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Set-Valued Analysis by Covering

Jean-Yves Larqué
jean-yves.larque@club-internet.fr

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Here is a presentation of an answer to the problem, deeply studied in the $70^s - 80^s$ and to a large extent unsolved then, of differential and integral calculus transfer to the framework of set-valued analysis. This transfer is achieved through the identification of the set-valued maps with some families, the *coverings*, made up of some (class of) special functions, the *representations*. The process is related to the one of the atlas local maps of the differential geometry [2], [5].

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Introduction

The concept of covering is the result of coupling of two models: on the first hand, the concret model of fluid mechanic where velocities field modelise the evolution of the dynamic system, on the other hand, the theoretical model of atlas local maps of differential geometry.

More precisely every covering of a set-valued map express an evolution point after point of the image as a whole. It is thus possible in a natural way to identify the set-valued map with all its coverings (or, if necessary, with all its coverings which have the minimum suitability required by the mathematical context):

$$\begin{aligned} & [t \multimap F(t) \subset E\,] \\ \equiv \quad & \left\{ t \sim> f_j\,(t) = [r_j(t| : b_j \to E]\,/\,(equiv.rel)\,, j \in J;\ \underset{j\in J}{\cup} r_j\,(t|b_j) = F\,(t) \right\}. \end{aligned}$$

Considering the strong compatibility of the equivalence relation based on transport theory with any usual concept of the classical algebraic, boolean and analytical theories, the study of the set-valued map $t \multimap F(t)$ comes down to the study of the *representation* $t \mapsto r(t|$. Actually it leads to write $r(t|$ under the form: $r(t| = i + e_T(t|$ where i and $e_T(t|$ belongs to suitable spaces as we will see later. This transfer process is then systematic and gives a very simple, flexible, and adaptable tool.

The theory obtained in this way answers to the main goals:

- the single-valued analysis fits into it, being a specific case of single valued set-valued maps,
- it is compatible with the set algebraic theories,
- a posteriori it is not at odds with the usual set-valued analysis theories,
- it enables us to consider the extension of the one-to-one distribution theory, taking of polydistributions while keeping the latter's character and perspective.

After to have defined the concept of covering and its relation with set-valued maps (§ 1), I will define the essential algebra bases to our subject (§ 2). Then I develop one to one the extensions of classical one-one analysis concepts and of their main properties in the following order: continuity, differentiability, measurability and integrability (§ 3, 4, 5). I end this presentation of my work by an introduction to the extension of the distributions theory through that I denote polydistributions (§ 6).

General notations and conventions. E a Banach space, T a topological vector space, O an open, connected set of T.

For $K \subset E$ and $\varepsilon > 0 : B(K, \varepsilon) = \{x \in E; d(x, K) < \varepsilon\}$; and $B = B(\{0\}, 1)$.

A set-valued map of domain O, and images in E is denoted $F : O \multimap E$.

The word "function" always means here single-valued map.

1. Coverings

I give here first a formal exposition of the concepts of covering and subjacent set-valued map as well as the generality of their application field which is enough to a first quick reading of the article in its globality. Then I give the theoretical support of the fundamental concept of covering and I end this part by some examples.

1.1. Formal Coverings framework

1.1.1. General definitions

Definition 1.1. 1. We call dynamic, or elementary covering, any (transfer) equivalence class of representations [see 1.3] $f : O \sim> E$ where O is the common domain of the representations of the class.

We call then subjacent set-valued map to f the set-valued map of domain O $\underline{f} : O \multimap E$ defined by

$$\widehat{[b, r(\,]} = class[b, r(\,] = f \;:\; \underline{f}(t) = r(t|b).$$

2. We call covering any family $f_J = (f_j)_{j\in J} : O \sim> E$ of dynamics with same domain O and order of f_J the (possibly infinite) number $\omega(f_J) = |J|$. The subjacent set-valued

map associated with f_J is then $\underline{f_J} : O \multimap E$ defined by

$$\underline{f_J}(t) = \underset{j \in J}{\cup} \underline{f_j}(t).$$

Definition 1.2. Let $\mathcal{H} = (P_1, \ldots, P_k)$ be k properties.

1. A dynamic is said to verify properties $\mathcal{H}$ if it has a representation which verifies properties $\mathcal{H}$.

2. A covering is said to verify properties $\mathcal{H}$ if all its elements verify properties $\mathcal{H}$.

3. To say that a set-valued map F has properties $\mathcal{H}$ is to say that F is subjacent to at least one covering which verifies properties $\mathcal{H}$.

The properties $\omega(F) = 1$, $\omega(F) < |\mathbb{N}|$, $\omega(F) \leqslant |\mathbb{N}|$ are said order properties.

The properties: F is simple, non singular, essentially unchanged, k-regular, are said structure properties (see Definition 1.7)

We call then "coverings of F" the coverings f_J of subjacent set valued map F and same structure, order, and analytical properties than these assumed for F. We will denote $\mathcal{C}(F)$ their set.

In the same way, we call "internal dynamics of F" the dynamics f such as $\underline{f} \subset F$ which have same structure and analytical properties than these assumed for F. We will denote $\mathcal{I}(F)$ their set.

1.1.2. Application field

Case of functions (See Theorem 1.15). Any function $f : O \mapsto E$ defines an only dynamic f. We can identify functions, single-valued dynamics and set-valued maps with singleton image.

General case (See Theorem 1.16). Any set-valued map has coverings.

1.2. Theoretical support

1.2.1. Bases and macro germs

Definition 1.3. We call base any triple $b = (n, V, \mu)$ such as:

- $n \in \mathbb{N}^*$, $V \subset \mathbb{R}^n$ compact, convex, / $\overset{\circ}{V} \neq \emptyset$
- $d\mu = f d\lambda$ / $f > 0, C^\infty$ on a neighbourhood of V and $\int_{\overset{\circ}{V}} d\mu = 1$.

n is the dimension of b and is denoted $dim(b)$. We will denote the set of bases by $\mathcal{B}$.

Comments. We have then $\overline{\overset{\circ}{V}} = V$ and $\mu(V - \overset{\circ}{V}) = 0$.

Definition 1.4. In a similar way of the germ notion [2], we say that two functions u and v, from neighbourhoods of V to E, define the same b-macro germ if they coincide on V. The set of macro germs of base b is, with the usual quotient rules, a vector space and we will denote it E^b. Only if necessary we identify a macro germ $\widehat{u}$ with one of its suitable representatives and we denotes it $u : b \mapsto E$ with $x \in b$ for $x \in V$.

Definition 1.5. Let $b = (n, V, \mu)$ be a base, $k \in \overline{\mathbb{N}}$.

We denote $L(b) = L(b, E)$ the subspace of E^b made up of b-macro germs $\widehat{u}$ which a representative u belongs to the Lebesgue space $L^1(\overset{\circ}{V}, d\mu, E)$.

Modulo μ almost everywhere (μ *a.e.*) equality, it is a Banach space for:

$$\|\widehat{u}\|_b = \inf_{v \in \widehat{u} \cap L^1(\overset{\circ}{V}, d\mu, E)} \|v\|_{L^1(\overset{\circ}{V}, d\mu, E)} = \|u\|_{L^1(\overset{\circ}{V}, d\mu, E)}.$$

For $\widehat{u} \in L(b)$, we define the integral $\int_b \widehat{u} d\mu$ as the common value, when v belongs to $\widehat{u} \cap L^1(\overset{\circ}{V}, d\mu, E)$, of $\int_{\overset{\circ}{V}} v d\mu$. We can then identify $\int_b \widehat{u} d\mu$ and $\int_{\overset{\circ}{V}} u d\mu$.

We denote $\mathcal{E}^k(b)$ the subspace of b-macro germs, said k-regular, which a representative is of C^k class on a neighbourhood of V. It is a normed space as subspace of $L(b)$.

1.2.2. Representations

Definition 1.6. We call parametrized macro germ of base b on O any function $u(: t \mapsto u(t|$ of domain O and images in E^b.

Therefore, if b' is another base, we call respectively right extension and left extension of $u($ by b' the parametrized macro germs defined by

$$\begin{aligned} u_{b'}(t| : (x, x') \in b \times b' \mapsto\ & u_{b'}(t|x, x') = u(t|x) \quad \text{and} \\ {}_{b'}u(t| : (x', x) \in b' \times b \mapsto\ & {}_{b'}u(t|x', x) = u(t|x). \end{aligned}$$

Definition 1.7. We call representation of base b any parametrized macro germ $r(: t \mapsto r(t|$ which can be written in the form:

$$\forall t : r(t| = i + e(t| + s(t| \ \text{ where } i \in E^b,\ e(t| \in \mathcal{E}^0(b),$$
$$s(t| = 0\ \mu \text{ a.e. and is bounded on } b.$$

$i, e($ and $s($ are respectively the invariant, the evolution and the singularity of $r($ for the breakdown $[b, i, e(, s(] \equiv r($.

Therefore a representation is said to be simple, non singular, essentially unchanged, k-regular, if it has a breakdown such as respectively: $i = 0$, $s(t| = 0$, $e(t| = 0$, $e(t| \in \mathcal{E}^k(b)$, $\forall t$.

1.2.3. Transfer equivalence relation

Definition 1.8. We call transfer from a base $b = (n, V, \mu)$ to a base $b' = (n', V', \mu')$ any C^∞ transport from $\mu_{b'}$ to μ_b, that is to say any function $\tau : \mathbb{R}^{n'} \mapsto \mathbb{R}^n$ such as

- τ is C^∞ on a neighbourhood of V' and $\tau(V') \subset V$,
- $\tau^{\#}. : u \mapsto u \circ \tau$ is such as: $\forall u \in L(b)$, $\tau^{\#}.u \in L(b')$ and $\int_{b'} \tau^{\#}.u d\mu' = \int_b u d\mu$.

We will denote $Tr(b, b')$ their set [6].

Theorem 1.9.

1. $Tr\,(b, b') \circ Tr\,(b', b'') \subset Tr\,(b, b'')$

2. The inverse image by a transfer from b to b' of a set with null μ_b-measure is of null $\mu_{b'}$-measure.

Proof. *1.* is obvious.

