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Weakly Convergent Sequence Coefficient in Musielak-Orlicz Sequence Spaces

Liying Fan
Department of Mathematics,
Harbin University of Science and Technology, Harbin 150080, China
frankying@yahoo.com.cn

Yunan Cui*
Department of Mathematics,
Harbin University of Science and Technology, Harbin 150080, China
yunan_ cui@yahoo.com.cn

Henryk Hudzik
Faculty of Mathematics and Computer Science,
Adam Mickiewicz University, Umultowska 87, 61-614 Poznan, Poland
hudzik@amu.edu.pl

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A formula for the weakly convergent sequence coefficient of Musielak-Orlicz sequence spaces l_Φ equipped with the Luxemburg norm is calculated. As a consequence of this result the cofficient for Nakano sequence spaces $l^{(p_i)}$ is found. Criteria for weakly uniformly normal structure of Musielak-Orlicz sequence spaces equipped with the Luxemburg are also given.

Keywords: Fixed point property, Musielak-Orlicz sequence spaces, weak convergent sequence coefficient

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1. Introduction

Let X be a Banach space. A mapping $T : C \subseteq X \to X$ is called nonexpansive if $||Tx - Ty|| \leq ||x - y||$ for all x, $y \in C$. The set of fixed points of T is $\mathrm{Fix}(T) := \{x \in C : Tx = x\}$. We say that the space X has the fixed point property (FPP) if for every nonempty closed, bounded and convex subset C of X and every nonexpansive mapping $T : C \to C$ we have $\mathrm{Fix}(T) \neq \phi$.

Similarly, X is said to have the weak fixed point property $(WFPP)$ if for every nonempty, weakly compact and convex subset C of X and every nonexpansive mapping $T : C \to C$ we have $\mathrm{Fix}(T) \neq \phi$.

It is well known that one of the central goals in the fixed point theory is obtaining a full characterization of those Banach spaces which have FPP or $WFPP$.

A Banach space X, or more generally, a closed convex subset K of X, is said to have

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normal structure if any bounded convex subset H of X (resp. of K) which has more than one point, contains a nondiametral point, i.e., there exists a point $x_0 \in H$ such that

$$\sup\{||x_0 - x|| : x \in H\} < \operatorname{diam}(H) := \sup\{||x - y|| : x, y \in H\}.$$

For $D \subset X$, $D \neq \phi$, let

$$r_x(D) = \sup\{||x - y|| : y \in D\},$$
$$r(D) = \inf\{r_x(D) : x \in D\}.$$

If X is reflexive and if D is a non-empty bounded closed subset of X, then weak compactness of the closed balls in X yields that the set

$$C(D) = \{z \in D : r_z(D) = r(D)\}$$

is a nonempty closed convex subset of D. The number $r(D)$ and the set $C(D)$ are called the *Chebyshev* radius and *Chebyshev* center of D, respectively.

We recall the following theorem proved by W. A. Kirk in [11] for reflexive Banach spaces, although clearly weak compactness of K suffices:

Let X be a reflexive Banach space and K be a non-empty bounded closed convex subset of X which has normal structure. Then any nonexpansive mapping $T : K \to K$ has a fixed point. In particular, if X has normal structure, then X has $WFPP$, and if X is reflexive and has normal structure, then X has FPP.

In order to know which kind of Banach spaces have normal structure, W. L. Bynum introduced in [3] geometric cofficients $WCS(X)$ and $N(X)$, and established their relationships to normal structure.

We set

$$N(X) = \inf\left\{\frac{\operatorname{diam}(A)}{r(A)}\right\},$$

where the infimum is taken over all bounded convex closed sets A in X with $\operatorname{diam}(A) > 0$. The number $N(X)$ is called the normal structure coefficient of X. A Banach space X has the uniformly normal structure iff $N(X) > 1$.

For a Banach space X without Schur property, the weakly convergent sequence coefficient ($WCS(X)$) is defined by

$$WCS(X) = \inf\left\{\frac{A(\{x_i\})}{r(\{x_i\})} : \{x_i\} \subset X \text{ is a sequence weakly but not strongly convergent to zero}\right\},$$

where $A(\{x_i\})$ is the asymptotic diameter of $\{x_i\}$, i.e.

$$A(\{x_i\}) = \lim_{n\to\infty}\{\sup[||x_i - x_j|| : i, j \geq n]\},$$

and $r(\{x_i\})$ is the Chebyshev radius of the set $\overline{co}(\{x_i\})$.

