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Some Quantitative Interpolation Theorems under Lions-Peetre's Methods

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We formulate the quantitative version of interpolation theorems on compactness and weak compactness, the improved estimate for the k-uniform rotundity, and even the stability of the nearly uniform convexity under Lions-Peetre's interpolation methods of constants and means associated with quasi-power function parameters.

Keywords: Interpolation, k-uniform rotundity, Lions-Peetre's method, nearly uniform convexity, noncompactness, weak noncompactness

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A basic result for the classical real interpolation methods is that the boundedness of a linear operator can be interpolated between Banach spaces with a logarithmically convex estimate for the interpolation norm. There are many generalizations of this kind of interpolation results on the behavior of other properties of Banach spaces and linear operators. For instance, the interpolation theorems on compactness, weak compactness, k-uniform rotundity, and nearly uniform convexity for Banach spaces or/and bounded linear operators under the classical real methods are valid in both qualitative and quantitative way [4], [7], [9], [12].

In [5 & 6], the author studied the more general interpolation methods of constants and means due to Lions and Peetre. We showed that, like the classical real methods, many properties of spaces and operators are stable under these methods. Among other things, the compactness can be inherited [6, Prop. 2.3]. In this paper, we intend to formulate the quantitative version of the above mentioned and some related properties under the generalized Lions-Peetre methods. In the first section, we collect some basic results for Lions-Peetre's interpolation methods. Section 2 is concerned with the estimates of the measures of noncompactness and weak noncompactness. Section 3 is devoted to the moduli of k-uniform rotundity. In Section 4, we consider the stability of the nearly uniform convexity under this kind of interpolation methods.

1. Lions-Peetre's interpolation methods with quasi-power functions

Throughout this paper, the notations $\subseteq$ and $=$ between Banach spaces stand for continuous inclusion and isomorphic equivalence respectively. For Banach space X, we denote

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by $\dot{U}_X$ and U_X the open and closed unit balls of X respectively. For a subset A of X, we denote $\operatorname{conv} A$ as the convex hull of A in X. For Banach spaces X and Y, we denote by $\mathcal{B}(X, Y)$ the Banach space of all bounded linear operators T from X to Y with the norm $\|T\|_{X,Y}$, and simply write $\mathcal{B}(X) = \mathcal{B}(X, X)$. We use standard interpolation theory notation as can be found in [1] and [2]. For a Banach couple $\overline{X} = (X_0, X_1)$ and an intermediate space X for $\overline{X}$, the regularization X^0 for $\overline{X}$ is the closure of $\Delta\overline{X}$ in X, and the dual space X' for $\overline{X}$ is the Banach space dual of X^0. For two Banach couples $\overline{X}$ and $\overline{Y}$, we denote by $\mathcal{B}\left(\overline{X}, \overline{Y}\right)$ the Banach space of all bounded linear operators T from $\Sigma\overline{X}$ to $\Sigma\overline{Y}$, for which $T \in \mathcal{B}(X_j, Y_j)$ with the norm

$$\|T\|_j = \|T\|_{X_j,Y_j} \quad (j = 0, 1),$$

and denote $\|T\|_{\overline{X},\overline{Y}} = \|T\|_0 \vee \|T\|_1$. We simply write $\mathcal{B}\left(\overline{X}\right) = \mathcal{B}\left(\overline{X}, \overline{X}\right)$. For $t > 0$, the J- and K-functionals on $\Delta\overline{X}$ and $\Sigma\overline{X}$, respectively, are given by

$$J(t, x) = \|x\|_0 \vee (t\, \|x\|_1)$$

for $x \in \Delta\overline{X}$, and

$$K(t, x) = \inf\left\{ \|x_0\|_0 + t\, \|x_1\|_1 \;\middle|\; x = x_0 + x_1,\, x_j \in X_j\, (j = 0, 1)\right\}$$

for $x \in \Sigma\overline{X}$.

Let ρ be a positive, strictly increasing and quasi-concave function defined on $\mathbf{R}^+ = (0, \infty)$, we denote

$$\rho^*(t) = \frac{1}{\rho\,(1/t)} \quad \text{and} \quad \bar{\rho}(t) = \sup_{s>0} \frac{\rho(st)}{\rho(s)}$$

for $t > 0$. A corresponding homogeneous function of two variables (again denoted by ρ) is defined by $(t_0, t_1) \longmapsto t_0\rho\,(t_1/t_0)$ for $t_0,\, t_1 > 0$. Function ρ is said to be submultiplicative if $\rho(st) \le \rho(s)\rho(t)$ for all $s, t > 0$, and of quasi-power if $\exists\, c > 0$ and $0 < \alpha < 1$ such that

$$\bar{\rho}(t) \le c\left(t^{1-\alpha} \vee t^{\alpha}\right)$$

for all $t > 0$. In the latter case, we may define $\rho(0) = \rho^*(0) = 0$, and assume that $c = 1$ throughout the paper. Thus $\bar{\rho}(2) < 2$. Given two non-negative functions f, g defined on an interval $[0, d)$, we denote $f \succ g$ (or $g \prec f$) if there exist positive constants a, b such that $f(t) \ge a g(bt)$ for t small enough.