2. follows from the property: if U is a borelian of $\mathbb{R}^n / U \subset \overset{\circ}{V}$: $\mu_{b'}(\tau^{-1}(U)) = \mu_b(U)$, [6]. □

We have then the following main statement:

Theorem 1.10.

$$\textit{If } dim(b) = dim(b') \textit{ then } Tr(b, b') \neq \varnothing.$$

Proof. Simple rewriting with the coverings terminology of the Caffarelli's theorem of the Transportation Theory [6]. □

Definition 1.11. We define the (transfer) equivalence relation between two parametrized macrogerms $u_1(: O \mapsto E^{b_1}$ and $u_2(: O \mapsto E^{b_2}$, denoted $u_1(\sim u_2($, by: $\exists b'_j \in \mathcal{B}, j = 1, 2 \,/\, dim(b_1 \times b'_1) = dim(b_2 \times b'_2)$, $\exists \tau_i \in Tr(b'_j \times b_j, b_i \times b'_i)$, $j \neq i = 1, 2$:

$$u_{i,b'_i}(t| = \tau_i^{\#} \cdot_{b'_j} u_j(t| \quad \text{on } b_i \times b'_i,\ \forall t \in O.$$

where $b \times b' = (n + n', V \times V', \mu_b \otimes \mu_{b'})$ denotes the product base.

Equivalence axioms check:

1. Reflexivity and symmetry are obvious.
2. Transitivity: Let $t \in O$, we put: $u(t| = u$, $v(t| = v$ and $w(t| = w$.

If $u_{b_1} =_{b'_1} v \circ \tau / \tau \in Tr\,(b'_1 \times b', b \times b_1)$, $\dim\,(b'_1 \times b') = \dim\,(b \times b_1)$ and $v_{b'_2} =_{b''_1} w \circ \tau' / \tau' \in Tr\,(b''_1 \times b'', b' \times b'_2)$, $\dim\,(b''_1 \times b'') = \dim\,(b' \times b'_2)$ then: $\dim\,(b'_1 \times b''_1 \times b'') = \dim\,(b \times b_1 \times b'_2)$ and:

$$\begin{aligned} u_{b_1 \times b'_2} = (u_{b_1})_{b'_2} &= \left({}_{b'_1}v\right)_{b'_2} \circ \left(\tau \times 1_{b'_2}\right) \\ &= {}_{b'_1}\left(v_{b'_2}\right) \circ \tau \times 1_{b'_2} \\ &= {}_{b'_1}\left({}_{b''_1}w\right) \circ \left(1_{b'_1} \times \tau'\right) \circ \left(\tau \times 1_{b'_2}\right) \\ &= {}_{b'_1 \times b''_1} w \circ \tau'' \end{aligned}$$

with: $\tau'' = \left(1_{b'_1} \times \tau'\right) \circ \left(\tau \times 1_{b'_2}\right) \in Tr\,(b'_1 \times b''_1 \times b'', b \times b_1 \times b'_2)$.

We have indeed:

Lemma. $\forall b, b', b'' \in \mathcal{B}$, $\forall \tau \in Tr\,(b, b')$: $\tau \times 1_{b''} \in Tr\,(b \times b'', b' \times b'')$ et $1_{b''} \times \tau \in Tr\,(b'' \times b, b'' \times b')$.

Proof. $\forall u \in L(b \times b)$:

$$\begin{aligned}\int_{b'\times b''} u\left(\tau(x'), x''\right) d\mu_{b'\times b''} &= \int_{b''}\left(\int_{b'} u\left(\tau(x'), x''\right) d\mu_{b'}\right) d\mu_{b''} \\ &= \int_{b''}\left(\int_{b} u\left(x, x''\right) d\mu_{b}\right) d\mu_{b''} \\ &= \int_{b\times b''} u\left(x, x''\right) d\mu_{b\times b''}\end{aligned}$$

because of Fubini's theorem $[x \to u(x, x'')] \in L(b)\mu_{b''}$ *a.e.* $x'' \in V''^{o}$ and similarly for $1_{b"} \times \tau$.

Theorem 1.12.

1. *For any b, b' and any $u($ we have: $u_{b'}(\sim u($ and ${}_{b'}u(\sim u($.*
2. *If $u_1(\sim u_2($, then: $u_1(t|b_1) = u_2(t|b_2)$, $\forall t$.*

Proof. Let be $t \in O$ and let's put: $u(t = u$ and $u_i(t = u_i$.

1. Let $b"$ be an arbitrary base, we have: $(u_{b'})_{b"} =_{b'\times b"} u \circ \tau_{2,3,1}$ and $u_{b'\times b"} =_{b"} (u_{b'}) \circ \tau_{3,2,1}$ with $\tau_{2,3,1} : (x, x', x") \to (x', x", x)$ and, $\tau_{3,2,1} : (x, x', x") \to (x", x', x)$. It follows that $u_{b'} \sim u$ and in a similar way ${}_{b'}u \sim u$.

2. Let be b_1, b_1' and $\tau : V \times V_1 \to V_1' \times V'$ suitable: for $(x, x_1) \in V \times V_1$, $u(x) = u_{b_1}(x, x_1) = {}_{b_1'}v\left(\tau(x, x_1)\right) = {}_{b_1'}v(x_1', x') = v(x') \in im(v)$, it follows then that $im(u) = im(v)$. □

Definition 1.13. We say that a representation has properties $\mathcal{H} = (P_1, \ldots, P_k)$ if it has a breakdown which has simultaneously properties $\mathcal{H}$. A set of properties $\mathcal{H}$ is said compatible if any representation equivalent to a representation which has properties $\mathcal{H}$ has also properties $\mathcal{H}$ and the compatibility is said strong if therefore the validity of $\mathcal{H}$ is actually independent from the considered breackdown.

Theorem 1.14. *If a parametrized macrogerm $u($ on O and a representation $r($ on O are equivalent, then $u($ is a representation. More precisely:*

let be $r(\equiv [b, e(, s( ]$, $u($ of base b' and $\tau \in Tr(b\times b_1, b_1' \times b')$ such as $u_{b_1'}(t| = \tau^{\#}\cdot_{b_1} r(t|$, $\forall t$, then

$$u(\equiv \left[b',\ \int_{b_1'} \tau^{\#}\cdot_{b_1} id\mu_{b_1'},\ \int_{b_1'} \tau^{\#}\cdot_{b_1} e(\, d\mu_{b_1'},\ \int_{b_1'} \tau^{\#}\cdot_{b_1} s(\, d\mu_{b_1'}\right].$$

Therefore it follows that structures properties are compatible properties.

Proof. First let's assume that $e(t|$ is C^k, then $(x', x_1') \mapsto {}_{b_1}e(t|\tau(x', x_1'))$ is C^k and as $b' \times b_1'$ is compact, Lebesgue's theorem on differentiability shows that the assignment $x' \mapsto \int_{b_1'} {}_{b_1}e(t|\tau(x', x_1'))d\mu_{b_1'}$ is C^k and then belongs to $\mathcal{E}^k(b')$.

On the other hand, let's assume that: $\exists A \,/\mu_{b'}(A) > 0$ and $\forall x' \in A$, $\int_{b_1'} {}_{b_1}s(t|\tau(x', x_1'))d\mu_{b_1'} \neq 0$ then: $\forall x' \in A$, $\exists A_1^{x'} \,/\mu_{b_1'}(A_1^{x'}) > 0$ and ${}_{b_1}s(t|\tau(x', x_1')) \neq 0$, $\forall x_1' \in A_1^{x'}$, therefore: $\mu_{b'\times b_1'}(\{(x', x_1') \,/{}_{b_1}s(t|\tau(x', x_1')) \neq 0\} > 0$, then: $\mu_{b'\times b_1'}(\tau^{-1}(x, x_1) \in b\times b_1 \,/{}_{b_1}s(t|x, x_1) \neq 0\}) > 0$, which following Theorem 1.9 contradicts the hypothesis on $s($. We have then: $\int_{b_1'} {}_{b_1}s(t|\tau(x', x_1'))d\mu_{b_1'} = 0$ $\mu_{b'}$ *a.e.* x'.

Therefore as $s(t|$ is bounded, $\forall x', x'_1 \mapsto {}_{b_1}s(t|\tau(x', x'_1))$ is also bounded, uniformely for $x' \in b'$. Then it is $\mu_{b'_1}$-integrable on b'_1 and its integral is bouded on b'.

It follows then from: $\forall x' \in b'_1$, $u(t|x') = u_{b'_1}(t|x', x'_1) =_{b_1} r(t|\tau(x', x'_1)) =_{b_1} i(\tau(x', x'_1)) + {}_{b_1}e(t|\tau(x', x'_1)) +_{b_1} s(t|\tau(x', x'_1))$ that $x' \mapsto {}_{b_1}i(\tau(x', x'_1))$ is $\mu_{b'_1}$-intergrable on $b'_1, \forall x' \in b'$: $[x \rightarrow \int_{b'_1} {}_{b_1}i(\tau(x', x'_1))d\mu_{b'_1}] \in E^{b'}$, and that: $u(t|x') = \int_{b'_1} u(t|x')d\mu_{b'_1} = \int_{b'_1} {}_{b_1}i(\tau(x', x'_1))d\mu_{b'_1}$ $+ \int_{b'_1} {}_{b_1}e(t|\tau(x', x'_1))d\mu_{b'_1} + \int_{b'_1} {}_{b_1}s(t|\tau(x', x'_1))d\mu_{b'_1}$ for any $x' \in b'$. The result on structure follows immediately. □

1.3. Examples

1.3.1. Case of functions

Theorem 1.15. *Any function $f : O \mapsto E$ defines an only dynamic f. This dynamic is therefore simple, non singular, infinitely regular, of arbitrary base.*

We can identify functions, single-valued dynamics and set-valued maps with singleton image.

Proof.

- Let b be an arbitrary base, $r(\equiv [b, e(]$, with $e(t|x) = f(t)$, $\forall t \in O$, $\forall x \in b$, is a non singular representation of $\{f\}$.
- Let $r'($ be an other representation of $\{f\}$ of base b'.
  We have: $r'(t|x') = f(t), \forall t \in O, \forall x' \in b'$. Then for example: $r_{b'}(t|x, x') = {}_b r'(t|1_{b\times b'}(x, x'))$ and $[1_{b\times b'} : (x, x') \rightarrow (x, x')] \in Tr(b \times b', b \times b')$, equivalence between $r($ and $r'($ follows immediately. □

1.3.2. General case

Theorem 1.16.

1. *Any set-valued map has simple, non singular, infinitely regular coverings.*
2. *If $F \subset G$ are two set-valued maps from O to E, any covering of F can be completed in a covering of G and therefore $\mathcal{I}(G) \subset \mathcal{I}(F)$.*

Proof. *1.* Let be $J = im(F) = \cup_{t\in O} F(t)$ and let $t \rightarrow y(t)$ be an arbitrary selection of F.