G. L. Zhang proved in [16] that if X does not have the Schur property, then

$$WCS(X) = \inf\left\{A(\{x_i\}) : x_i \in S(X) \text{ for all } i \in \mathbb{N}, x_i \xrightarrow{w} 0\right\},$$

where $S(X)$ is the unit sphere of X, as usual.

A Banach space X has weakly uniformly normal structure (so the weak fixed point property as well) whenever $WCS(X) > 1$. For the notions concerning the fixed point property and the weak fixed point property as well as the coefficients $N(X)$ and $WCS(X)$ we refer to [1], [2], [3], [4], [5], [11], [12], [16] and [18]. In this paper, we will give a formula how to compute the coefficient $WCS(X)$ for Musielak-Orlicz sequence spaces.

A map $\Phi : R \to [0,\infty]$ is said to be an Orlicz function if Φ is vanishing only at 0, even, left continuous in the extended sense (which means that infinite limits are admitted) on $[0,\infty)$ and convex.

A sequence $\Phi = (\Phi_i)$ of Orlicz functions is called a *Musielak-Orlicz* function. By $\Psi = (\Psi_i)$ we denote the complementary function of Φ in the sense of Young, that is,

$$\Psi_i(v) = \sup\{|v|\,u - \Phi_i(u) : u \geq 0\},\ i = 1, 2, \cdots, v \in \mathbb{R}.$$

Given a *Musielak-Orlicz* function Φ, we define a convex modular by $I_\Phi(x) = \sum_{i=1}^{\infty} \Phi_i(x(i))$ for any $x \in l^0$ (the space of all real sequences). The linear space l_Φ defined by

$$l_\Phi = \left\{x \in l^0 :\ I_\Phi(kx) < \infty \text{ for some } k > 0\right\}$$

is called the *Musielak-Orlicz sequence space* generated by Φ. We consider l_Φ equipped with the *Luxemburg norm*

$$\|x\| = \inf\left\{k > 0 :\ I_\Phi\left(\frac{x}{k}\right) \leq 1\right\}.$$

To simplify notations, we put $l_\Phi = (l_\Phi, \|\cdot\|)$. The space l_Φ is a Banach space (see [10], [13], [14], [15] and [17]).

We say a *Musielak-Orlicz function* Φ satisfies the δ_2-condition ($\Phi \in \delta_2$ for short) if there exist constants $k \geq 2$ and $a > 0$, and a sequence (c_i) of positive numbers such that $\sum_{i=i_0}^{\infty} c_i < \infty$ for some natural i_0 and the inequality

$$\Phi_i(2u) \leq k\Phi_i(u) + c_i$$

holds for every $i \in \mathcal{N}$ and $u \in \mathbb{R}$ satisfying $\Phi_i(u) \leq a$ (see [8] and [15]).

2. Results

Lemma 2.1. *A set B in a Musielak-Orlicz space ℓ_Φ is l_Ψ-weakly sequentialy compact if and if* $\lim_{j\to\infty} \sup_{x\in B} \sum_{i>j} |x(i)\,y(i)| = 0$ *for any* $y \in l_\Psi$.

Proof. This is a general result in Köthe sequence lattices (see [18]). □

Another characterization of ℓ_Ψ-weakly sequentially compact subsets of ℓ_Φ which will be used in the proof of Theorem 2.3 gives the next lemma.

Lemma 2.2. *A set B in a Musielak-Orlicz space ℓ_Φ is l_Ψ-weakly sequentialy compact if and only if* $\lim_{j\to\infty} \lim_{\xi\to 0} \sup_{x\in B} \sum_{i>j} \frac{1}{\xi}\Phi_i(\xi\,|x(i)|) = 0$.

Proof. *Sufficiency.* Take any $y \in l_\Psi$ and choose $\eta > 0$ such that $I_\Psi(\eta y) < \infty$. For any $\varepsilon > 0$, take $j_0 > 0$ and $\xi_0 > 0$ such that $\sum_{i>j_0} \frac{1}{\xi_0} \Phi_i(\xi_0 |x(i)|) < \frac{\varepsilon\eta}{2}$ for any $x \in B$. Next take $i_0 > j_0$ such that $\sum_{i>i_0} \Psi_i(\eta |y(i)|) < \frac{\xi_0 \eta \varepsilon}{2}$. Then for all $x \in B$,

$$\begin{aligned} \sum_{i>i_0} \frac{1}{\xi_0} |x(i)\, y(i)| &= \frac{1}{\xi_0 \eta} \sum_{i>i_0} |\xi_0 x(i)\, \eta y(i)| \\ &\le \frac{1}{\eta} \sum_{i>i_0} \frac{\Phi_i(\xi_0 x(i))}{\xi_0} + \frac{1}{\eta \xi_0} \sum_{i>i_0} \Psi_i(\eta y(i)) < \varepsilon. \end{aligned}$$

By Lemma 2.1, we conclude that B is l_Ψ-weakly sequentialy compact.