Let $\rho\colon \mathbf{R}^+ \to \mathbf{R}^+$ be a quasi-power function with $\rho(1) = 1$, and let $1 \le p_0, p_1 \le \infty$. We define K_{ρ,p_0,p_1} and J_{ρ,p_0,p_1} to be Lions-Peetre's interpolation methods of constants and means associated with the function parameter ρ respectively. More precisely, the space $K_{\rho,p_0,p_1}\left(\overline{X}\right)$ consists of all those $x \in \Sigma\overline{X}$ such that there exist strongly measurable functions $x_j\colon \mathbf{R}^+ \to X_j$ $(j = 0, 1)$ satisfying $x = x_0 + x_1$ and $t^j\, \|x_j(t)\|_j / \rho(t) \in L^{p_j}\left(\mathbf{R}^+, dt/t\right)$ $(j = 0, 1)$ with the norm

$$\|x\|_{K_{\rho,p_0,p_1}} = \inf\left\{ \left\| \|x_0(t)\|_0 / \rho(t) \right\|_{L^{p_0}(dt/t)} + \left\| t\, \|x_1(t)\|_1 / \rho(t) \right\|_{L^{p_1}(dt/t)} \right\};$$

and the space $J_{\rho,p_0,p_1}\left(\overline{X}\right)$ consists of all those $x \in \Sigma\overline{X}$ such that there exists a strongly measurable function $u\colon \mathbf{R}^+ \to \Delta\overline{X}$ satisfying $x = \int_0^\infty u(t)\,dt/t$ and $t^j \left\|u(t)\right\|_j/\rho(t) \in L^{p_j}\left(\mathbf{R}^+, dt/t\right)$ $(j = 0, 1)$ with the norm

$$\|x\|_{J_{\rho,p_0,p_1}} = \inf\left\{\max_{j=0,1}\left\|t^j\left\|u(t)\right\|_j/\rho(t)\right\|_{L^{p_j}(dt/t)}\right\}.$$

In [5], the author described these interpolation methods in terms of the Brudnyi-Krugljak methods associated with the quasi-power parameters. Now we define $\varphi\colon \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ by

$$\varphi^{-1}(t) = t^{1/p_0}\rho\left(t^{-1/q}\right), \tag{1}$$

where $1/q = 1/p_0 - 1/p_1$. Then φ is a Young function satisfying both Δ_2 and ∇_2 conditions. Let Φ be the weighted Orlicz space of all measurable functions $f\colon \mathbf{R}^+ \longrightarrow \mathbf{R}$ such that $\int_0^\infty \varphi\left(t^{-q/p_0}|f(t)|\right)t^q\,dt/t < \infty$, which is equipped with the Luxemberg norm. If $p_0 \neq p_1$, then

$$K_\Phi\left(\overline{X}\right) = K_{\rho,p_0,p_1}\left(\overline{X}\right) = J_{\rho,p_0,p_1}\left(\overline{X}\right) = J_\Phi\left(\overline{X}\right), \tag{2}$$

where the real interpolation methods K_Φ and J_Φ associated with Φ are given in [2, (3.3.1) & (3.4.3)] as follows

$$K_\Phi\left(\overline{X}\right) = \left\{x \in \Sigma\overline{X} \;\middle|\; \|x\|_{K_\Phi} = \|K(t,x)\|_\Phi < \infty\right\}$$

and

$$J_\Phi\left(\overline{X}\right) = \left\{x \in \Sigma\overline{X} \;\middle|\; x = \int_0^\infty u(t)\,dt/t,\ \|x\|_{J_\Phi} = \inf_u \|J(t,u(t))\|_\Phi < \infty\right\}.$$

If $p_0 = p_1 = p$, then

$$K_{\rho,p,p}\left(\overline{X}\right) = J_{\rho,p,p}\left(\overline{X}\right) = J_{\rho,p}\left(\overline{X}\right) = K_{\rho,p}\left(\overline{X}\right), \tag{3}$$

where $J_{\rho,p} = J_{L^p_\rho}$ and $K_{\rho,p} = K_{L^p_\rho}$, for which $f \in L^p_\rho$ iff $f/\rho \in L^p\left(\mathbf{R}^+, dt/t\right)$.

Let us now turn to the discrete version of the K_{ρ,p_0,p_1}- and J_{ρ,p_0,p_1}-methods. The space $K^d_{\rho,p_0,p_1}\left(\overline{X}\right)$ consists of all those $x \in \Sigma\overline{X}$, for which

$$x = x_0(k) + x_1(k), \quad k \in \mathbf{Z},$$

satisfying $x_j(k) \in X_j$ and $\left(2^{jk}\left\|x_j(k)\right\|_j/\rho(2^k)\right)_k \in l^{p_j}$ $(j = 0, 1)$ with the norm

$$\|x\|_{K^d_{\rho,p_0,p_1}} = \inf\left\{\left\|\left(\left\|x_0(k)\right\|_0/\rho(2^k)\right)_k\right\|_{l^{p_0}} + \left\|\left(2^k\left\|x_1(k)\right\|_1/\rho(2^k)\right)_k\right\|_{l^{p_1}}\right\};$$

and the space $J^d_{\rho,p_0,p_1}\left(\overline{X}\right)$ consists of all those $x \in \Sigma\overline{X}$, for which there exists a sequence $(u(k))_{k\in\mathbf{Z}}$ in $\Delta\overline{X}$ with $x = \sum_k u(k)$ in $\Sigma\overline{X}$ and $\left(2^{jk}\left\|u(k)\right\|_j/\rho(2^k)\right)_k$ in l^{p_j} $(j = 0, 1)$ with the norm

$$\|x\|_{J^d_{\rho,p_0,p_1}} = \inf\left\{\max_{j=0,1}\left\|\left(2^{jk}\left\|u(k)\right\|_j/\rho(2^k)\right)_k\right\|_{l^{p_j}}\right\}.$$

It is easy to obtain the following equivalence

$$K^d_{\rho,p_0,p_1}\left(\overline{X}\right) = K_{\rho,p_0,p_1}\left(\overline{X}\right) = J_{\rho,p_0,p_1}\left(\overline{X}\right) = J^d_{\rho,p_0,p_1}\left(\overline{X}\right). \tag{4}$$

In the sequel, we will choose suitable interpolation norms to estimate the different moduli/measures because of this equivalence.