Let $\forall y \in J, f_y : O \rightarrow E$ be defined by $f_y(t) = \begin{cases} ly & \text{if } y \in F(t), \\ y(t) & \text{if not.} \end{cases}$

$(f_y)_{y\in J}$ following the previous result can be identified with a simple, non singular, infinitely regular, covering of arbitrary basis b and has for subjacent set-valued map F.

2. Immediate. □

1.3.3. σ-compact varieties

Definition 1.17. We call $C^k\sigma$-compact variety of E, $k \in \overline{\mathbb{N}}$, any subset W of E which can be written in form of a union, at most countable, of compact C^k varieties W_i of finite dimension such as:

$$dimW = \sup_i \dim W_i < +\infty.$$

We have then:

Theorem 1.18. *Let $F : O \multimap E$ be a set-valued map such as:*

$$\exists M, K > 0, \forall t, F(t)\ C^k\ \sigma\text{-compact variety} \ / \ K \leq k \text{ and } dim\, F(t) \leq M$$

then F has a simple, non singular, K-regular covering of order at most countable.

Proof. Let be: $\forall t \in O$, $m(t) = \dim F(t)$ and $m = \max_{t\in O} m(t) \leq M$, $B^m = \{x \in \mathbb{R}^n / \|x\| < 1\}$, and $B_k^m = \left(1 - \frac{1}{k}\right) \overline{B_k^m}$.

Let be t fixed: $F(t) = \cup W_j$, $\dim W_j = m_j$, $j = 1, ...j(t)$.

For any y of W_j there is a C^K-diffeomorphism $\varphi^j y$ from an open ball $B_j(x_y, 2\varepsilon_y)$ of $\mathbb{R}^m$ into an open neighbourhood of y in W_j. Let's put: $O_y = \varphi_y^j\left(B_j(x_y, \varepsilon_y)\right)$. The family $\{O_y\}$ made up an open cover of W_j and consequently, by compacity, we can extract from $\{O_y\}$ a finit cover$\{O_{y_i}\} = \{O_{j,i}\}_{i=1,\ldots,k_j}$.

By homothetie-translation $\zeta_{i,j}$ we can then assume that: $\forall i = 1, ...k_j\ B(x_{y_i}, \varepsilon_{y_i}) = \pi_j \circ \zeta_{i,j}\left(B^m\right)$ where π_j is the canonical projector from $\mathbb{R}^m$, identified to $\mathbb{R}^{m_j} \times \mathbb{R}^{m-m_j}$, on $\mathbb{R}^{m_j}$. Let then $\phi_{i,j}$ defined by $\phi_{i,j} = \varphi_i^j \circ \pi_j \circ \zeta_{i,j}$ on a neighbourhood of B^m and:

$$[b_k, r_{i,j,k}(]_{i,j} : b_k = (m, B_k^m, \tfrac{1}{vol(B_k^m)}\lambda) \quad \text{and} \quad \forall t,\ r_{i,j,k}(t| = \phi_{i,j}|_{B_{k+1}^m}$$

where λ is the Lebesgue-measure on $\mathbb{R}^m$ and with the convention: $r_{i,j,k}(t| = r_{1,1,k}(t|$ if $\phi_{i,j}$ is not defined, ($j > j(t)$ or $i \notin [\![1, k_j]\!]$). Then $r_{l,j,k}(t|$ is C^k on a neighbourhood of B_k^m, $|\{(i,j,k)\}| \leq |\mathbb{N}|$ and $\cup_{i,j,k}\, r_{i,j,k}(t|b_k) = \cup_{i,j}\, O_{j,i} = F(t)$. The proof is complete. $\square$

1.3.4. Caratheodory parametrization

Definition 1.19. A representation $[b, r(]$ is said to be k(t)-Lipschitz on its base if:

$$\forall t,\ \|r(t|x_1) - r(t|x_2)\| \leq k(t)\,\|x_1 - x_2\|,\ \forall x_1, x_2 \in b.$$

Theorem 1.20. *Let be: $F :]a,b[\times O \multimap \mathbb{R}^n$ with convex images, $C :]a,b[\multimap O$, B^n the closed unit ball of $\mathbb{R}^n$.*

1. *If F is measurable-Lipschitz or of Carathéodory on $\{C(t\alpha\}_{\alpha\in]a,b[}$ with compact images, then F has a simple, non singular covering of order 1, of base $b = (n, B^n, \frac{1}{vol(B^n)}\lambda)$, $k(\alpha,t)$- Lipschitz on its base for any $(\alpha,t) \in]a,b[\times O$.*
2. *If F is measurable-Lipschitz on $\{C(\alpha)\}_{\alpha\in]a,b[}$ with closed images, then F is of order at most countable and has a simple, non singular covering $f_{\mathbb{N}}$, of base b, uniformly c-Lipschitz $(*)$ on its base for any $(\alpha,t) \in]a,b[\times O$.*
   $(*)$ c independent of $k \in \mathbb{N}$.

Proof. The proofs are simple re-writing of the classical statements on Caratheodory parametrization given in [1], [3] $\square$

2. Algebra

To can tackle the core of my work, it is beforehand necessary to define in coverings space elementary algebra rules straight related to set-valued maps rules algebra. We only consider here the most usual operations, but nevertheless, most operations on set-valued maps can be dealed in similar ways to which follows.

Let $f, g : O \sim> E$ be dynamics, $f_I, g_J : O \sim> E$ coverings, and $F, G : O \multimap E$ set-valued maps. Let be $[b, r(] = [b, i, e(, s(] \in f$ and $[b', i', r'(] = [b', e'(, s'(] \in g$.

2.1. Vector combinations

Theorem 2.1. *Let α, β, be two scalars, we define the vector combination $\alpha f + \beta g$ as the dynamic of which a representation is $\alpha r_{b'}(+ \beta_b r'($ of breakdown $[b \times b', \alpha i_{b'} + \beta_b i', \alpha e_{b'}(+ \beta_b e'(, \alpha s_{b'}(+ \beta_b s'(]$. We have then:*

$$\underline{f + g} = \underline{f} + \underline{g}.$$

Therefore if f_I and g_J are respectively coverings of F and G, $\alpha f_I + \beta g_J = (\alpha f_i + \beta g_j)_{(i,j) \in I \times J}$ is a covering of $\alpha F + \beta G$.

Common stucture and order properties are kept.

The theorem follows directly of:

Lemma (Compatibility).

$$\textit{If } u \sim u_1 \textit{ and } v \sim v_1, \textit{ then: } u_{b'} +_b v \sim u_{1b'_1} +_{b_1} v_1$$

with the usual convention: $[b, u(t] = u$, $[b', v(t] = v, ...$

Proof. We have: $u_{1\breve{b_1}} = {}_{\breve{b}}u \circ \tau \;/\; \tau : b_1 \times \breve{b_1} \mapsto \breve{b} \times b$ and $v_{1\breve{b'_1}} = {}_{\breve{b'}}v \circ \tau' \;/\; \tau' : b'_1 \times \breve{b'_1} \mapsto \breve{b'} \times b'$.

Let be the canonical projectors $\pi : \breve{b} \times b \mapsto b$, $\pi' : \breve{b'} \times b' \mapsto b'$, $\breve{\pi} : b_1 \times \breve{b_1} \times \breve{b'_1} \mapsto b_1 \times \breve{b_1}$ and $\breve{\pi}' : b'_1 \times \breve{b'_1} \times \breve{b_1} \mapsto b'_1 \times \breve{b'_1}$.

π and π' belong to $Tr(b, \breve{b} \times b)$ and $Tr(b', \breve{b'} \times b')$ respectively, therefore $\delta = \pi \circ \tau$ and $\delta' = \pi' \circ \tau'$ belong to $Tr(b, b_1 \times \breve{b_1})$ and $Tr(b', b'_1 \times \breve{b'_1})$, $\phi = \delta \circ \breve{\pi}$,$\phi' = \delta' \circ \breve{\pi}'$ belong to $Tr(b, b_1 \times b'_1 \times \breve{b_1} \times \breve{b'_1})$ and $Tr(b', b_1 \times b'_1 \times \breve{b_1} \times \breve{b'_1})$. Let's show that (ϕ, ϕ') belongs to $Tr(b \times b', b_1 \times b'_1 \times \breve{b_1} \times \breve{b'_1})$:

Let be $\zeta \in L(b \times b')$, we put $\Omega = {V_1}^o \times {V'_1}^o \times \breve{V_1}^o \times \breve{V'_1}^o$. We have then

$$\begin{aligned}
\int_\Omega \zeta\,(\phi, \phi') d\mu_\Omega &= \int_{\overset{o}{V'_1} \times \overset{o}{\breve{V'_1}}} \int_{\overset{o}{V_1} \times \overset{o}{\breve{V_1}}} \zeta\,(\delta, \delta') d\mu_{b_1 \times \breve{b_1}}\, d\mu_{b'_1 \times \breve{b'_1}} \\
&= \int_{\overset{o}{V'_1} \times \overset{o}{\breve{V'_1}}} \int_{\overset{o}{V}} \zeta\,(x, \delta') d\mu_b\, d\mu_{b'_1 \times \breve{b'_1}} \\
&= \int_{\overset{o}{V'}} \int_{\overset{o}{V}} \zeta\,(x, x') d\mu_b\, d\mu_{b'} \\
&= \int_{\overset{o}{V} \times \overset{o}{V'}} \zeta\,(x, x') d\mu_{b \times b'}.
\end{aligned}$$

Then we verify immediately that:

$$\left(u_{1b'_1} + {}_{b_1}v_1\right)_{\breve{b}_1 \times \breve{b}'_1} = (u_{b'} + {}_b v) \circ (\phi, \phi') \,.$$

The equality of dimensions is obvious and the result follows by symmetry. Therefore if $u($ and $v($ are representations, it is immediate that $u_{b'} + {}_b v$ is a representation of same structure. $\square$

Theorem 2.2 (Convexity). *If a set-valued map $F : O \multimap E$ has convex images, the set of its coverings of fixed order and structure properties and the set $\mathcal{I}(F)$ of its internal dynamics are convex.*

Proof. We have first: $\forall F : O \multimap E$, $\forall f_J$, $\widetilde{f}_{\widetilde{J}}$ covering of F, $\forall \lambda \in [0,1] : F(t) \subset \lambda \underline{f_J}(t) + (1-\lambda)\underline{\widetilde{f}_{\widetilde{J}}}(t)$.