Necessity. Since B is l_Ψ-weakly sequentialy compact, so B is norm bounded. Without loss of generality, we may assume that $I_\Phi(x) \le 1$ for any $x \in B$ since if we can prove that

$$\lim_{j\to\infty} \lim_{\xi\to 0} \sup_{x\in B(\ell_\Phi)} \sum_{i>j} \frac{1}{\xi} \Phi_i(\xi |x(i)|) = 0,$$

then

$$\lim_{j\to\infty} \lim_{\xi\to 0} \sup_{x\in B} \frac{\|x\|_\Phi}{\xi} \sum_{i>j} \Phi_i\left(\xi \frac{|x(i)|}{\|x\|_\Phi}\right) = 0,$$

which means that $\lim_{j\to\infty} \lim_{\xi\to 0} \sup_{x\in B} \sum_{i>j} \frac{1}{\xi} \Phi_i(\xi |x(i)|) = 0$. If the necessity does not hold, then there exist $\varepsilon > 0$, $i_k \uparrow \infty$, $\xi_k \downarrow 0$ and a sequence $\{x_k\}$ in B satisfying $\sum_{i>i_k} \frac{1}{\xi_k} \Phi_i(\xi_k |x_k(i)|) > \varepsilon$ $(k = 1, 2, ...)$. We may also assume here that $\xi_1 \le \frac{1}{2}$ and $\sum_{k=1}^\infty \xi_k < \infty$. Put

$$y = (y(i))_{i=1}^\infty, \quad \text{where } y(i) = \sup_k p_i(\xi_k |x_k(i)|)\, (i = 1, 2, ...).$$

By the Young equality $tp_i(t) = \Phi_i(t) + \Psi_i(p_i(t))$ and the inequality $tp_i(t) \le \Phi_i(2t)$ for all natural i and $t \ge 0$, where p_i denotes the right hand side derivative of Φ_i, we have

$$\begin{aligned} I_\Psi(y) &= \sum_{i=1}^\infty \Psi_i\Big(\sup_k (p_i(\xi_k |x_k(i)|))\Big) \le \sum_{i=1}^\infty \sum_{k=1}^\infty \Psi_i(p_i(\xi_k |x_k(i)|)) \\ &\le \sum_{i=1}^\infty \sum_{k=1}^\infty \xi_k |x_k(i)|\, p_i(\xi_k |x_k(i)|) \le \sum_{i=1}^\infty \sum_{k=1}^\infty \Phi_i(2\xi_k |x_k(i)|) \\ &\le 2 \sum_{i=1}^\infty \sum_{k=1}^\infty \xi_k \Phi_i(x_k(i)) = 2 \sum_{k=1}^\infty \xi_k I_\Phi(x_k) \le 2 \sum_{k=1}^\infty \xi_k < \infty. \end{aligned}$$

This shows that $y \in l_\Psi$. By Lemma 2.1, we get the following contradiction:

$$\begin{aligned} 0 \leftarrow \sum_{i>i_k} x_k(i)\, y(i) &\ge \sum_{i>i_k} |x_k(i)|\, p_i(\xi_k |x_k(i)|) \\ &= \sum_{i>i_k} \frac{1}{\xi_k} \xi_k |x_k(i)|\, p_i(\xi_k |x_k(i)|) \ge \sum_{i>i_k} \frac{\Phi_i(\xi_k |x_k(i)|)}{\xi_k} > \varepsilon. \end{aligned}$$

□

If $\Phi \notin \delta_2$, then $WCS\,(l_\Phi) = 1$ because in this case $(\ell_\Phi)^+$ contains a sequence $(x_n)_{n=1}^{\infty}$ with pairwise disjoint supports and with$\|x_n\| = 1$ for any $n \in \mathbb{N}$ and $\|\sup_n x_n\| = 1$. So we assume in the following that $\Phi \in \delta_2$.

Theorem 2.3. *If Φ is an Orlicz function satisfying condition δ_2, we have the following formula*

$$WCS\,(l_\Phi) = \liminf_{n\to\infty} \Big\{ \inf \Big[c_{x,n} > 0 : x \in S\,(l_\Phi)\,,\ \mathrm{supp}\,(x) \ \textit{is finite,} \\ m\,(x) \geq n,\ I_\Phi\left(\frac{x}{c_{x,n}}\right) \leq \frac{1}{2}, \lim_{\lambda\to 0} \frac{I_\Phi\,(\lambda x)}{\lambda} \leq \frac{1}{n} \Big] \Big\},$$

where $m\,(x) = \min\,\{n : n \in \mathrm{supp}\,(x)\}\,.$

Proof. Let us denote the right hand side of the equality in Theorem 2.3 by d.

