial f(x_0)$.*

Theorem 4.2. *Suppose that $h : X \times X^* \to \bar{\mathbb{R}}$ is proper, convex, lower semicontinuous and*

$$h(x, x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*$$
$$h^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

Define

$$T = \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle\}.$$

Then

1. $T = \{(x, x^*) \in X \times X^* \mid h^*(x^*, x) = \langle x, x^* \rangle\}.$
2.   *T is maximal monotone.*
3.   *Let φ_T be the Fitzpatrick function associated with T, as defined in (1), that is,*

$$\varphi_T(x, x^*) = \sup_{(y, y^*) \in T} \langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle.$$

   Then

$$\varphi_T(x, x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*$$
$$\varphi_T^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}. \tag{28}$$

4.   *The maximal monotone operator T satisfies a* strict Brøndsted-Rockafellar property: *If $\eta > \varepsilon$ and $x^* \in T^\varepsilon(x)$, that is,*

$$\langle x - y, x^* - y^* \rangle \geq -\varepsilon, \quad \forall (y, y^*) \in T,$$

   then, for any $\lambda > 0$ there exists $(\bar{x}_\lambda, \bar{x}_\lambda^*) \in X \times X^*$ such that

$$\bar{x}_\lambda^* \in T(\bar{x}_\lambda), \qquad \|x - \bar{x}_\lambda\| < \lambda, \quad \|x^* - \bar{x}_\lambda^*\| < \frac{\eta}{\lambda}.$$

Proof. To prove item *1.*, denote by $\pi : X \times X^* \to \mathbb{R}$ the duality product. This function is everywhere differentiable and

$$\pi'(x, x^*) = (x^*, x).$$

Suppose that $h(x, x^*) = \langle x, x^* \rangle = \pi(x, x^*)$. Then, by Lemma 4.1

$$(x^*, x) \in \partial h(x, x^*), \quad \text{that is,} \quad h(x, x^*) + h^*(x^*, x) = \langle (x, x^*), (x^*, x) \rangle,$$

which implies $h^*(x^*, x) = \langle x, x^* \rangle$. Conversely, if $h^*(x^*, x) = \langle x, x^* \rangle$, then by the same reasoning $h^{**}(x, x^*) = \langle x, x^* \rangle$. As h is proper, convex and lower semicontinuous, $h(x, x^*) = h^{**}(x, x^*)$, which concludes the proof of item *1.*

Take $(x, x^*), (y, y^*) \in T$. Then, as proved above

$$(x^*, x) \in \partial h(x, x^*), \qquad (y^*, y) \in \partial h(y, y^*).$$

As ∂h is monotone,

$$\langle (x, x^*) - (y, y^*), (x^*, x) - (y^*, y) \rangle \geq 0,$$

which gives $\langle x - y, x^* - y^* \rangle \geq 0$. Hence, T is monotone.

To prove maximal monotonicity of T, take $(z, z^*) \in X \times X^*$ and assume that

$$\langle x - z, x^* - z^* \rangle \geq 0, \quad \forall (x, x^*) \in T. \tag{29}$$

Using Theorem 3.1 and Proposition 3.2 we know that

$$\inf h_{(z,z^*)}(u, u^*) + \frac{1}{2}\|u\|^2 + \frac{1}{2}\|u^*\|^2 = 0.$$

Therefore, there exists a minimizing sequence $\{(u_k, u_k^*)\}$ such that

$$h_{(z,z^*)}(u_k, u_k^*) + \frac{1}{2}\|u_k\|^2 + \frac{1}{2}\|u_k^*\|^2 < \frac{1}{k^2}, \quad k = 1, 2, \ldots \tag{30}$$

Note that the sequence $\{(u_k, u_k^*)\}$ is bounded and

$$
\begin{aligned}
h_{(z,z^*)}(u_k, u_k^*) - \langle u_k, u_k^* \rangle &\leq h_{(z,z^*)}(u_k, u_k^*) + \|u_k\|\,\|u_k^*\| \\
&\leq h_{(z,z^*)}(u_k, u_k^*) + \frac{1}{2}\|u_k\|^2 + \frac{1}{2}\|u_k^*\|^2.
\end{aligned}
$$

Combining the two above inequalities we obtain

$$h_{(z,z^*)}(u_k, u_k^*) < \langle u_k, u_k^* \rangle + \frac{1}{k^2}.$$

Now applying Theorem 3.4, we conclude that there for each k there exists some $(\bar{u}_k, \bar{u}_k^*)$ such that

$$h_{(z,z^*)}(\bar{u}_k, \bar{u}_k^*) = \langle \bar{u}_k, \bar{u}_k^* \rangle, \quad \|\bar{u}_k - u_k\| < 1/k, \quad \|\bar{u}_k^* - u_k^*\| < 1/k.$$

Then,

$$(\bar{x}_k, \bar{x}_k^*) := (\bar{u}_k + z, \bar{u}_k^* + z^*) \in T,$$

and from (29)

$$\langle \bar{u}_k, \bar{u}_k^* \rangle = \langle \bar{x}_k - z, \bar{x}_k^* - z^* \rangle \geq 0.$$