Actually: $\forall y \in F(t)$, $\exists j, \widetilde{j}\ /\ y \in \underline{f_j}(t)$ and $y \in \underline{\widetilde{f}_{\widetilde{j}}}(t)$, then: $y = \lambda y + (1-\lambda) y \in \left(\lambda \underline{f_j} + (1-\lambda)\underline{\widetilde{f}_{\widetilde{j}}}\right)(t) = \underline{\left(\lambda f_j + (1-\lambda)\widetilde{f}_{\widetilde{j}}\right)}(t) \subset \underline{\left(\lambda f_J + (1-\lambda)\widetilde{f}_{\widetilde{J}}\right)}(t)$.

On the other hand, if $F(t)$ is convex:

$$\underline{\left(\lambda f_J + (1-\lambda)\widetilde{f}_{\widetilde{J}}\right)}(t) = \lambda \underline{f_j}(t) + (1-\lambda)\underline{\widetilde{f}_{\widetilde{j}}}(t) = \lambda F(t) + (1-\lambda) F(t) \subset F(t).$$

The stability of order and structure is obvious. $\square$

Theorem 2.3 (Convexity and order). *Let's assume that the set-valued map $F : O \multimap E$ is of finite order and has one of the following properties:*

- *F has convex images*
- *F has C-convex images: $F = \underline{f_I}$ with $\forall i$, $\underline{f_i}$ has convex images*

then $co(F)$ is of order 1.

Proof. Let $\widehat{[b_i, r_i(\,]}_{i=1,\dots,m}$ be a covering of F.

Let be $V = \{\alpha = (\alpha_j)_{j=1,\dots,m-1} /\ \forall j, \alpha_j \geq 0,\ |\alpha| = \alpha_1 + \cdots + \alpha_{m-1} \leq 1\}$. V is compact, convex, of non empty interior, and $b_o = (m-1, V, \frac{1}{vol(V)}\lambda)$ is a base.

It follows that $[b, r(\,] = [b_o \times b_1 \times \cdots \times b_m, r(\,] : r(\alpha, x_1, \dots, x_m) = \alpha_1 r_1(t|x_1) + \cdots + \alpha_{m-1} r_{m-1}(t|x_{m-1}) + (1-|\alpha|) r_m(t|x_m)$ defines a macrogerm of subjacent set valued map F.

We have sill to prove that $r($ is a representation. Let be $m = 2$, $[b, r(\,] = [(1, [0,1], d\lambda) \times b_1 \times b_2, r(\,]$ with $r(t|\alpha, x, x') = \alpha r_1(t|x) + (1-\alpha) r_2(t|x')$.

We have $r(= i + e(+ s($ with $i = \alpha i_1 + (1-\alpha) i_2$ and so on... If $e_j(t|,\ j = 1,2$ are C^k so is $e(t|$.

As $s(t|\alpha, x, x') = \alpha s_1(t|x) + (1-\alpha) s_2(t|x')$, $s(t|$ is bounded. Therefore: $s(t|^{-1}(E - \{0\}) \subset [0,1] \times s_1(t|^{-1}(E - \{0\}) \times b_2 \cup [0,1] \times b_1 \times s_2(t|^{-1}(E - \{0\})$, then: $\lambda \otimes \mu_{b_1} \otimes \mu_{b_2}(s(t|^{-1}(E - \{0\})) \leq \mu_{b_1}(s_1(t|^{-1}(E - \{0\})) + \mu_{b_2}(s_2(t|^{-1}(E - \{0\})) = 0 + 0$.

It follows that $s(t| = 0 \mu_b$ *a.e.*; $r($ is then a representation. The general case follows by immediate recursion on m. $\square$

Theorem 2.4 (Relaxation). *If we call convexified of a covering f_I the covering $co(f_I)$ defined by*

$$co(f_I) = \left\{ \sum_{k=1,\dots,m} \alpha_{i_k} f_{i_k} \,;\, i_k \in I,\, \alpha_{i_k} \geqslant 0,\, \sum_{k=1,\dots,m} \alpha_{i_k} = 1 \right\}$$

we have then: if f_I is a covering of F, then $co(f_I)$ is a covering of $co(F)$.

Structure properties are kept.

Proof. It follows from Theorem 2.1 that $co(f_I)$ is a covering and that structure properties are kept. As it is possible to have $f_{i_1} = \cdots = f_{i_k}$, we have therefore $\underline{co(f_I)} = co(F)$. $\square$

2.2. Algebraic product

Theorem 2.5. *If E is an algebra, we define the algebraic product $f.g$ of two simple dynamics as the simple dynamic of which a representation is $r_{b'}(\,_b r'($, of breakdown $[b \times b', e_{b'}(\,_b e'(, s_{b'}(\,_b r'(+_b s'(\, r_{b'}(\,]$. We have then:*

$$\underline{f.g} = \underline{f}.\underline{g}\,.$$

Therefore if f_I and g_J are respectively coverings of simple set-valued maps F and G, $f_I.g_J = (f_i.g_j)_{(i,j)\in I\times J}$ is a covering of $F.G$.

Common stucture and order properties are kept.

Proof. The proof of compatibility is a strict transcription of proof of Theorem 2.1 with same transfer operators. The check of characteristics of representations follows immediately from the compacity of the basis and the k-regularity. $\square$

2.3. Cartesian product

Theorem 2.6. *We define the cartesian product of f by g as the dynamic $f \times g$ which representation is $(t \to r_1(t| \times r_2(t|\,) = [b_1 \times b_2, t \to e_1(t| \times e_2(t|, t \to s_1(t| \times s_2(t|]$ and we have then:*

$$\underline{f \times g} = \underline{f} \times \underline{g}\,.$$

Therefore if f_I and g_J are respectively coverings of set-valued maps F and G, $f_I \times g_J = (f_i \times g_j)_{(i,j)\in I\times J}$ is a covering of $F \times G$.

Structure and order proprieties are kept.

The theorem follows directly of:

Lemma (Compatibility). *If $(\tau_1, \tau_2) \in Tr(b_1', b_1) \times Tr(b_2', b_2)$, then $[\tau_1 \times \tau_2 : (x_1, x_2) \to (\tau_1(x_1), \tau_2(x_2))] \in Tr(b_1' \times b_2', b_1 \times b_2)$.*

Proof. Following Fubini's theorem we have:

$$\begin{aligned}
\forall \psi \geq 0, \int_{b_1 \times b_2} \psi(\tau_1, \tau_2) d\mu_{b_1 \times b_2} &= \int_{b_2} \Big(\int_{b_1} \psi(\tau_1, \tau_2) d\mu_{b_1} \Big) \, d\mu_{b_2} \\
&= \int_{b_2} \Big(\int_{b'_1} \psi(x'_1, \tau_2) d\mu_{b'_1} \Big) \, d\mu_{b_2} \\
&= \int_{b'_1} \Big(\int_{b_2} \psi(x'_1, \tau_2) d\mu_{b_2} \Big) \, d\mu_{b'_1} \\
&= \int_{b'_1} \Big(\int_{b'_2} \psi(x'_1, x'_2) d\mu_{b'_2} \Big) \, d\mu_{b'_1} \\
&= \int_{b'_1 \times b'_2} \psi d\mu_{b'_1 \times b'_2}
\end{aligned}$$

and the result follows [6]. Validity of other transfer axioms is immediate. □

Theorem 2.7. *If we call k^{th}-coordinate-covering of a covering $f_I : O \sim> E = \times_{j=1,\dots n} E_j$ the covering $f_I^k = (\pi_k \circ f_i)_{i \in I}$ where π_k is the k^{th} projector of E, and if f_I is a covering of a set-valued map F, then f_I^k is a covering of $F^k = \pi_k \circ F$.*

Common structure and order properties are kept.

Proof. That follows immediately of general definitions. □

2.4. Boolean structure

It is obvious that:

1. If f_I and g_J are respectively coverings of F and G, $f_I \cup g_J$ is a covering of $F \cup G$. Common stucture and order properties are kept and therefore: $\mathcal{I}(F) \cup \mathcal{I}(G) \subset \mathcal{I}(F \cup G)$.
2. If $F \cap G$ is of domain O, any of its covering can be written as intersection of coverings of F and G and in particular: $\mathcal{I}(F) \cap \mathcal{I}(G) = \mathcal{I}(F \cap G)$.

2.5. Chain

Notations. Let be $F : O \subset T \multimap T'$, $G : O' \subset T' \multimap E$ / $F(O) \subset O'$ and $(G \circ F)(t) = \cup_{s \in F(t)} G(s)$, f_I and g_J coverings of F and G respectively.

Theorem 2.8 (Re-indexing process).
If $f_I = \widehat{[b_i, r_i(\,]}_{i \in I}$, then $\{t \mapsto (g_j(r_i(t|x))_{i \in I, j \in J, x \in b_i}\}$ defines a covering of $G \circ F$ independent of the system of representations of f_I used. Structure properties of G are kept.

Proof. Only the independence of $[b_i, r_i(\,]_{i \in I}$ require to be proved.

Let $[b'_i, r'_i(\,]_{i \in I}$ be an other system of representations of f_I. Let be $i \in I$, for suitable $\breve{b}_i, \breve{b}'_i$ and $\tau_i : b'_i \times \breve{b}'_i \mapsto \breve{b}_i \times b_i$, we have: $r'_i(t|x') = r(t|\pi_{b_i} \circ \tau_i(x', x'_o))$, $\forall t, x' \in b'_i$, with x'_o arbitrary fixed in $\breve{b}'_i$. Then : $\{t \mapsto (g_j(r'_i(t|x'))_{i \in I, j \in J, x' \in b'_i}\} \subset \{t \mapsto (g_j(r_i(t|x))_{i \in I, j \in J, x \in b_i}\}$. The result follows by symmetry. □

Definition 2.9 (C-chain process). If $[b, r(\,]$ and $[b', r'(\,]$ are two representations respectively on O and O' such as $\forall t, r(t|b) \subset O'$, we say that $r'($ is C-chainable by $r($ if the parametrized macro germ on O defined by $(x, x') \in b \times b' \mapsto (r' \odot r)(t|x, x') = r'(r(t|x)|x')$, $\forall t \in O$, is a representation.