First, we will show that $WCS\,(l_\Phi) \leq d$. Given any $\varepsilon > 0$, by the definition of the lower limit, there exists $n_1 \in N$ with

$$\inf \Big[c_{x,2x_1} > 0 : x \in S\,(l_\Phi)\,,\ \mathrm{supp}\,(x)\ \text{ is finite, } m\,(x) \geq 2x_1, \\ I_\Phi\left(\frac{x}{c_{x,2x_1}}\right) \leq \frac{1}{2},\ \lim_{\lambda\to 0} \frac{I_\Phi\,(\lambda x)}{\lambda} \leq \frac{1}{2x_1} \Big] \\ \leq d + \varepsilon \quad (\forall\ n \geq n_1).$$

Then by the definition of the infimum, there exists $x_1 \in S\,(l_\Phi)$ with finite supp(x_1) and $m\,(x_1) \geq n_1$ such that $\lim_{\lambda\to 0} \frac{I_\Phi(\lambda x_1)}{\lambda} \leq \frac{1}{n_1}$ and

$$\inf \left\{ c_{x_1,n_1} > 0 : I_\Phi\left(\frac{x_1}{c_{x_1,n_1}}\right) \leq \frac{1}{2} \right\} \leq d + \varepsilon.$$

Since supp(x_1) is finite, there exists $n_2 > n_1$ satisfying supp$(x_1) \cap [n_2, +\infty) = \emptyset$.

Now we can get $x_2 \in S\,(l_\Phi)$ with finite supp(x_2), $m\,(x_2) \geq n_2$ and $\lim_{\lambda\to 0} \frac{I_\Phi(\lambda x_2)}{\lambda} \leq \frac{1}{n_2}$, satisfying $\inf \Big[c_{x_2,2x_2} > 0 : I_\Phi\left(\frac{x_2}{c_{x_2,2x_2}}\right) \leq \frac{1}{2} \Big] \leq d + \varepsilon$ for all $n \geq n_2$.

Continuing the above process, we get that there exist a sequence of natural numbers $n_1 < n_2 < \cdots$, satisfying for all $i \in N$:

(1) $x_i \in S\,(l_\Phi)\,;$

(2) supp(x_i) is finite, and supp$(x_i) \cap$ supp $(x_j) = \phi$;

(3) $m\,(x_i) \geq n_i$, and $\lim_{\lambda\to 0} \frac{I_\Phi(\lambda x_i)}{\lambda} \leq \frac{1}{n_i}$;

(4) $\inf \left\{ c_{x_i,n_i} > 0 : I_\Phi\left(\frac{x_i}{c_{x_i,n_i}}\right) \leq \frac{1}{2} \right\} \leq d + \varepsilon.$

We claim that

$$\lim_{j\to\infty} \lim_{\lambda\to 0} \sup_{i\in N} \frac{1}{\lambda} I_\Phi(\lambda x_i \chi_{\{j+1,j+2,\ldots\}}) = 0.$$

Take any $\varepsilon > 0$. Since $1/n_i \searrow 0$ as $i \to \infty$, there is i_ε such that $1/n_i < \varepsilon/3$ for all $i > i_\varepsilon$. Since $\lim_{\lambda\to 0}(I_\Phi(\lambda x_i)/\lambda) \le 1/n_i$ for all $i \in \mathbb{N}$, there is $\lambda_\varepsilon > 0$ such that

$$\frac{1}{\lambda} I_\Phi(\lambda x_i) < \frac{\varepsilon}{2}$$

for all $i > i_\varepsilon$ and $0 < \lambda \le \lambda_\varepsilon$. Since $\Phi \in \delta_2$, we have that $I_\Phi(\lambda x_i) < \infty$ for any $\lambda > 0$ and $i \in \mathbb{N}$. Consequently, there is $j_\varepsilon \in \mathbb{N}$ such that

$$\frac{1}{\lambda_\varepsilon} I_\Phi(\lambda_\varepsilon x_i \chi_{\{j+1,j+2,\ldots\}}) < \frac{\varepsilon}{2}$$

for any $i \in \{1, \ldots, i_\varepsilon\}$ and all natural $j \ge j_\varepsilon$. Since the function $\Phi(u)/u$ is nondecreasing, we have that