For $1 \le p \le \infty$ and for $j = 0, 1$, let l^p_j be the sequence space over $\mathbf{Z}$ consisting of all $\lambda = (\lambda_\nu)_\nu$ such that

$$\|\lambda\|_{l^p_j} = \left(\sum_\nu \left(2^{-j\nu}|\lambda_\nu|\right)^p\right)^{1/p} < \infty.$$

For $1 \le p_0, p_1 \le \infty$ and for φ given in (1), let l^φ_* be the weighted Orlicz sequence space consisting of all $\lambda = (\lambda_\nu)_\nu$ such that

$$\sum_\nu \varphi\left(2^{-\nu q/p_0}|\lambda_\nu|\right) 2^{\nu q} < \infty,$$

which is equipped with the Luxemberg norm. Like the continuous version [5], we have

$$l^\varphi_* \subseteq J^d_{\rho,p_0,p_1}\left(l^1_0, l^1_1\right) \subseteq K^d_{\rho,p_0,p_1}\left(l^\infty_0, l^\infty_1\right) \subseteq l^\varphi_*.$$

This implies that

$$l^\varphi_* = J^d_{\rho,p_0,p_1}\left(l^1_0, l^1_1\right) = K^d_{\rho,p_0,p_1}\left(l^\infty_0, l^\infty_1\right), \tag{5}$$

and hence, by (2), (3) and (4),

$$K^d_{\rho,p_0,p_1}\left(\overline{X}\right) = K_{l^\varphi_*}\left(\overline{X}\right) = J_{l^\varphi_*}\left(\overline{X}\right) = J^d_{\rho,p_0,p_1}\left(\overline{X}\right), \tag{6}$$

where

$$K_{l^\varphi_*}\left(\overline{X}\right) = \left\{x \in \Sigma\overline{X} \;\middle|\; \|x\|_{K_{l^\varphi_*}} = \left\|\left(K(2^k, x)\right)_k\right\|_{l^\varphi_*} < \infty\right\}$$

and

$$J_{l^\varphi_*}\left(\overline{X}\right) = \left\{x \in \Sigma\overline{X} \;\middle|\; x = \sum_\nu u(k),\ \|x\|_{J_{l^\varphi_*}} = \inf_u \left\|\left(J\left(2^k, u(k)\right)\right)_k\right\|_{l^\varphi_*} < \infty\right\}.$$

2. On measures of noncompactness and weak noncompactness

In this section, we study the measures of noncompactness and weak noncompactness for bounded linear operators interpolated by Lions-Peetre's methods. We present first the discrete version of [6, Prop. 2.2].

Lemma 2.1. *Let* $X = K^d_{\rho,p_0,p_1}\left(\overline{X}\right)$ *and let* $Y = K^d_{\rho,p_0,p_1}\left(\overline{Y}\right)$.

(*i*) *If* $x \in X$ *with the decomposition* $x = x_0(k) + x_1(k)$, $k \in \mathbf{Z}$, *for which* $x_j(k) \in X_j$ *and* $\left(2^{jk}\|x_j(k)\|_j/\rho(2^k)\right)_k \in l^{p_j}$ $(j = 0, 1)$, *and if we denote*

$$M_j = \left\|\left(2^{jk}\|x_j(k)\|_j/\rho(2^k)\right)_k\right\|_{l^{p_j}} \quad (j = 0, 1),$$

then

$$\|x\|_X \le 4\,\bar\rho\left(M_0, M_1\right).$$

(*ii*) *If* $T \in \mathcal{B}\left(\overline{X}, \overline{Y}\right)$*, then* $\|T\|_{X,Y} \le 4\,\bar{\rho}\left(\|T\|_0, \|T\|_1\right)$.

Proof. If we choose $n \in \mathbf{Z}$ such that $2^{n-1} < M_1/M_0 \le 2^n$, then

$$x = x_0(k+n) + x_1(k+n).$$

This implies that

$$\begin{aligned} \|x\|_X &\le \left\| \left(\frac{\|x_0(k+n)\|_0}{\rho(2^k)} \right)_k \right\|_{l^{p_0}} + \left\| \left(\frac{2^k \|x_1(k+n)\|_1}{\rho(2^k)} \right)_k \right\|_{l^{p_1}} \\ &\le \left(M_0 + 2^{-n} M_1\right) \bar{\rho}(2^n) \le 4\,\bar{\rho}\left(M_0, M_1\right), \end{aligned}$$

which completes the proof of part (*i*), and implies part (*ii*). □

Let X and Y be Banach spaces. For a closed bounded subset A of X, we define the (ball) measure of noncompactness for the set A by

$$\chi_X(A) = \inf\left\{\eta > 0 \,|\, A \subseteq F + \eta U_X \text{ for some finite subset } F \text{ of } X\right\}. \tag{7}$$

For $T \in \mathcal{B}(X, Y)$, we define the (ball) measure of noncompactness for the operator T by

$$\chi(T) = \chi\left(T\colon X \to Y\right) = \chi_Y\left(T(U_X)\right).$$

Observe that $\chi(T) \le \|T\|_{X,Y}$ and, if $S, T \in \mathcal{B}(X, Y)$, then

$$\chi(S+T) \le \chi(S) + \chi(T).$$

For $T \in \mathcal{B}\left(\overline{X}, \overline{Y}\right)$, let $\chi_j(T) = \chi\left(T\colon X_j \to Y_j\right)$ $(j = 0, 1)$, and let

$$\chi_{\rho,p_0,p_1}(T) = \chi\left(T\colon K^d_{\rho,p_0,p_1}\left(\overline{X}\right) \to K^d_{\rho,p_0,p_1}\left(\overline{Y}\right)\right).$$

We establish now a formula concerning the measure of noncompactness under Lions-Peetre's methods of interpolation by following the approach of [3].