Theorem 2.10. *If $r'($ is C-chainable by $r($, any representation equivalent to $r'($ is C-chainable by any representation equivalent to $r($.*

Proof. Let be $r_1(\sim r($ and $r'_1(\sim r'($: $r_1\widetilde{b_1}(t|x_1, \widetilde{x_1}) = {}_{\widetilde{b}}r(t|\tau(x_1, \widetilde{x_1}))$ and $r'_1\widetilde{b'_1}(t'|x'_1, \widetilde{x'_1}) = {}_{\widetilde{b'}}r'(t'|\tau'(x'_1, \widetilde{x'_1}))$. We have:

$$\begin{aligned}
(r'_1 \odot r_1)_{\widetilde{b_1} \times \widetilde{b'_1}}(t|x_1, x'_1, \widetilde{x_1}, \widetilde{x'_1}) &= (r'_{1\widetilde{b'_1}} \odot r_{1\widetilde{b_1}})(t|x_1, \widetilde{x_1}, x'_1, \widetilde{x'_1}) \\
&= r'_{1\widetilde{b'_1}}(r_{1\widetilde{b_1}}(t|x_1, \widetilde{x_1})|x'_1, \widetilde{x'_1}) \\
&= {}_{\widetilde{b'}}r'({}_{\widetilde{b}}r(t|\tau(x_1, \widetilde{x_1}))|\tau'(x'_1, \widetilde{x'_1})) \\
&= ({}_{\widetilde{b'}}r' \odot_{\widetilde{b}} r)(t|(\tau \otimes \tau')(x_1, \widetilde{x_1}, x'_1, \widetilde{x'_1})) \\
&= {}_{\widetilde{b'} \times \widetilde{b}}(r' \odot r)(t|(\tau \otimes \tau') \circ \sigma(\widetilde{x'_1}, \widetilde{x_1}, x_1, x'_1)
\end{aligned}$$

with obviously $\sigma = (3, 2, 4, 1)$ and $(\tau \otimes \tau') \circ \sigma \in Tr(\widetilde{b'} \times \widetilde{b} \times b \times b'; \widetilde{b'_1} \times \widetilde{b_1} \times b_1 \times b'_1)$.

Equivalence of macro germs follows by symmetry and we deduce then the result from Theorem 1.14. □

Definition 2.11. We say that a dynamic $g : O' \sim> E$ is C-chainable by a dynamic $f : O \sim> O'$ if their respective representations are C-chainable (in same order) and we define the C-chain $g \circ f : O \sim> E$ as the associated equivalence class.

Therefore we define the C-chain $g_J \circ f_I$ of suitable coverings by $g_J \circ f_I = (g_j \circ f_i)_{(i,j) \in I \times J}$.

Theorem 2.12. *If $g_J : O' \sim> E$ is C-chainable by $f_I : O \sim> O'$ and are respectively coverings of set valued maps $G : O' \multimap E$ and $F : O \multimap O'$, the C-chain $g_J \circ f_I$ is a covering of the chain $G \circ F$.*

Proof.

$$\begin{aligned}
\underline{g_J \circ f_I}(t) = \bigcup_{(i,j) \in I \times J} \underline{g_j \circ f_i}(t) &= \bigcup_{(i,j) \in I \times J} r^j_g(r^i_f(t|b)|b') \\
&= \bigcup_{j \in J} \bigcup_{i \in I} \bigcup_{s \in \underline{f_i}(t)} \underline{g_j}(s) \\
&= \bigcup_{s \in F(t)} G(s) \\
&= G \circ F(t).
\end{aligned}$$

□

Theorem 2.13. *Let be $f_I : O \sim> O' \subset T'$ simple k-regular and $g_J : O' \sim> E$. If we assume that:*

- *f_I is non singular or $dim T' < \infty$,*
- *g_J has a a system of representations $[b'_j, i'_j, e'_j(\,, s'_j(\,]_{j \in J}$ such as $(\zeta, x') \mapsto e'(\zeta, x')$ is C^k and $(\zeta, x') \mapsto s'(\zeta, x')$ is bounded on every $K \times b'$ with $K \subset O'$ compact,*

then g_J is C-chainable by f_I and the C-chain $g_J \circ f_I$ is k-regular.

Proof. Let be $|I| = |J| = 1$, $f_I = f = \widehat{[b, 0, e(\,, s(\,]}$, $g_J = g = \widehat{[b', i', e'(\,, s'(\,]}$. We have then: $E(t| : (x, x') \mapsto e'(e(t|x)|x')$ and $S(t| = S^1(t| + S^2t|$ with $S^1(t| : (x, x') \mapsto e'(r(t|x)|x') - e'(e(t|x)|x')$ and $S^2(t| : (x, x') \mapsto s'(r(t|x)|x')$.

As $x \mapsto e(t|x)$ and $(\zeta, x') \mapsto e'(\zeta, x')$ are C^k so is $E(t|$.

We have also: if f is non singular, that is to say $s(= 0$, then: $S^1(= 0$; if not: $\mu_{b\times b'}\{(x, x')/ S^1(t|x, x') \neq 0\} \leq (\mu_b \otimes \mu_{b'})(\{x/ s(t|x) \neq 0\} \times b') = 0$ and, as then $dimT' < \infty$, $e(t|$ continuous and $s(t|b)$ bounded: $e(t|b)$ and $e(t|b) - \overline{s(t|b]}$ are compact. Then, as $(\zeta, x') \mapsto e'(\zeta, x')$ is continuous, $e'(e(t|b) - \overline{s(t|b]}|b') - e'(e(t|b)|b')$ is compact and then its subset $S^1(t|b \times b')$ is bounded. It follows that $S^1($ is a singularity.

In the same way $S^2(t|b \times b') \subset s'((e(t|b) + \overline{s(t|b)}|b')$ is bounded by hypothesis on $s'($ because $e(t|b) + \overline{s(t|b)}$ is compact.

Therefore let be: $\beta(x, x') = 1$ if $S^2(t|x, x') \neq 0$, 0 if not. We have:

$$\begin{aligned}
\mu_{b\times b'}\{(x, x')/ S^2(t|x, x') \neq 0\} &= \int_{b\times b'} \beta(x, x') d\mu_b \otimes d\mu_{b'} \\
&= \int_b \int_{b'} \beta(x, x') d\mu_{b'} d\mu_b \\
&= \int_b \mu_{b'}\{x'/ s'(r(t|x)|x') \neq 0\} d\mu_b = 0.
\end{aligned}$$

Then $S^2($ is also a singularity and so is therefore $S($.

It follows that $[b \times b',_b i', E(\,, S(\,]$ is a k-regular representation of $g \circ f$.

The transition to the general case comes down to: $r_i(= r($ and $r'_j(= r'($ for any (i, j). □

Theorem 2.14.

1. *Any covering of any set-valued map $G : O' \multimap E$ is C-chainable by any function $F = f : O \mapsto O'$ and structure properties of G are kept.*
2. *Let $G = g : O' \mapsto E$ be a C^k-function and $F : O \multimap O'$ be a k-regular set-valued map, if one of the following hypthesies is verified:*
   - *g is affine,*
   - *F simple non singular,*
   - *F simple and T' of finite dimension,*

   then g is C-chainable with any covering of F and structure properties of F are kept.

Proof. *1.* If $g_J = \widehat{[b_j, r_j(\,]}_{j\in J}, \widehat{[b_j, t \mapsto r_j(f(t)|\,]}_{j\in J}$ is a covering of $G \circ F$ and the result is then immediate.

2. $g(t') = l(t') + c$ with $l \in \mathcal{L}(T', E)$, $c \in E$, $f_j = \widehat{[b_j, i_j, e_j(\,, s_j(\,]}$: $g \circ r_j(t|x) = l.(i_j(x) + c) + l \circ e_j(t|x) + l \circ s_j(t|x)$ and $l \circ e_j(t|$ is C^k if $e_j(t|$ is C^k. Independently we have: $l \circ s_j(t|x) \neq 0 \Rightarrow s_j(t|x) \neq 0$ and then $\mu_j\,\{x\,/\,l \circ s_j(t|x) \neq 0\} \leq \mu_j\,\{x\,/\,s_j(t|x) \neq 0\} = 0$. Therefore $l \circ s_j(t|$ is bounded.

The two other cases are immediate applications of Theorem 2.13. □

3. Continuity

The fundamental idea used here, then systematically in all followings parts, is to characterize each coverings analytical concept by the corresponding classical one-one concept on the (class of its) "total evolution", function from O to Lebesgue's space. This unicity for all analytical domains and the simplicity of the process is, in itself, one of its main properties. After to have treated the stict extension point of view, I will study relations with Kuratowsky semi-continuity then the continuous selections existence and, to end, I will give some results on chains.

3.1. Basics

Theorem 3.1. *Any representation $[b', r'(\,]$ equivalent to a representation $[b, r(\,]$ of total evolution $e_T(: O \mapsto L(b)$ continuous at t_o is of total evolution $e'_T(: O \mapsto L(b')$ continuous at t_o.*

Remark. Let be $r(\equiv [b, i, e_T(\,] : e_T(t| - e_T(t_o| = r(t| - r(t_o|$ and then is independent of the breakdown.

Proof. $r(\equiv [b, i, e_T(\,] : e_T(t| - e_T(t_o| = r(t| - r(t_o|$ is independent from the breakdown.

Let be $e_T(: O \mapsto L(b)$ continuous at t_o and $r'(\equiv [b', i', e'_T(\,]$ such as $r'(\sim r($. We have for any suitable b_1, b'_1 and $\tau : r'_{b'_1}(t| = \tau^{\#}._{b'_1} r(t|\, \forall t$. Then: $e'_{T,b'_1}(t| - e'_{T,b'_1}(t_o| = r'_{b'_1}(t| - r'_{b'_1}(t_o| = \tau^{\#}._{b'} r(t| - \tau^{\#}._{b'} r(t_o| = \tau^{\#}.\,({}_{b_1}e_T(t| -_{b_1} e(t_o|)$ and by transportation:

$$\begin{aligned}
\|e'_T(t| - e'_T(t_o|\,\|_{b'} &= \int_{b'\times b'_1} \left\| e'_{T,b'_1}(t| - e'_{T,b'_1}(t_o| \right\| d\mu_{b'\times b'_1} \\
&= \int_{b'\times b'_1} \|\,{}_{b_1}e_T(t| -_{b_1} e_T(t_o|\,\| \circ \tau d\mu_{b'\times b'_1} \\
&= \int_{b_1\times b} \|\,{}_{b_1}e_T(t| -_{b_1} e_T(t_o|\,\|\, d\mu_{b_1\times b} \\
&= \|e_T(t| - e_T(t_o|\,\|_b \,.
\end{aligned}$$

□

Definition 3.2. We say that a representation $r(= [b, i, e_T(\,]$ is continuous at t_o (respectively on O) if it has the strong compatible property: $t \mapsto e_T(t$ is continuous at t_o (resp. on O) as function from O to $L(b)$.