$$\frac{1}{\lambda} I_\Phi(\lambda x_i \chi_{\{j+1,j+2,\ldots\}}) \le \frac{1}{\lambda_\varepsilon} I_\Phi(\lambda_\varepsilon x_i \chi_{\{j+1,j+2,\ldots\}}) < \frac{\varepsilon}{2}$$

for all $j \ge j_\varepsilon$, $0 < \lambda \le \lambda_\varepsilon$ and $i \in \{1, \ldots, i_\varepsilon\}$. In consequence,

$$\frac{1}{\lambda} I_\Phi(\lambda x_i \chi_{\{j+1,j+2,\ldots\}}) < \frac{\varepsilon}{2}$$

for all $i \in \mathbb{N}, j \ge j_\varepsilon$ and $0 < \lambda \le \lambda_\varepsilon$, whence

$$\frac{1}{\lambda} \sup_{i\in\mathbb{N}} |I_\Phi(\lambda x_i \chi_{\{j+1,j+2,\ldots\}}) \le \frac{\varepsilon}{2} < \varepsilon$$

for all $j \ge j_\varepsilon$ and $0 < \lambda \le \lambda_\varepsilon$, and the claim is proved.

Therefore, by Lemma 2.2, we have that $x_i \xrightarrow{w} 0$.

It is obvious that

$$I_\Phi\left(\frac{x_i + x_j}{d+\varepsilon}\right) = I_\Phi\left(\frac{x_i}{d+\varepsilon}\right) + I_\Phi\left(\frac{x_j}{d+\varepsilon}\right) \le \frac{1}{2} + \frac{1}{2} = 1,$$

whence $||x_i - x_j|| = ||x_i + x_j|| \le d+\varepsilon$. This shows that $A(\{x_n\}) \le d+\varepsilon$, so $WCS(l_\Phi) \le d$, by the arbitrariness of $\varepsilon > 0$.

In the following, we will prove that $WCS(l_\Phi) \ge d$. Let us take any sequence $\{x_n\}$ in $S(\ell_\Phi)$ that is weakly convergent to zero. Since $x_n \xrightarrow{w} 0$ we have that $x_n \to 0$ coordinatewise and, by Lemma 2.2,

$$\lim_{i\to\infty} \lim_{\lambda\to 0} \sup_n \frac{1}{\lambda} I_\Phi\left(x_n \chi_{\{i_1+1,\, i_1+2,\ldots\}}\right) = 0. \tag{1}$$

By $\Phi \in \delta_2$, we have for any $x \in \ell_\Phi$ that

$$\|(0, ..., 0, x(i), x(i+1), ...)\|_\Phi \to 0 \quad \text{as } i \to \infty. \tag{2}$$

Moreover, since $x_n \to 0$ coordinatewise, we have for any natural j:

$$\left\|\sum_{i=1}^{j} x_n(i)\, e_i\right\|_\Phi \to 0 \quad \text{as } n \to \infty. \tag{3}$$

On the base of (1), (2) and (3), there are natural numbers n_1, i_1 and j_1 such that $i_1 < j_1$ and the element $y_{n_1} := \sum_{i=i_1+1}^{j_1} x_{n_1}(i)\, e_i$ satisfies the inequalities

$$1 \geq \|y_{n_1}\|_\Phi \geq 1 - \frac{1}{2} \quad \text{and} \quad \lim_{\lambda \to 0} \frac{1}{\lambda} I_\Phi(\lambda y_{n_1}) \leq \frac{1}{i_1}.$$

Next, we can find natural numbers n_2, i_2 and j_2 such that $n_2 > n_1$, $j_1 < i_2 < j_2$ and the element $y_{n_2} := \sum_{i=i_2+1}^{j_2} x_{n_2}(i)\, e_i$ satisfies the inequalities

$$1 \geq \|y_{n_2}\|_\Phi \geq 1 - \frac{1}{3} \quad \text{and} \quad \lim_{\lambda \to 0} \frac{1}{\lambda} I_\Phi(\lambda y_{n_2}) \leq \frac{1}{i_2}.$$

Continuing this construction by induction, one can find three sequences $\{n_k\}$, $\{i_k\}$ and $\{j_k\}$ of natural numbers such that $n_{k+1} > n_k$, $i_k < j_k < i_{k+1}$ and the elements $y_{n_k} := \sum_{i=i_k+1}^{j_k} x_{n_k}(i)\, e_i$ satisfy the equalities

$$1 \geq \|y_{n_k}\|_\Phi \geq 1 - \frac{1}{k+1} \quad \text{and} \quad \lim_{\lambda \to 0} \frac{1}{\lambda} I_\Phi(\lambda y_{n_k}) \leq \frac{1}{i_k}$$

for any k. It is obvious that $m(y_{n_k}) > i_k$ for any k.