The duality product is uniformly continuous on bounded sets. Since $\{(u_k, u_k^*)\}$ is bounded and $\lim_{k\to\infty} \|u_k - \bar{u}_k\| = \lim_{k\to\infty} \|u_k^* - \bar{u}_k^*\| = 0$ we conclude that

$$\liminf_{k\to\infty} \langle u_k, u_k^* \rangle \geq 0.$$

Using (30) and the fact that h majorizes the duality product, we have

$$0 \leq \langle u_k, u_k^* \rangle + \frac{1}{2}\|u_k\|^2 + \frac{1}{2}\|u_k^*\|^2 \leq h_{(z,z^*)}(u_k, u_k^*) + \frac{1}{2}\|u_k\|^2 + \frac{1}{2}\|u_k^*\|^2 < \frac{1}{k^2}.$$

Hence, $\langle u_k, u_k^* \rangle < 1/k^2$ and $\limsup_{k\to\infty} \langle u_k, u_k^* \rangle \leq 0$, which implies $\lim_{k\to\infty} \langle u_k, u_k^* \rangle = 0$. Combining this result with the above inequalities we conclude that

$$\lim_{k\to\infty} (u_k, u_k^*) = 0.$$

Therefore, $\lim_{k\to\infty} (\bar{u}_k, \bar{u}_k^*) = 0$ and $\{(\bar{x}_k, \bar{x}_k^*)\}$ converges to (z, z^*). As $h(\bar{x}_k, \bar{x}_k^*) = \langle \bar{x}_k, \bar{x}_k^* \rangle$ and h is lower semicontinuous,

$$h(z, z^*) \leq \langle z, z^* \rangle.$$

which readily implies $h(z, z^*) = \langle z, z^* \rangle$. Therefore $(z, z^*) \in T$ and T is maximal monotone.

For proving item *3.*, note that as T is maximal monotone, Fitzpatrick function φ_T is minimal in the family of functions which majorizes the duality product and at T are equal to the duality product. In particular, the first inequality in item *3.* holds and $h \geq \varphi_T$. Hence,

$$\varphi_T^* \geq h^*,$$

which readily implies the second inequality in item *3.*

For proving item *4.*, assume that $\eta > \varepsilon > 0$ and

$$\langle x - y, x^* - y^* \rangle \geq -\varepsilon, \quad \forall (y, y^*) \in T.$$

Fitzpatrick function of T is

$$
\begin{aligned}
\varphi_T(x, x^*) &= \sup_{(y,y^*)\in T} \langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle \\
&= \sup_{(y,y^*)\in T} -\langle x - y, x^* - y^* \rangle + \langle x, x^* \rangle.
\end{aligned}
$$

Therefore

$$\varphi_T(x, x^*) \leq \langle x, x^* \rangle + \varepsilon < \langle x, x^* \rangle + \eta.$$

Now, use item *3.* and Theorem 3.4 to conclude that there exists $(\bar{x}_\lambda, \bar{x}_\lambda^*)$ such that

$$\varphi_T(\bar{x}_\lambda, \bar{x}_\lambda^*) = \langle \bar{x}_\lambda, \bar{x}_\lambda^* \rangle, \qquad \|x - \bar{x}_\lambda\| < \lambda, \quad \|x^* - \bar{x}_\lambda^*\| < \frac{\eta}{\lambda}.$$

The firs equality above says that $(\bar{x}_\lambda, \bar{x}_\lambda^*) \in T$, which ends the proof of the theorem. $\quad\square$

Corollary 4.3. *Let $T : X \rightrightarrows X^*$ be maximal monotone. If there exists $h \in \mathcal{F}_T$, that is, $h : X \times X^* \to \bar{\mathbb{R}}$ proper, convex, lower semicontinuous and*

$$h(x, x^*) \geq \langle x, x^* \rangle \quad \forall (x, x^*) \in X \times X^*$$

with equality in $(x, x^*) \in T$, such that

$$h^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle \quad \forall (x^*, x^{**}) \in X^* \times X^{**},$$

then, T has the strict Brøndsted-Rockafellar property *and the conjugate of φ_T, the Fitzpatrick function associated to T, majorizes the duality product in $X^* \times X^{**}$*

$$\varphi_T^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

The duality product is continuous in $X \times X^*$. Therefore, if a convex function majorizes the duality product then the convex closure of this function also majorizes the duality product and has the same conjugate. This fact can be used to remove the assumption of lower semicontinuity of h in Theorem 4.2.

Corollary 4.4. *Suppose that $h : X \times X^* \to \bar{\mathbb{R}}$ is convex and*

$$
\begin{aligned}
h(x, x^*) &\geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^* \\
h^*(x^*, x^{**}) &\geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.
\end{aligned}
$$

Define

$$T = \{(x, x^*) \in X \times X^* \mid h^*(x^*, x) = \langle x, x^* \rangle\}.$$

Then

1.   *T is maximal monotone.*

2.   *Let φ_T be Fitzpatrick function associated with T. Then*

$$\varphi_T(x, x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*$$
$$\varphi_T^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}. \tag{31}$$

3.   *The maximal monotone operator T satisfies a* strict Brøndsted-Rockafellar property*: If $\eta > \varepsilon$ and $x^* \in T^\varepsilon(x)$, that is,*

$$\langle x - y, x^* - y^* \rangle \geq -\varepsilon, \quad \forall (y, y^*) \in T,$$

then, for any $\lambda > 0$ there exists $(\bar{x}_\lambda, \bar{x}_\lambda^*) \in X \times X^*$ such that

$$\bar{x}_\lambda^* \in T(\bar{x}_\lambda), \qquad \|x - \bar{x}_\lambda\| < \lambda, \quad \|x^* - \bar{x}_\lambda^*\| < \frac{\eta}{\lambda}.$$

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