The continuity of dynamics, coverings, and set-valued maps, follows then general rules (Definition 1.2).

Theorem 3.3. *If $f : O \mapsto E$ is a function, it is equivalent to say:*

1. *the function f is continuous at t_o (resp. on O),*
2. *the dynamic $f : O \sim> E$ is continuous at t_o (resp. on O),*
3. *the set-valued map $\{f\} : O \multimap E$ is continuous at t_o (resp. on O).*

Proof. We have actually for a function $f : r(t|x) = e_T(t|x) = f(t),\ \forall t \in O,\ x \in b$ and then: $\|e_T(t| - e_T(t_o|\,\|_b = \|f(t) - f(t_o)\|$. □

Theorem 3.4.

1. *The continuity of coverings and set-valued maps is stable by vector combination, union and, in case of simplicity, by algebraic product.*
2. *The continuity is stable by relaxation.*
3. *If $E = \times_{k=1,\ldots n} E_k$, it is equivalent to say f_I (resp. F) is continuous and for any $k = 1, \ldots n$ its k^{th} coordinate f_I^k (resp. F^k) is continuous.*

Proof. The results follow respectively from Theorem 2.1 for combinations, Theorem 2.4 for relaxation and Theorems 2.6 and 2.7 for cartesian product.

For algebraic product, as representations are simple, we have: $r(t|x)r'(s|x') - r(t_o|x)r'(s_o|x') = e_T(t|x)\left(e'_T(s|x') - e'_T(s_o|x')\right) + e'_T(s_o|x')\left(e_T(t|x) - e_T(t_o|x)\right)$.

The result follows then from Theorem 2.5.

Union is obvious. $\square$

3.2. Semi-continuity in sense of Kuratowski [1], [3]

Theorem 3.5. *Any simple non singular set-valued map $F : O \multimap E$ which is continuous at t_o is lower semi-continuous (l.s.c.) at t_o.*

Proof. $\omega(F) = 1$: f *simple non singular* $/\underline{f} = F$. Let be y $\in \underline{f}(t_o)$. Let's assume that $d(y, \underline{f}(t)) \nrightarrow 0$ when $t \to t_o$: $\exists \varepsilon > 0$, $\exists t_n \to t_o \,/\, d(y, \underline{f}(t_n)) \geq \varepsilon$, $\forall n$, or again: $d(y, y_n) \geq \varepsilon$, $\forall n$, $\forall y_n \in \underline{f}(t_n)$.

Let $r(\equiv [b, e(\,]$ be a representation of continuous evolution of $\underline{f}$: $\|e(t_n| - e(t_o|\|_b \to 0$. Therefore: there is an extracted $t_\nu \,/\, e(t_\nu|x) \to e(t_o|x)\mu_b$ *a.e.* x. As $r(t_o| = e(t_o|$ is continuous at any point of b the set $\left\{z \in \underline{f}(t_o) \,/\, \exists x_z \in r(t_o|^{-1}z),\ z_\nu = r(t_\nu|x_z) \to r(t_o|x_z) = z\right\}$ is dense in $\underline{f}(t_o)$.

Therefore: $\beta > 0$, $\exists z_\beta \in \underline{f}(t_o) \,/\, \|z_\beta - y\| < \beta$ and $y_{\beta,\nu} = r(t_\nu|x_\beta) \in \underline{f}(t_\nu) \to r(t_o|x_\beta) = z_\beta$.

Then $\forall \beta < \varepsilon/2$, $\exists \nu_\beta \,/\, \nu \geq \nu_\beta \Rightarrow \|y_{\beta,\nu} - z_\beta\| < \beta$ and therefore: $\|y_{\beta,\nu} - y\| \leq \|y_{\beta,\nu} - z_\beta\| + \|z_\beta - y\| < 2\beta < \varepsilon$ which contradicts the definition.

We have therefore [1], [3]: $\forall y \in \underline{f}(t_o)$, $\lim_{t \to_O t_o} d(y, \underline{f}(t)) = 0$, it is to say $\underline{f}(t_o) \subset \liminf_{t \to_O t_o} \underline{f}(t)$ and then $\underline{f} = F$ l.s.c. at t_o.

General case: f_J simple non singular / $\underline{f_J} = F$. It follows from the case $\omega(F) = 1$ that $\underline{f_J}$ is l.s.c. at t_o, $\forall j \in J$. Therefore: $\forall$ open set $U / U \cap (t_o) \neq \emptyset$, $\underline{f_j}^{-1}(U)$ is open.

Let U be an open set $/\, U \cap F(t_o) \neq \emptyset$: $F^{-1}(U) = \cup_{j \in J \,/\, U \cap \underline{f_j}(t_o) \neq \emptyset}\, \underline{f_j}^{-1}(U)$ is then open and therefore F is l.s.c. at t_o. $\square$

Theorem 3.6. *Any set-valued map $F : O \multimap E$ of finite order which is continuous at t_o is stochastically upper semi-continuous at t_o in the sense that:*

$$\forall\, \widehat{[b, r_j(\,]}_{j \in J} \in \mathcal{C}(F),\ \forall \varepsilon > 0,\ \lim_{t \to t_o} \mu_b \left\{x \in b \,/\, \exists j, r_j(t|x) \notin B\left(F(t_o), \varepsilon\right)\right\} = 0.$$

Proof. Let be $t_o = 0$ and let's assume that the property is false: $\exists \widehat{[b, r_j(\,]}_{j\in J} \in \mathcal{C}(F)$, $\exists \varepsilon > 0$, $\exists \alpha > 0$, $\exists t_n \to 0$, $\mu_b\,\{x \in b\,/\,\exists j, r_j(t_n|x) \notin B\,(F(0), \varepsilon)\} \geq \alpha$, $\forall n$.

Let be $J = [\![1, k]\!]$: $A_n = \{x \in b\,/\,\exists j, r_j(t_n|x) \notin B\,(F(0), \varepsilon)\} = \cup_{j\in J} A_n^j\,/\,A_n^j = \{x \in b, r_j(t_n|x) \notin B\,(F(0), \varepsilon)\}$.

Let's assume that: $\forall j$, $\lim_n \mu_b(A_n^j) = 0$ then $\exists N\,/\,\forall n \geq N : \mu_b(A_n^j) \leq \alpha/2k$, $\forall j$. It follows: $\mu_b(A_n) \leq \sum_j \mu_b(A_n^j) \leq k.\alpha/2k < \alpha$ which is absurd.

Therefore, let be $j_o\,/\,\lim_n \mu_b(A_n^{j_o}) = 0$. We have then: $\|r_{j_o}(t_n| - r_{j_o}(0|\,\|_b \geq \int_{A_n^{j_o}} \|r_{j_o}(t_n|\,x) - r_{j_o}(0|x)\|\,d\mu_b \geq \int_{A_n^{j_o}} \varepsilon.d\mu_b \geq \varepsilon.\alpha/2k, \forall n$, which contradicts the hypotheses $\lim_{t\to 0} \|r_{j_o}(t_n| - r_{j_o}(0|\,\|_b = 0$. □

3.3. Continuous selections

Theorem 3.7.

1. *Any simple non singular continuous set-valued map $F : O \multimap E$ of locally compact domain and closed convex images has a continuous selection φ such as $\varphi(t_o) = yo$ at any t_o and any y_o in $F(t_o)$.*
2. *A set-valued map $F : O \multimap E$ of locally compact domain has a continuous selection if, and only if, it has a simple non singular continuous internal dynamic $f : O \sim> E$ with convex images.*

Proof. The direct properties are immediate consequences of Michael's Theorem [1], [3] and Theorem 3.5. For the reciprocal proof of *2.* we put $f = \varphi$, the result follows then from Theorem 1.15. □

Theorem 3.8. *A set-valued map $F : O \multimap E$ has a continuous selection φ at t_o such as $\varphi(t_o) = y_o$ if, and only if, y_o is a uniformity point of F at t_o in the sense that F has an internal dynamic $f : O \sim> E$ which is continuous at t_o and such as:*

$$f = \widehat{[b, r(\,]}\,/\,\exists x_o \in b : r(t_o|x_o) = y_o \text{ and } r(t| : b \mapsto E$$
$$\text{continuous at } x_o \text{ uniformly for } t \text{ near } t_o.$$

Therefore $U(t_o) = \{x \in b/\, r(t_o|x_o)$ uniformity point at $t_o\}$ is open in b.

Proof. Let be $f = \widehat{[b, r(\,]}$, $\underline{f} \subset F$, $x_o \in b : r(t_o|x_o) = y_o$, W a neighbourhood of t_o such as:

$$\forall \varepsilon > 0,\ \exists \eta > 0,\ \forall t \in W,\ \forall x \in b\ \|x - x_o\| < 2\eta : \|r(t|x) - r(t|x_o)\| < \frac{\varepsilon}{6}.$$

$\forall x_1 \in b$, $\|x_1 - x_o\| < \eta$, $\forall x \in b$, $\|x - x_1\| < \eta$, $\forall t \in W$, then $\|r(t|x) - r(t|x_1)\| \leq \|r(t|x) - r(t|x_o)\| + \|r(t|x_o) - r(t|x_1)\| < \frac{\varepsilon}{3}$. It follows that $U(t_o)$ is open in b.

Therefore as b is convex of nonempty interior we have $\mu_b(b \cap \{x, \|x - x_o\| < \eta\}) > 0$.

Let's show then that $t \mapsto r(t|x_o)$ is continuous at t_o:

let be $t_\nu \to t_o$ and let's assume that there are $\varepsilon > 0$ and an extracted $t_\beta : \|r(t_\beta|x_o) - r(t_o|x_o)\| \geq \varepsilon$.