Theorem 2.2. *Assume that* $1 < p_0, p_1 < \infty$*. If* $T \in \mathcal{B}\left(\overline{X}, \overline{Y}\right)$*, then*

$$\chi_{\rho,p_0,p_1}(T) \le c\bar{\rho}\left(\chi_0(T), \chi_1(T)\right),$$

where c *is a positive constant only depending on* p_0*,* p_1*,* ρ*.*

Proof. If we choose $0 < \theta_0 < \alpha \wedge (1-\alpha)$, $\theta_1 = 1 - \theta_0$ and

$$\frac{1}{r_j} = \frac{1-\theta_j}{p_0} + \frac{\theta_j}{p_1} \quad (j = 0, 1),$$

then by [10, Ex. 5.3], l^{φ}_{*} is an interpolation space for the couple $\left(l^{r_0}_{\theta_0}, l^{r_1}_{\theta_1}\right)$. Thus, the Calderón transform

$$\Omega\left((\lambda_\nu)_\nu\right) = \left(\sum_k \left(1 \wedge 2^{\nu-k}\right) \lambda_k \right)_k$$

is bounded on l^{φ}_{*}. For $\kappa \in \mathbf{Z}$, let τ_κ be the shift operator on l^{φ}_{*} given by

$$\tau_\kappa\left((\lambda_\nu)_\nu\right) = (\lambda_{\nu+\kappa})_\nu$$

for $(\lambda_\nu)_\nu \in l^{\varphi}_{*}$. Now we have

$$\|\tau_\kappa\|_{l^{\varphi}_{*},l^{\varphi}_{*}} = O\left(\|\tau_\kappa\|_{l^{r_0}_{\theta_0},l^{r_0}_{\theta_0}} \vee \|\tau_\kappa\|_{l^{r_1}_{\theta_1},l^{r_1}_{\theta_1}}\right) = O\left(2^{\kappa\theta_0} \vee 2^{\kappa\theta_1}\right),$$

which implies that

$$\lim_{\kappa\to\infty} 2^{-\kappa}\|\tau_\kappa\|_{l^{\varphi}_{*},l^{\varphi}_{*}} = \lim_{\kappa\to\infty} \|\tau_{-\kappa}\|_{l^{\varphi}_{*},l^{\varphi}_{*}} = 0.$$

Moreover, by (5), (6) and Lemma 2.1 (ii), we have

$$\|\tau_\kappa\|_{l^{\varphi}_{*},l^{\varphi}_{*}} \leq c\,\bar{\rho}\left(\|\tau_\kappa\|_{l^{\infty}_{0},l^{\infty}_{0}}, \|\tau_\kappa\|_{l^{\infty}_{1},l^{\infty}_{1}}\right) = c\,\bar{\rho}\left(2^{\kappa}\right)$$

for a positive constant c depending on p_0, p_1 and ρ. By applying [3, Th. 5.1] on the sequence space l^{φ}_{*} and by rewriting the constant, we obtain the estimate

$$\chi_{\rho,p_0,p_1}(T) \leq c\bar{\rho}\left(\chi_0(T), \chi_1(T)\right),$$

which completes the proof. □

Let us consider now the measure of weak noncompactness introduced by Kryczka, Prus and Szczepanik [7], and apply their idea on Lions-Peetre's methods. Let X be a Banach space and let $(x_\nu)_{\nu\geq 1}$ be a sequence in X. A sequence $(y_\nu)_{\nu\geq 1}$ in X is said to be successive convex combinations (scc for short) for $(x_\nu)_\nu$ if there exists a sequence of integers $0 = k_0 < k_1 < \ldots$ such that $y_\nu \in \operatorname{conv}\{x_i\}_{i=k_\nu+1}^{k_{\nu+1}}$. In particular, vectors u_1, u_2 are said to be a pair of scc for $(x_\nu)_\nu$ if $u_1 \in \operatorname{conv}\left(\{x_\nu\}_{\nu=1}^{k}\right)$ and $u_2 \in \operatorname{conv}\left(\{x_\nu\}_{\nu=k+1}^{\infty}\right)$ for some integer $k \geq 1$. By the convex separation of $(x_\nu)_\nu$ we mean

$$\operatorname{csep}\left((x_\nu)_\nu\right) = \inf\left\{\|u_1 - u_2\|_X \,\middle|\, u_1,\, u_2 \text{ is a pair of scc for } (x_\nu)_\nu\right\}.$$

For each nonempty and bounded subset A of X, we define the measure of weak noncompactness of A by

$$\gamma(A) = \gamma_X(A) = \sup\left\{\operatorname{csep}\left((x_\nu)_\nu\right) \,\middle|\, (x_\nu)_\nu \subseteq \operatorname{conv} A\right\}.$$

For Banach spaces X and Y, let $T \in \mathcal{B}(X, Y)$. We define the measure of weak noncompactness of operator T by

$$\gamma(T) = \gamma\left(T\colon X \to Y\right) = \gamma_Y\left(T(\dot{U}_X)\right).$$

We formulate now the following generalization of [7, Th. 3.8].

Theorem 2.3. *Assume that* $1 < p_0, p_1 < \infty$. *If* $T \in \mathcal{B}\left(\overline{X}, \overline{Y}\right)$, *then*

$$\gamma_{\rho,p_0,p_1}(T) \leq 4\,\bar{\rho}\left(\gamma_0(T), \gamma_1(T)\right),$$

where $\gamma_j(T) = \gamma_{X_j,Y_j}(T)$ $(j = 0, 1)$ *and*

$$\gamma_{\rho,p_0,p_1}(T) = \gamma\left(T\colon K^d_{\rho,p_0,p_1}\left(\overline{X}\right) \to K^d_{\rho,p_0,p_1}\left(\overline{Y}\right)\right),$$

respectively.