By the definition of d, there exists $n_0 \in N$ such that

$$\inf \Big[c_{x,n} > 0 : x \in S(l_\Phi),\ \text{supp}(x) \ \text{is finite},\ m(x) \geq n, \\ I_\Phi\left(\frac{x}{c_{x,n}}\right) \leq \frac{1}{2},\ \lim_{\lambda \to 0} \frac{I_\Phi(\lambda x)}{\lambda} \leq \frac{1}{n} \Big] > d - \varepsilon,$$

whenever $n > n_0$.

Hence

$$I_\Phi\left(\frac{y_{n_k} - y_{n_l}}{d - \varepsilon}\right) = I_\Phi\left(\frac{y_{n_k}}{d - \varepsilon}\right) + I_\Phi\left(\frac{y_{n_l}}{d - \varepsilon}\right) > \frac{1}{2} + \frac{1}{2} = 1$$

for $k \neq l$; k, l being large enough.

Consequently, $||y_{n_k} - y_{n_l}|| > d - \varepsilon$ for $k \neq l$; k and l being large enough, that is, $A(\{y_{n_k}\}) > d - \varepsilon$. Since $\|x_{n_k} - x_{n_l}\| \geq \|y_{n_k} - y_{n_l}\| - \frac{1}{k} - \frac{1}{l}$, so $A(\{x_{n_k}\}) = A(y_{n_k}) > d - \varepsilon$. Hence $WCS(l_\Phi) > d - \varepsilon$. By the arbitrariness of $\varepsilon > 0$, we have $WCS(l_\Phi) \geq d$ and the proof of the theorem is finished. □

Example 2.4. Take for example the l^p space $(1 < p < \infty)$ and caculate $WCS(l^p)$. Take any $x \in S(\ell^p)$. If $c > 0$ satisfies the equality $I_\Phi\left(\frac{x}{c}\right) = \frac{1}{2}$, i.e. $\frac{1}{c^p} \sum_{i=1}^{\infty} |x(i)|^p = \frac{1}{2}$, then $c^p = 2$, and so $c = 2^{\frac{1}{p}}$. Therefore $WCS(l^p) = 2^{\frac{1}{p}}$. This is the result that has been proved originally by Bynum in [4].

In the following let $\{p_k\}_{k=1}^{\infty}$ be an increasing sequence of real numbers with $p_1 > 1$ and $\lim_{n \to \infty} p_n = p < \infty$.

Example 2.5. Let $l^{(p_i)}$ be the Nakano sequence space equipped with the Luxemburg norm. Then $WCS\left(l^{(p_i)}\right) = 2^{\frac{1}{p}}$.

Proof. For any weakly null sequence $\{x_n\}$ in $S\left(l^{(p_i)}\right)$ and any $x \in S\left(l^{(p_i)}\right)$ with finite $m(x)$, we may assume without loss of generality that $m(x) \cap m(x_n) = \emptyset$ for any $n \in N$. We now consider the equation

$$I_\Phi\left(\frac{x}{c}\right) + I_\Phi\left(\frac{x_n}{c}\right) = 1,$$

that is,

$$\sum_{i=1}^{\infty}\left|\frac{x(i)}{c}\right|^{p_i} + \sum_{i=1}^{\infty}\left|\frac{x_n(i)}{c}\right|^{p_i} = 1.$$

Since $c > 0$ satisfying the last equality is greater than 1, we have

$$\frac{1}{c^p}\sum_{i=1}^{\infty}|x(i)|^{p_i} + \frac{1}{c^p}\sum_{i=1}^{\infty}|x_n(i)|^{p_i} = \frac{2}{c^p} \leq 1,$$

i.e. $c \geq 2^{\frac{1}{p}}$. This shows the inequality $WCS\left(l^{(p_i)}\right) \geq 2^{\frac{1}{p}}$.