As $t_\beta \to t_o$: $\exists \beta_o / \forall \beta \geq \beta_o,\ t_\beta \in W$ and then $\forall x \in b/\ \|x - x_o\| < \eta, / \forall \beta \geq \beta_o$:

$$\begin{aligned} & \|r(t_\beta|x) - r(t_o|x)\| \\ \geq\ & \|r(t_\beta|x_o) - r(t_o|x_o)\| - \|r(t_\beta|x) - r(t_\beta|x_o)\| - \|-r(t_o|x) + r(t_o|x_o)\| \\ \geq\ & \varepsilon - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} = \frac{\varepsilon}{3} \end{aligned}$$

and:

$$\|r(t_\beta| - r(t_o|\|_b \geq \int_{b\cap\|x-x_o\|<\eta} \|r(t_\beta|x) - r(t_o|x)\|\, d\mu_b \geq\ \mu_b(b \cap \{x, \|x - x_o\| < \eta\}) < \frac{\varepsilon}{3}.$$

Then $t_\beta \to t_o$ and $\|r(t_\beta| - r(t_o|\|_b \nrightarrow 0$ which contradicts the continuity of f.

Therefore $t \mapsto \varphi(t) = r(t|x_o)$ defines a continuous selection of F at t_o.

The reciprocal proof is actually identical to the reciprocal proof of Theorem 3.7. □

3.4. Chain rules

Let be $f_I : O \sim> O' \subset T'$ of subjacent set-valued map F and $g_J : O' \sim> E$ of subjacent set-valued map G.

Theorem 3.9.

1. *Let's assume that:*
   - *f_I is a "continuous functions stream" at t_o in the sense it has a system of representations $[b_i, r_i(\,]_{i\in I}$ which verifies the strongly compatible property: $t \to r_i(t|x)$ is continuous at t_o for any $x \in b_i$ and any $i \in I$*
   - *g_J is continuous on its domain,*

   then $G \circ F$ is continuous at t_o.
2. *Let's assume that f_I and g_J verify hypthesies of Theorem 2.13 and that:*
   - *f_I has a system of representations $[b_i, r_i(\,]_{i\in I}$ of evolution $e_i($ such as $(t, x) \mapsto e(t|x)$ wich is continuous at t_o uniformly for $x \in b$, $\forall i \in I$,*
   - *g_J is uniformly continuous on O',*

   then g_J is C-chainable by f_I and therefore $g_J \circ f_I$ and $G \circ F$ are continuous at t_o.

Proof. *1.* The compatibility is obvious. The result on continuity follows immediately from the re-indexing process Theorem 2.8,

2. g_J is C-chainable by f_I following Theorem 2.13. Therefore following Theorem 2.10 we can then choose to use systems of representations which verify specific hypothesies of the proof.

Let be $(i, j) \in I \times J$ and $t_\nu \to t_o$.

Following hypothesies: $\forall \varepsilon > 0$, $\exists \eta > 0$, $\exists \nu_o / \forall \nu \geq \nu_o$, $\|r_i(t_\nu|x) - r_i(t_o|x)\| < \eta$, $\forall x \in b_i$ and then: $\|e'_j(e_i(t_\nu|x) - e'_j(e_i(t_o|x)\|_{b'_j} < \varepsilon$, $\forall x \in b_i$. Then we have for representations defined in Theorem 2.13:

$$\begin{aligned}\|E_{i,j,T}(t_\nu| - E_{i,j,T}(t_o|\,\|_{b\times b'} &= \|E_{i,j}(t_\nu| - E_{i,j}(t_o|\,\|_{b\times b'}\\ &= \int_b\int_{b'}\|e'_j(e_i(t_\nu|x)|x') - e'_j(e_i(t_o|x)|x')\|d\mu_{b'}d\mu_b\\ &= \int_b\|e'_j(e_i(t_\nu|x) - e'_j(e_i(t_o|x\|_{b'}d\mu_b\\ &\le \int_b\varepsilon d\mu_b \le \varepsilon\end{aligned}$$

□

Theorem 3.10.

1. *Let be $G : O' \multimap E'$ a continuous set valued map and $F = f : O \mapsto O'$ a continuous function, then $G \circ f$ is continuous.*
2. *Let be $G = g : O' \mapsto E'$ a Lipschitz function near t_o and $F : O \multimap O'$ a continuous set valued map. Let's assume that F and g verify hypothesies of Theorem 2.14 2., then $g \circ F$ is continuous at t_o.*

Proof. *1.* It is enough to consider the covering obtained in the proof of Theorem 2.14 *1.* and the result follows.

2. Following Theorem 2.14 *2.* only the continuity is still to be proved. For that we consider the covering $\widehat{[b_i, E_{i,T}(\,]}_{i\in I} = [b_i, g\circ e_i(\,, \widehat{g\circ e_{i,T}(-}g\circ e_i(\,]_{i\in I}$ of $g\circ F$ where $\widehat{[b_i, e_{i,T}(\,]}_{i\in I}$ is a suitable covering of F and we have:

$$\|E_{i,T}(t|\,\|_b = \int_b \|g(e(t|x) - g(e(t_o|x)\|d\mu_b \le k\|e(t| - e(t_o|\,\|_b.$$

The result follows. □

4. Differentiability

The obvious difficulty due to the extension process used in differentiability context is purely apparent as it is showed in the coherence theorem given in first. To start I enlist basics of the extended differentiability concept then, in a second time, I will shortly study relation between differentiability and contingent derivatives. To end I present extensions of partial and high order differentials then, in a way related to the Thom's "multijets" idea, extension of differential operators and some of their applications such as Taylor's theorem and chains formulae.

T a normed space.

4.1. Basics

Let be $k \in \overline{\mathbb{N}}$. The coherence of the notion is based on the following theorem:

Theorem 4.1.

$$\mathcal{E}^k(b, \mathcal{L}(T, E)) \subset_> \mathcal{L}(T, \mathcal{E}^k(b, E)).$$

Therefore, if $dim(T) < \infty$*:* $\mathcal{E}^k(b, \mathcal{L}(T, E)) \simeq \mathcal{L}(T, \mathcal{E}^k(b, E))$ *where* $\mathcal{L}$ *denotes the space of continuous linear operators.*

Proof. First let be $u. \in \mathcal{E}^k(b, \mathcal{L}(T, E)) : \forall t, [x \mapsto u.t|x)] \in \mathcal{E}^k(b, E)$ because $x \mapsto u.\|x)$ is C^k uniformly on every bounded set of T and therefore: $\|u.\|_{\mathcal{L}(T,\mathcal{E}^k(b,E))} = \sup_{\|t\|=1} \int_b \|u.t|x)\| \, d\mu_b \leq \int_b \sup_{\|t\|=1} \|u.t|x)\| \, d\mu_b = \int_b \|u.|x)\| \, d\mu_b = \|u.\|_b < \infty$.

On the other hand, if $dim(T) = m < \infty$, we have: $\mathcal{L}(T, E) \simeq E^m$ and then $\mathcal{E}^k(b, \mathcal{L}(T, E)) \simeq \mathcal{E}^k(b, E^m)$ and similarly $\mathcal{L}(T, \mathcal{E}^k(b, E)) \simeq \mathcal{E}^k(b, E)^m$. The topological equivalence follows. $\square$

Definition 4.2. We say that a k-regular representation $[b, r(\,]$ is k-regular differentiable at t_o if:

$$\exists u. \in \mathcal{E}^k(b, \mathcal{L}(T, E)),\ \exists h \mapsto \epsilon(h| \in L(b, E) :$$
$$e_T(t_o + h| = e_T(t_o| + u.h| + \|h\| \, \epsilon(h| \text{ and } \lim_{h \to 0} \|\varepsilon(h|\,\|_b = 0.$$

$u.$ is then said differential of $r($ at t_o and will be denoted $r^{(1)}(t_o)$.

$[b, r(\,]$ is said differentiable if it exists $k' \leq k$ such as $r($ is k'-regular differentiable.

Comments. 1. As $r(= i + e_T($ and $\|\varepsilon(h| + s(t| - s(t+h|\,\|_b = \|\varepsilon(h|\,\|_b$, the differentiability property is equivalent each of the two properties:

$$\exists u. \in \mathcal{E}^k(b, \mathcal{L}(T, E)),\ \exists h \mapsto \epsilon(h| \in L(b, E) :$$
$$r(t_o + h| = r(t_o| + u.h| + \|h\| \, \epsilon(h| \text{ and } \lim_{h \to 0} \|\varepsilon(h|\,\|_b = 0,$$

$$\exists u. \in \mathcal{E}^k(b, \mathcal{L}(T, E)),\ \exists h \mapsto \epsilon(h| \in L(b, E) :$$
$$e(t_o + h| = e(t_o| + u.h| + \|h\| \, \epsilon(h| \text{ and } \lim_{h \to 0} \|\varepsilon(h|\,\|_b = 0.$$

2. The unicity of the differential $r^{(1)}(t_o)$ follows then from the identity of this differential with the derivative of $e_T(: O \mapsto \mathcal{E}^k(b)$ at t_o.

Theorem 4.3.

1. *The differentiability is a strong compatible property.*
2. *The differentials of two equivalent representations are equivalent and the transfers of the equivalence are the same.*

Proof. *1.* Let's put $u. = r^{(1)}(t_o)$ and let's assume that $[b', r'(\,] \sim [b, r(\,]$.

For suitable τ, b_1, b_1', we have: $r'_{b_1'}(t+h| = \tau^{\#}._{b_1} r(t+h| = \tau^{\#}._{b'}(r(t| + u.h| + \|h\| \, \varepsilon(h|) = \tau^{\#}._{b_1} r(t| + \tau^{\#}._{b_1} u.h| + \|h\| \, \tau^{\#}._{b_1} \varepsilon(h|$, then: $r'_{b_1'}(t+h| = r'_{b_1'}(t + \tau^{\#}._{b_1} u.h| + \|h\| \, \tau^{\#}._{b_1} \varepsilon(h|$

Let $[b', u'.]$ and $[b', \varepsilon'(\,]$ be defined by: $u'.h|x') = \int_{b_1'} {}_{b_1} u.h|\tau(x', x_1'))d\mu_{b_1'}$ and $\varepsilon'(h|x') = \int_{b_1'} {}_{b_1} \varepsilon(h|\tau(x', x_1'))d\mu_{b_1'}$, $\forall x' \in b'$.

As $x \mapsto u.|x)$ is continuous from b compact to $\mathcal{L}(T, E)$: $\sup_{x \in b} \|u.|x)\|_{\mathcal{L}(T,E)} = M < \infty$ and then for $x' \in b'$: $\sup_{\|h\|=1} \|u'.h|x'\| \leq \int_{b_1'} \sup_{\|h\|=1} \|{}_{b_1} u.h|\tau(x', x_1'))\| \, d\mu_{b_1'} \leq \int_{b_1'} M d\mu_{b_1'}$

$= M$. It follows that $u.|x') \in \mathcal{L}(T,E)$. Therefore $[x \mapsto u.| \circ \pi_b \circ \tau$ is C^k and $b' \times b'_1$ is compact and following Lebesgue's theorem $[x' \mapsto u'.|x')] \in \mathcal{E}^k(b, \mathcal{L}(T,E))$ is then C^k.