Proof. For $j = 0, 1$, the operator $T \in \mathcal{B}\left(\overline{X}, \overline{Y}\right)$ induces operators $\widetilde{T}_j \in \mathcal{B}\left(l^{p_j}[X_j], l^{p_j}[Y_j]\right)$ $(j = 0, 1)$ by the formula

$$\widetilde{T}_j x = (Tx(k))_k \quad \text{for } x = (x(k))_k \in l^{p_j}[X_j].$$

According to [7, Th. 3.6 & Rmk. 3.7], we have

$$\gamma\left(\widetilde{T}_j\right) = \gamma_j(T) \quad (j = 0, 1). \tag{8}$$

Now we assume that $X = K^d_{\rho,p_0,p_1}\left(\overline{X}\right)$ and $Y = K^d_{\rho,p_0,p_1}\left(\overline{Y}\right)$. Let $\epsilon > 0$ be fixed and let $(x_\nu)_\nu$ be a sequence in $\dot{U}_X$. For each x_ν, there exists a decomposition $x_\nu = x_{0,\nu}(k) + x_{1,\nu}(k)$, $k \in \mathbf{Z}$, such that $\left(2^{jk} x_{j,\nu}(k)/\rho(2^k)\right)_k \in \dot{U}_{l^{p_j}[X_j]}$ $(j = 0, 1)$. Let $y_\nu = Tx_\nu$ and $y_{j,\nu} = \left(2^{jk} T x_{j,\nu}(k)/\rho(2^k)\right)_k$ $(j = 0, 1)$. A similar argument as in the proof of [7, Th. 2.3 & Th. 3.8] shows that one can find a sequence of integers $0 = n_1 < n_2 < \dots$ and nonnegative numbers λ^ν_l, for which $\sum_{l=n_\nu+1}^{n_{\nu+1}} \lambda^\nu_l = 1$, and the sequences $(z_{j,\nu})_\nu$ given by

$$z_{j,\nu} = \sum_{l=n_\nu+1}^{n_{\nu+1}} \lambda^\nu_l y_{j,l} \quad (j = 0, 1)$$

satisfying

$$\|z_{j,k} - z_{j,m}\|_{l^{p_j}} \le \operatorname{csep}\left((z_{j,\nu})_\nu\right) + \epsilon. \tag{9}$$

Let $z_\nu = \sum_{l=n_\nu+1}^{n_{\nu+1}} \lambda^\nu_l y_l$. Then, by combining (8), (9) and Lemma 2.1 (i), we obtain

$$\begin{aligned}
\operatorname{csep}\left((y_\nu)_\nu\right) &\le \|z_1 - z_2\|_{K^d_{\rho,p_0,p_1}} \le 4\,\bar{\rho}\left(\|z_{0,1} - z_{0,2}\|_{l^{p_0}}, \|z_{1,1} - z_{1,2}\|_{l^{p_1}}\right) \\
&\le 4\,\bar{\rho}\left(\operatorname{csep}\left((z_{0,\nu})_\nu\right) + \epsilon, \operatorname{csep}\left((z_{1,\nu})_\nu\right) + \epsilon\right) \\
&\le 4\,\bar{\rho}\left(\gamma\left(\widetilde{T}_0\right) + \epsilon, \gamma\left(\widetilde{T}_1\right) + \epsilon\right) = 4\,\bar{\rho}\left(\gamma_0(T) + \epsilon, \gamma_1(T) + \epsilon\right).
\end{aligned}$$

The estimate for $\gamma_{\rho,p_0,p_1}(T)$ follows by letting $\epsilon \to 0$ and by taking the superimum of $\operatorname{csep}((y_\nu)_\nu)$. □

Remark 2.4. If we denote $X = K^d_{\rho,p_0,p_1}\left(\overline{X}\right)$, and choose T to be the identity operator, then

$$\gamma_X\left(U_X\right) \le 4\,\bar{\rho}\left(\gamma_{X_0}\left(U_{X_0}\right), \gamma_{X_1}\left(U_{X_1}\right)\right).$$

In particular, if X_0 or X_1 is reflexive, then X is also reflexive.

Remark 2.5. Recall that an operator T is compact iff $\chi(T) = 0$, and T is weakly compact iff $\gamma(T) = 0$. Assume that $T \in \mathcal{B}\left(\overline{X}, \overline{Y}\right)$. As a consequence of Theorem 2.1 and Theorem 2.2, we obtain that, if $T\colon X_0 \to Y_0$ or $T\colon X_1 \to Y_1$ is compact, resp., weakly compact, then

$$T\colon K^d_{\rho,p_0,p_1}\left(\overline{X}\right) \to K^d_{\rho,p_0,p_1}\left(\overline{Y}\right)$$

is also compact, resp., weakly compact.

Remark 2.6. For a Banach space X, let $T \in \mathcal{B}(X)$. We denote by $r(T, X)$ and $r_e(T, X)$ the spectral radius and the essential spectral radius of T on X, respectively. It is known that

$$r(T, X) = \lim_{k \to 0} \left(\|T^k\|_{X,X}\right)^{1/k}. \tag{10}$$

and

$$r_e(T,X) = \lim_{k\to 0} \chi\left(T^k\right)^{1/k}. \tag{11}$$

Let $T \in \mathcal{B}\left(\overline{X}\right)$, and let $X = K^d_{\rho,p_0,p_1}\left(\overline{X}\right)$. By combining the submultiplicativity of $\bar{\rho}$ with (10), (11), (3), Lemma 2.1 *(ii)* and Theorem 2.2, we have

$$r(T,X) \le \bar{\rho}\left(r(T,X_0), r(T,X_1)\right),$$

and

$$r_e(T,X) \le \bar{\rho}\left(r_e(T,X_0), r_e(T,X_1)\right).$$

3. On moduli of k-uniform rotundity

In this and next section, we establish some interpolation formulae concerning the k-uniform rotundity and nearly uniform convexity for Bacach spaces. These results are known for the classical real interpolation methods, and can be easily carried to Lions-Peetre's methods with quasi-power function parameters. Let us start with a useful lemma.

Lemma 3.1. *Let f_0 and f_1 be concave and increasing functions defined on $[0,1)$ with $f_0(0) = f_1(0) = 0$. If t_0, t_1, $s \in (0,1)$ with $s \le \bar{\rho}(t_0,t_1)$, then*

$$\bar{\rho}\left(f_0(1-t_0)^{1/k}, f_1(1-t_1)^{1/k}\right) \le c^{1/k}\bar{\rho}\left(f_0(1-s)^{1/k}, f_1(1-s)^{1/k}\right),$$

where $c = 1\big/\left(\alpha \wedge (1-\alpha)\right)$.