Take the sequence $\{e_n\}$, where $e_n = (\underbrace{0,\ldots,0}_{n-1}1, 0, 0, \ldots)$. Then $e_n \in l^{(p_i)}$ for any $n \in N$ and $\lim_{\lambda\to 0}\frac{I_\Phi(\lambda e_n)}{\lambda} = 0 < \frac{1}{n}$. Since $I_\Phi\left(e_n/2^{\frac{1}{p}}\right) = \left(2^{-\frac{1}{p}}\right)^{p_i} = 2^{-\frac{p_i}{p}} \leq \frac{1}{2}$, we get $\inf\left\{c > 0 : I_\Phi\left(\frac{x}{c}\right) \leq \frac{1}{2}\right\} \leq 2^{\frac{1}{p}}$. Consequently $WCS\left(l^{(p_i)}\right) \leq 2^{\frac{1}{p}}$, whence $WCS\left(l^{(p_i)}\right) = 2^{\frac{1}{p}}$. □

Theorem 2.6. *A Musielak-Orlicz sequence space l_Φ corresponding to a finitely valued and vanishing only at zero Musielak-Orlicz function Φ with $\sup_{i\in\mathbb{N}}\Phi_i(2a_i) < \infty$, where a_i are the positive numbers satisfying $\Phi_i(a_i) = 1$ for all $i \in \mathbb{N}$, has the weakly uniformly normal structure if and only if $\Phi \in \delta_2$.*

Proof. We need only to prove that if $\Phi \in \delta_2$, then the Musielak-Orlicz sequence space l_Φ has the weakly uniformly normal structure because if $\Phi \notin \delta_2$, then l_Φ contains a linearly isometric copy of l_∞ and so l_Φ has not the weakly uniformly normal structure. By virtue of the Kamińska considerations in [9], we may assume that $\Phi_i(1) = 1$ for all $i \in \mathcal{N}$. We will divide our proof into two steps.

Step 1. We will prove that if $I_\Phi(x_n) \to 0$ then $||x_n|| \to 0$ as $n \to \infty$. Although the proof of this implication was given in [6] we present here another proof. Suppose that the implication does not hold. Then passing to a subsequence if necessary we may assume that there exists $\varepsilon_0 > 0$ such that $\lim_{n\to\infty}||x_n|| = \varepsilon_0$. Hence there exists $n_0 \in N$ such that

$$||x_n|| > \frac{\varepsilon_0}{2} \text{ for } n > n_0.$$

By using $\Phi \in \delta_2$, we have

$$I_\Phi\left(\frac{x_n}{||x_n||}\right) = 1 \text{ for all } n \in N.$$

Using again $\Phi \in \delta_2$ and the fact that all Φ_i vanish only at zero, we get that there exist $a > 0$, $k > 0$ and $c_i > 0$ such that $\sum_{i=1}^{\infty} c_i < \frac{1}{3}$ and

$$\Phi_i\left(\frac{2u}{\varepsilon_0}\right) \leq k\Phi_i(u) + c_i$$

for all $i \in N$ and $u \in R$ satisafying $\Phi_i(u) \leq a$.

Hence, we have for all $n > n_0$,

$$\begin{aligned}
1 = I_\Phi\left(\frac{x_n}{||x_n||}\right) &\leq I_\Phi\left(\frac{2x_n}{\varepsilon_0}\right) \\
&= \sum_{\Phi_i(x_n(i))\leq a} \Phi_i(\frac{2x_n(i)}{\varepsilon_0}) + \sum_{\Phi_i(x_n(i))>a} \Phi_i(\frac{2x_n(i)}{\varepsilon_0}) \\
&\leq \sum_{i=1}^{\infty} k\Phi_i(x_n(i)) + \sum_{i=1}^{\infty} c_i + \sum_{\Phi_i(x_n(i))>a} \Phi_i(\frac{2x_n(i)}{\varepsilon_0}) \\
&= kI_\Phi(x_n) + \sum_{i=1}^{\infty} c_i + \sum_{\Phi_i(x_n(i))>a} \Phi_i(\frac{2x_n(i)}{\varepsilon_0}).
\end{aligned}$$

Since $I_\Phi(x_n) \to 0$ as $n \to \infty$ by the assumption, there exists $n_1 \geq n_0$ such that $kI_\Phi(x_n) < \frac{1}{3}$ for $n > n_1$. Therefore, we have the inequality

$$1 \leq \frac{1}{3} + \frac{1}{3} + \sum_{\Phi_i(x_n(i))>a} \Phi_i(\frac{2x_n(i)}{\varepsilon_0})$$

for $n > n_1$, which gives that the value of the last sum is $\geq \frac{1}{3}$ for any $n > n_1$.

This shows that for each $n > n_1$ there exists $i \in N$ such that $\Phi_i(x_n(i)) > a$, i.e., $I_\Phi(x_n) > a$ for any $n > n_1$. A contradiction, which finishes the proof of the implication.