Proof. Without loss of generality, we may assume that $0 < \alpha \le 1-\alpha < 1$. If $t_0 \le t_1$, then

$$s \le t_0\bar{\rho}\left(t_1/t_0\right) \le t_0^{\alpha} \cdot t_1^{1-\alpha} \le \alpha t_0 + (1-\alpha)t_1,$$

and hence $1 - s \ge \alpha(1-t_0) + (1-\alpha)(1-t_1)$. This implies that

$$\begin{aligned} f_0(1-s) &\ge \alpha f_0(1-t_0) + (1-\alpha)f_0(1-t_1) \ge \alpha f_0(1-t_0), \\ f_1(1-s) &\ge \alpha f_1(1-t_0) + (1-\alpha)f_1(1-t_1) \ge \alpha f_1(1-t_1). \end{aligned}$$

Similarly, if $t_0 \ge t_1$, then we also have

$$f_0(1-s) \ge \alpha f_0(1-t_0) \quad \text{and} \quad f_1(1-s) \ge \alpha f_1(1-t_1).$$

Therefore,

$$\begin{aligned} \bar{\rho}\left(f_0(1-t_0)^{1/k}, f_1(1-t_1)^{1/k}\right) &\le \bar{\rho}\left(\left(f_0(1-s)/\alpha\right)^{1/k}, \left(f_1(1-s)/\alpha\right)^{1/k}\right) \\ &= c^{1/k}\bar{\rho}\left(f_0(1-s)^{1/k}, f_1(1-s)^{1/k}\right), \end{aligned}$$

which completes the proof. □

Let X be a Banach space, and let $x_0, x_1, \ldots, x_k \in X$. The k-dimensional volume enclosed by $x_0, x_1, \ldots, x_k$ is defined by

$$\mathcal{A}_X\left(\{x_i\}_i\right) = \sup_{x_l^* \in X', \|x_l^*\|_{X'} \le 1} \left\{ \begin{vmatrix} 1 & \cdots & 1 \\ \langle x_1^*, x_0\rangle & \cdots & \langle x_1^*, x_k\rangle \\ \vdots & \ddots & \vdots \\ \langle x_k^*, x_0\rangle & \cdots & \langle x_k^*, x_k\rangle \end{vmatrix} \right\}.$$

The modulus of k-rotundity of X is defined by

$$\delta_X^{(k)}(\epsilon) = \inf\left\{1 - \left\|\frac{x_0 + x_1 + \cdots + x_k}{k+1}\right\|_X \,\middle|\, \|x_i\|_X \le 1,\, \mathcal{A}_X\left(\{x_i\}_i\right) \ge \epsilon\right\}$$

for $0 \le \epsilon \le (k+1)^{(k+1)/2}$. The space X is k-uniformly rotund (k-UR in short), or equivalently k-uniformly convex, if $\delta_X^{(k)}(\epsilon) > 0$ for $\epsilon > 0$.

For $1 < p < \infty$ and for a Banach space X, let us denote $L^p[X]$ the X-valued L^p-space over $\left(\mathbf{R}^+, dt/t\right)$. If X is k-UR, then $L^p[X]$ is also k-UR [12]. In [6, Prop. 6.2], the author showed that if X_0 or X_1 is k-UR, then the interpolation space $J_{\rho,p_0,p_1}\left(\overline{X}\right)$ is also k-UR with the estimate

$$\delta_X^{(k)}(\epsilon) \ge 1 - \bar{\rho}\left(1 - \delta_{L^{p_0}[X_0]}^{(k)}\left(c\,\epsilon^{1/\alpha} \wedge \epsilon^{1/(1-\alpha)}\right),\; 1 - \delta_{L^{p_1}[X_1]}^{(k)}\left(c\,\epsilon^{1/\alpha} \wedge \epsilon^{1/(1-\alpha)}\right)\right)$$

for a positive constant c and for ϵ small enough. We can now improve this estimate for the modulus of k-rotundity in case that both X_0 and X_1 are k-UR.

Theorem 3.2. *Assume that $1 < p_0, p_1 < \infty$. Let $X = J_{\rho,p_0,p_1}\left(\overline{X}\right)$, and let $\delta_j = \delta_{L^{p_j}[X_j]}^{(k)}$ $(j = 0, 1)$. If both X_0 and X_1 are k-UR, then*

$$\delta_X^{(k)} \succ \left(\bar{\rho}\left(\left(\delta_0^{-1}\right)^{1/k}, \left(\delta_1^{-1}\right)^{1/k}\right)^k\right)^{-1}.$$

Proof. By [12, Prop. 8 & Prop. 1], $L^{p_j}[X_j]$ are k-UR, and hence we may assume that δ_j are strictly increasing and convex $(j = 0, 1)$. Let $0 < \epsilon < (k+1)^{(k+1)/2}$, and let $x_i \in U_X$, $i = 0, 1, \ldots, k$, with $\mathcal{A}_X\left(\{x_i\}_i\right) \ge \epsilon$. If $\eta > 0$ fixed, then by [6, Prop. 6.1], we may find decompositions $x_i = \int_0^\infty u_i(t)\,dt/t$, for $i = 0, 1, \ldots, k$, such that $v_i^j(t) = t^j u_i(t)/\left((1+\eta)\rho(t)\right) \in L^{p_j}[X_j]$ $(j = 0, 1)$, and

$$\frac{\epsilon}{(1+\eta)^k} \le \mathcal{A}_X\left(\left\{\frac{x_i}{1+\eta}\right\}_i\right) \le k^{k/2}\bar{\rho}\left(\epsilon_0^{1/k}, \epsilon_1^{1/k}\right)^k, \tag{12}$$

where $\epsilon_j = \mathcal{A}_{L^{p_j}[X_j]}\left(\{v_i^j\}_i\right)$ $(j = 0, 1)$. Let

$$f_j = \delta_j^{-1}, \quad t_j = \frac{1}{k+1}\left\|\sum_{i=0}^k v_i^j\right\|_{L^{p_j}[X_j]} \qquad (j = 0, 1),$$

and let

$$s = \frac{1}{(1+\eta)(k+1)}\left\|\sum_{i=0}^k x_i\right\|_X.$$

Then we have $\epsilon_j \le f_j(1 - t_j)$ $(j = 0, 1)$, and $s \le \bar{\rho}(t_0, t_1)$ by [6, Prop. 2.2]. This, together with (12) and Lemma 3.1, implies that

$$\epsilon \le c\,k^{k/2}(1+\eta)^k\bar{\rho}\left(f_0(1-s)^{1/k}, f_1(1-s)^{1/k}\right)^k.$$

Consequently,

$$\epsilon \leq c\, k^{k/2} \bar{\rho}\left(\delta_0^{-1}\left(\delta_X^{(k)}(\epsilon)\right)^{1/k}, \delta_1^{-1}\left(\delta_X^{(k)}(\epsilon)\right)^{1/k}\right)^k$$

by letting $\eta \to 0$. Therefore,

$$\delta_X^{(k)} \succ \left(\bar{\rho}\left((\delta_0^{-1})^{1/k}, (\delta_1^{-1})^{1/k}\right)^k\right)^{-1},$$

which completes the proof. □

Example. Let $\varphi\colon \mathbf{R}^+ \to \mathbf{R}^+$ be a submultiplicative Young function with $\varphi(1) = 1$, and let

$$\underline{p}_\varphi = \inf_{t>0} \left(t\varphi'(t)/\varphi(t)\right) \quad \text{and} \quad \overline{p}_\varphi = \sup_{t>0} \left(t\varphi'(t)/\varphi(t)\right)$$

satisfying $2 < \underline{p}_\varphi \leq \overline{p}_\varphi < \infty$. We may choose p_0, p_1 such that

$$2 < p_0 < \underline{p}_\varphi \leq \overline{p}_\varphi < p_1 < \infty,$$

and define $\rho\colon \mathbf{R}^+ \to \mathbf{R}^+$ by

$$\rho(t) = t^{q/p_0} \varphi^{-1}\left(t^{-q}\right).$$

Then ρ is of quasi-power with $\bar{\rho} = \rho$ and $\rho(1) = 1$ [6, Sec. 3]. Let now (Ω, μ) be a complete σ-finite measure space. By [11, Ex. 3.13], [12, Th. 2] and the complex interpolation, we have

$$\delta^{(k)}_{L^{p_j}[L^{p_j}(\Omega)]}(\epsilon) \succ \delta^{(k)}_{L^2[L^2(\Omega)]}\left(\epsilon^{p_j/2}\right) \succ \epsilon^{p_j/k} \quad (j = 0, 1)$$

for $0 \leq \epsilon \leq (k+1)^{(k+1)/2}$. Now let us consider $X = J_{\rho, p_0, p_1}\left(L^{p_0}(\Omega), L^{p_1}(\Omega)\right)$, which is isomorphic to the Orlicz sequence space $L^\varphi(\Omega)$. According to Theorem 3.1, we may obtain

$$\begin{aligned}
\left(\delta_X^{(k)}\right)^{-1}(\epsilon) &\prec \rho\left(\left(\delta^{(k)}_{L^{p_0}[L^{p_0}(\Omega)]}\right)^{-1}(\epsilon)^{1/k}, \left(\delta^{(k)}_{L^{p_1}[L^{p_1}(\Omega)]}\right)^{-1}(\epsilon)^{1/k}\right)^k \\
&= \rho\left(\epsilon^{1/p_0}, \epsilon^{1/p_1}\right) \prec \varphi(\epsilon)^k,
\end{aligned}$$

and hence $\delta_X^{(k)}(\epsilon) \succ \varphi\left(\epsilon^{1/k}\right)$.

4. On nearly uniform convexity

Let X be an infinite dimensional Banach space. Recall that the modulus of noncompact convexity of X is defined by

$$\begin{aligned}
\Delta_X(\epsilon) = \inf\Big\{1 - \inf\left\{\|x\|_X \,\middle|\, x \in A\right\} \,\Big|\, & A \text{ is a closed} \\
& \text{convex subset of } U_X \text{ with } \chi_X(A) \geq \epsilon\Big\}
\end{aligned}$$

for $0 \leq \epsilon \leq 1$, where $\chi_X(A)$ is the measure of noncompactness for set A given in (7). The Banach space X is said to be nearly unifomly convex (NUC in short) if $\Delta_X(\epsilon) > 0$ for all $\epsilon > 0$. According to [9, Cor. 4.2], the nearly uniform convexity was stable under the classical real interpolation methods. By a similar way, we have the following result concerning Lions-Peetre's interpolation methods.

Theorem 4.1. *Let $1 < p_0, p_1 < \infty$ and let $X = K^d_{\rho,p_0,p_1}\left(\overline{X}\right)$. If X_j is NUC for $j = 0$ or 1, then X is also NUC.*

Proof. For $0 < \epsilon < 1$, let A be a closed convex subset of U_X with $\chi_X(A) \geq \epsilon$. Then there exists a sequence $(x_\nu)_{\nu\geq 1}$ in A such that

$$\|x_m - x_n\|_X \geq \epsilon/2$$

for $m \neq n$. Fix an arbitrary $\eta \in (0,1)$, one may now decompose x_ν into $x_\nu = x^0_\nu(k) + x^1_\nu(k)$, $k \in \mathbf{Z}$, satisfying

$$\left\|\left(\left\|x^0_\nu(k)\right\|_0 / \rho(2^k)\right)_k\right\|_{l^{p_0}} + \left\|\left(2^k \left\|x^1_\nu(k)\right\|_1 / \rho(2^k)\right)_k\right\|_{l^{p_1}} \leq 1 + \eta.$$