Step 2. Suppose that the assumptions about Φ are satisfied but l_Φ has not the weakly uniformly normal structure, i.e. $WCS(l_\Phi) = 1$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $S(l_\Phi)$ for which the sets $\operatorname{supp}(x_n)$ are finite, $m(x_n) > n$, $\lim_{\lambda\to 0} \frac{I_\Phi(\lambda x_n)}{\lambda} < \frac{1}{n}$ and $\overline{c}_{x_n} \searrow 1$, where $\overline{c}_{x_n} = c_{x_n,n}$ and $c_{x_n,n}$ are from the formula for $WCS(\ell_\Phi)$ in Theorem 2.3. By $\Phi \in \delta_2$, there are a constant $K > 0$ and a sequence (c_i) of nonnegative numbers such that

$$\sum_{i=1}^{\infty} \Phi_i(c_i) < \infty \quad \text{and} \quad \Phi_i(2u) \leq K\Phi_i(u) \tag{$*$}$$

for $u \in [c_i, 1]$, $n \in \mathcal{N}$. Condition $(*)$ follows from condition δ_2 for Φ and the assumption that $\sup_{i\in N} \Phi_i(2) < \infty$. Namely, condition δ_2 is equivalent to the fact that there exist constants $a \in (0,1)$ and $K \geq 2$ and two sequences of positive numbers $(c_i)_{i=1}^{\infty}$ and $(d_i)_{i=1}^{\infty}$ with $\Phi_i(d_i) = a$ and $c_i < d_i$ for all $i \in N$ such that

$$\sum_{i=1}^{\infty} \Phi_i(c_i) < \infty \quad \text{and} \quad \Phi_i(2u) \leq K_1\Phi_i(u)$$

for all $i \in N$ and $u \in [c_i, d_i]$. Then for all $i \in N$ and $u \in [d_i, 1]$, we have

$$\Phi_i(2u) \leq \Phi_i(2) = \frac{\Phi_i(2)}{\Phi_i(d_i)}\Phi_i(d_i) \leq \frac{\Phi_i(2)}{a}\Phi_i(u).$$

So, it is enough to define $K = \max(K_1, \sup_i \Phi_i(2))$.

Put $\overline{x}_n = (\overline{x}_n(1), \overline{x}_n(2), \cdots\cdots, \overline{x}_n(i), \cdots\cdots\cdots)$, where $\overline{x}_n(i) = x_n(i)$ if $|x_n(i)| \leq \overline{c}_{x_n} c_i$ and $\overline{x}_n(i) = \overline{c}_{x_n} c_i \operatorname{sgn}(x_n(i))$ if $|x_n(i)| > \overline{c}_{x_n} c_i$ for each $i \in \mathcal{N}$. Then $I_\Phi\left(\frac{\overline{x}_n}{\overline{c}_{x_n}}\right) \to 0$ as $n \to \infty$. By the result from Step 1, we have that $||\overline{x}_n|| \to 0$. Moreover, we also have $||x_n - \overline{x}_n|| \to 1$.

Notice that thanks to condition $(*)$, we have $I_\Phi(2\frac{(x_n - \overline{x}_n)}{\overline{c}_{x_n}}) \leq K I_\Phi(\frac{x_n}{\overline{c}_{x_n}}) + k\sum_{i=1}^\infty c_i \leq K + k\sum_{i=1}^\infty c_i < \infty$ for all $n \in N$. Let us introduce the following function $g : R^+ \to [0, \infty]$:

$$g(\lambda) = \sup\left\{I_\Phi\left(\lambda\frac{x_n - \overline{x}_n}{\overline{c}_{x_n}}\right) : n = 1, 2, \cdots\right\}$$

for $\lambda \in R^+$. Then g is convex, $g(0) = 0$, $g(1) \leq \frac{1}{2}$ and $g(2) \leq k + \sum_{i=1}^\infty c_i < \infty$. Therefore, g is continuous on the interval $[0, 2)$. Thus, by the Darboux property of g, there exists $\lambda_0 \in (1, 2)$ such that $g(\lambda_0) \leq 1$. This means that $I_\Phi\left(\lambda_0\frac{x_n - \overline{x}_n}{\overline{c}_{x_n}}\right) \leq 1$ for all $n \in N$, whence we have $||x_n - \overline{x}_n|| \leq \overline{c}_{x_n}/\lambda_0$ for all $n \in N$. Since $\overline{c}_{x_n} \searrow 1$ and $1 < \lambda_0 < 2$, we conclude that there exists $m \in N$ such that $\sup_{n \geq m}(\overline{c}_{x_n}/\lambda_0) < 1$, which, by the above inequality, contradicts the condition $\|x_n - \overline{x}_n\| \to 1$ as $n \to \infty$. $\square$

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