Without loss of generality, we may assume that X_0 and hence $l^{p_0}[X_0]$ are NUC. Let $Y = l^{p_0}[X_0] \oplus_1 l^{p_1}$, the l^1-direct sum of $l^{p_0}[X_0]$ and l^{p_1}, and let $y_\nu \in Y$ given by

$$y_\nu = \left(x^0_\nu(k) / \left(\rho(2^k)(1+\eta)\right),\ 2^k \left\|x^1_\nu(k)\right\|_1 / \left(\rho(2^k)(1+\eta)\right)\right)_k.$$

Then $\|y_\nu\|_Y \leq 1$, and hence the closed convex hull A of the set $\left\{y_\nu \middle| \nu \in \mathbf{Z}\right\}$ is a subset of U_Y $(j = 0, 1)$. By Lemma 2.1 (i), we have

$$\begin{aligned} \epsilon/2 &\leq \|x_m - x_n\|_X \\ &\leq 4\,\bar{\rho}\left(\left\|\left(\frac{\|x^0_m(k) - x^0_n(k)\|_0}{\rho(2^k)}\right)_k\right\|_{l^{p_0}}, \left\|\left(\frac{2^k\,\|x^1_m(k) - x^1_n(k)\|_1}{\rho(2^k)}\right)_k\right\|_{l^{p_1}}\right). \end{aligned}$$

This implies that $\epsilon \leq 8(1+\eta)\,\|y_m - y_n\|_Y$, and hence

$$\delta = \chi_Y(A) \geq \frac{\epsilon}{16(1+\eta)} > 0.$$

By [8, Th. 1], Y is NUC and hence $\Delta_Y(\delta) > 0$. Then there exist $\lambda_\nu \geq 0$, $1 \leq \nu \leq l$, with $\sum_{\nu=1}^l \lambda_\nu = 1$ such that

$$\left\|\sum_{\nu=1}^l \lambda_\nu y_\nu\right\|_Y \leq (1+\eta)\left(1 - \Delta_Y(\delta)\right).$$

Let $x = \sum_{\nu=1}^l \lambda_\nu x_\nu$. Then

$$x = \sum_{\nu=1}^l \lambda_\nu x^0_\nu(k) + \sum_{\nu=1}^l \lambda_\nu x^1_\nu(k)$$

for each $k \in \mathbf{Z}$, and hence

$$\|x\|_X \leq \left\|\sum_{\nu=1}^l \lambda_\nu y_\nu\right\|_Y \leq (1+\eta)\left(1 - \Delta_Y(\delta)\right).$$

By taking the infimum of the left-hand side and by letting $\eta \to 0$, we obtain

$$\Delta_X(\epsilon) \geq \Delta_Y(\delta) > 0,$$

which implies that X is also NUC. $\square$

Remark 4.2. For Banach spaces X and Y, let $T \in \mathcal{B}(X,Y)$. The operator T is said to be NUC whenever, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $(x_\nu)_{\nu \geq 1}$ is a sequence in U_X with $\|Tx_n - Tx_m\|_Y \geq \epsilon$ for $m \neq n$, then

$$\inf\left\{\|x\|_X \;\middle|\; x \in \text{conv }\{x_\nu\}\right\} < 1 - \delta.$$

In particular, the identity operator on X is NUC iff the Banach space X is NUC. Assume that $T \in \mathcal{B}\left(\overline{X}, \overline{Y}\right)$. By a slight modification of Theorem 4.1, we may obtain that, if $T\colon X_0 \to Y_0$ or $T\colon X_1 \to Y_1$ is NUC, then

$$T\colon K^d_{\rho,p_0,p_1}\left(\overline{X}\right) \to K^d_{\rho,p_0,p_1}\left(\overline{Y}\right)$$

is also NUC.

Remark 4.3. A Banach space X is said to have the property (β) if, for any $\epsilon > 0$, there exists $\delta > 0$ such that, whenever $1 < \|x\|_X < 1 + \delta$,

$$\chi_X\left(\text{conv}\left(\{x\} \cup U_X\right) \setminus U_X\right) < \epsilon$$

holds. According to [9, Sec. 2], X has the property (β) iff, for any $\epsilon > 0$, there exists $\delta > 0$ such that, whenever $x \in U_X$ and $(x_\nu)_\nu$ is a sequence in U_X with

$$\inf\left\{\|x_n - x_m\|_X \;\middle|\; n \neq m\right\} \geq \epsilon,$$

we have

$$\frac{1}{2}\|x + x_\nu\|_X \leq 1 - \delta$$

for some ν. Let $1 < p_0, p_1 < \infty$ and let $X = K^d_{\rho,p_0,p_1}\left(\overline{X}\right)$. If X_j has (β)-property for $j = 0$ or 1, then by a slight modification on the proof of Theorem 4.1 and by [9, Prop. 4.5], we can show that X also has (β)-property.

Remark 4.4. It is known that if Banach space X is k-UR or has (β)-property, then X is NUC. These properties are so called metric properties which are invariant under isometries in contrast to topological properties which are invariant with respect to isomorphisms. In Theorem 3.1, we obtained an estimate of the modulus of k-UR for the interpolation space $J_{\rho,p_0,p_1}\left(\overline{X}\right)$. The proof of this result is partially based on the fact that, if Banach space is k-UR, so is $L^p[X]$ for $1 < p < \infty$. This property, however, is not valid for the nearly uniformly convex Banach spaces by [11, Th. 2]. On the other hand, for the discrete K^d_{ρ,p_0,p_1}-method, there exists a constant > 1 in the inequality given in Lemma 2.1 (i). Therefore, it seems difficult to establish a quantitative formula for the estimate of the modulus of noncompact convexity for either continuous or discrete versions of Lions-Peetre's interpolation spaces.

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