Independently we have: $\lim_{h \mapsto 0} \|\varepsilon'(h|\,\|_{b'} \leq \lim_{h \mapsto 0} \int_{b' \times b'_1} \|_{b_1} u.h|\tau(x', x'_1))\| \, d\mu_{b' \times b'_1} = \lim_{h \mapsto 0} \|\varepsilon(h|\,\|_b = 0$, that achieved the proof of the differentiability of $r'($.

The compatibility is therefore strong as every property compatible on $r(t'| - r(t|$.

2. We have: $r'_{b'_1}(t+h| - r'_{b'_1}(t| = \tau^{\#}._{b_1} r(t+h| - \tau^{\#}._{b'}(r(t|$ and then $\forall \beta_n > 0, \beta_n \to 0, \forall h_o \in T$: $u'_{b'_1}.\beta_n h_o| + \|\beta_n h_o\| \, \varepsilon'(\beta_n h_o|) = \tau^{\#}._{b_1} u.\beta_n h_o| + \|\beta_n h_o\| \, \tau^{\#}._{b_1} \varepsilon(\beta_n h_o|)$, that can be written: $u'_{b'_1}.h_o| + \|h_o\| \, \varepsilon'(\beta_n h_o|) = \tau^{\#}._{b_1} u.h_o| + \|h_o\| \, \tau^{\#}.b_1 \varepsilon(\beta_n h_o|)$. Then, as $\lim_{\beta_n h_o \to 0} \|\varepsilon(\beta_n h_o|\,\|_b = 0$ and $\lim_{\beta_n h_o \to 0} \|\varepsilon'(\beta_n h_o|\,\|_b = 0$, there is β_ν, extracted / $\varepsilon(\beta_n h_o| \to 0$ and $\varepsilon'(\beta_n h_o| \to 0$ $\mu_{b' \times b'_1}$ *a.e.*.

Then: $u'_{b'_1}.h_o| = \tau^{\#}._{b_1} u.h_o|$, $\mu_{b' \times b'_1}$ *a.e.* and therefore, by continuity, the egality holds on $b' \times b'_1$. The result follows by symmetry. □

Definition 4.4. The definitions of differentiability for dynamics, coverings and set-valued maps follow from general rules (see 1.2) and we have therefore:

1. If the dynamic $f : O \sim> E$ is the class of $[b, r(\,]$ the differential of f is the dynamic $df : O \sim> \mathcal{L}(T,E)$ defined by:
$$df(t_o) = \widehat{[b, r^{<1>}(t_o).]}\,.$$
2. The differential of a covering $f_I : O \sim> E$ is then the covering $df_I : O \sim> \mathcal{L}(T,E)$ defined by:
$$df_I = (df_i)_{i \in I}\,.$$
3. The differential of a set-valued map $F : O \multimap E$ is the set-valued map $dF : O \multimap \mathcal{L}(T,E)$ defined by:
$$dF = \underset{f_J \in \mathcal{C}(F)}{\cup} \underline{df_J} = \underset{f \in \mathcal{I}(F)}{\cup} \underline{df}\,.$$
4. A dynamic, a covering, a set-valued map, is said continuously differentiable on O if its differential is continuous on O.

Comments and general convention. The value of the differential $dF(t)$ of a set valued map is function of the structure and analytical properties set $\mathcal{H}$ assumed for F and then for every of its coverings (resp. internal dynamics). We have obviously if $\mathcal{H}_1 \subset \mathcal{H}_2$ then $d_{\mathcal{H}_2}F(t) \subset d_{\mathcal{H}_1}F(t)$ because $\mathcal{C}_{\mathcal{H}_2}(F) \subset \mathcal{C}_{\mathcal{H}_1}(F)$ (resp. $\mathcal{I}_{\mathcal{H}_2}(F) \subset \mathcal{I}_{\mathcal{H}_1}(F)$). It must denote than if the very restrictive structure property $\omega = 1$ is included in the reference properties set $\mathcal{H}$, and only in this case, the definition of dF relative to $\mathcal{I}(F)$ is not usable.

Except particular cases, in any situation where several coverings or set valued maps are considered the properties set $\mathcal{H}$ will be only made up of their common properties.

Theorem 4.5. *Any covering and any set valued map diffrenentiable at t_o (resp. on O) is continuous at t_o (resp. O).*

Proof. Actually we have for a dynamic $\widehat{[b, r(\,]}$: $\|e_T(t| - e_T(t_o|\,\|_b \leq \|t - t_o\| \left(\left\|r^{(1)}(t_o\right\|_{\mathcal{L}} + \|\varepsilon(t - t_o|\,\|_b\right)$. The result follows. □

Theorem 4.6 (Case of functions). *Let $f : O \mapsto E$ be a function, it is equivalent to say:*

- *The function f is Frechet-differentiable at t_o (resp. on O).*
- *The dynamic f is differentiable at t_o (resp. on O).*
- *The set-valued map $\{f\}$ is differentiable at t_o (resp. on O).*

Therefore: $d\{f\}(t_o) = \{f'(t_o)\}$.

Proof. Let be f Frechet-differentiable at t_o: $\exists u. \in \mathcal{L}(T, E)/\, f(t_o + h) = f(t_o) + u.h + \|h\|\, \varepsilon(h)$, $\lim_{h\to 0} \varepsilon(h) = 0$.

Let be b a base and $r(\equiv [b, 0, e(, 0], e(t|x) = f(t), U. : x \in b \mapsto U.|x) = u.$, $\epsilon(h| : x \in b \mapsto \epsilon(h|x) = \varepsilon(h)$. Then $U. \in \mathcal{E}^\infty(b, \mathcal{L}(T, E))$ and $\forall h$, $\epsilon(h| \in L(b)$. Therefore $r(t_o + h| = r(t_o| + U.h| + \|h\|\, \varepsilon(h|$ and $\lim_{h\to 0} \|\epsilon(h|\,\|_b = \lim_{h\to 0} \|\varepsilon(h)\| = 0$. It follows that $r($ is differentiable at t_o.

Let be $f = \widehat{[b, r(]}$ differentiable at t_o: $\exists U. \in \mathcal{E}(b, \mathcal{L}(T, E))$, $\exists h \mapsto \varepsilon(h| \in L(b), /\ r(t_o + h| = r(t_o| + U.h| + \|h\|\, \varepsilon(h|$, $\lim_{h\to 0} \|\epsilon(h|\,\|_b = 0$. Then $f(t_o + h) = f(t_o) + U.h|x) + \|h\|\, \varepsilon(h|x)\, \forall x \in b$.

Let be h_o fixed, $\beta_n > 0$, $\beta_n \to 0$: there is β_ν, extracted, $A \subset b/\, \mu_b(A) = 0$ and $\forall x \in b - A, \varepsilon(\beta_\nu h_o|x) \to 0$. Therefore $\forall x, x_o \in b - A, U.h_o|x) - U.h_o|x_o) = \|h_o\|\, (\epsilon(\beta_\nu h_o|x) - \epsilon(\beta_\nu h_o|x_o)) \to 0$ when $\beta_\nu \to \infty$ and then $U.h_o|x) - U.h_o|x_o) = 0$. It follows by k-regularity that the egality holds for any $x \in b$. We have then $U.h|b) = \{u.h\}$ with $u. \in \mathcal{L}(T, E)$ because $\|u.h\| = \|U.h\|_b$, $\epsilon(h|b) = \{\varepsilon(h)\}$ with $\|\varepsilon(h)\| = \|\epsilon(h|\,\|_b \to 0$ if $\|h\| \to 0$ and $f(t_o + h) = f(t_o) + u.h + \|h\|\, \varepsilon(h)$. f is then Frechet-differentiable and therefore: $d\{f\}(t) = \{f'(t)\}$. □

Theorem 4.7.

1. *The spaces of differentiable dynamics, coverings, set-valued maps, are stable by linear combination and union. Therefore:*
   (a) *The operator $f_J \mapsto df_J$ is*
   - *linear: $d(\alpha f_I + g_J) = \alpha df_I + dg_J$,*
   - *boolean: $d(f_I \cup g_J) = df_I \cup dg_J$ and, subject to existence, $d(f_I \cap g_J) = df_I \cap dg_J$.*

   (b) *The operator $F \mapsto dF(t)$ is*
   - *sublinear: $\alpha dF(t) = d(\alpha F)(t)$, $dF(t) + dG(t) \subset d(F + G)(t)$,*
   - *subboolean: $dF(t) \cup dG(t) \subset d(F \cup G)(t)$ and, subject to existence, $d(F \cap G)(t) \subset dF(t) \cap dG(t)$,*
   - *increasing: If $F \subset G$ then $dF(t) \subset dG(t)$.*
2. *The differentiability is stable by relaxation and:*
   - *$co(df_J) = dco(f_J)$,*
   - *$co(dF)(t) \subset d(co(F))(t)$ and therefore if F has convex images, dF has convex images.*
3. *If E is an algebra, the spaces of simple differentiable dynamics, coverings, set-valued maps, are stable by algebraic product.*

Proof. The results *1.* and *2.* follow immediately from the definitions of differentials and corresponding theorems of the section Algebra.

For *3.*, let be $f = \widehat{[b, e(, s(]}$ and $g = \widehat{[b', e'(, s'(]}$. We have then: $d(fg) = \widehat{[b \times b', E_T(]}$ with: $E_T(t| = r^{(1)}(t| \otimes e'(t| + e(t| \otimes r'^{(1)}(t|$ where $\otimes$ denotes the analytical tensorial product [7]. We have actually, with $r^{(1)} = u$ and $r'^{(1)} = v$:

$$\begin{aligned} & e_T(t+h|x)e'_T(t+h|x') \\ &= e_T(t|x)e'_T(t|x') + u.h|x)e'(t|x') + e(t|x)v.h|x') \\ &\quad + \|h\| \Big(e_T(t+h|x)\varepsilon'(h|x') + \varepsilon(h|x)e'_T(t|x') + u.h|x)v.h|x') \\ &\quad + \frac{1}{\|h\|}(u.h|x)s'(t|x') + s(t|x)v.h|x')) \Big) \end{aligned}$$

$e(t|b), e'(t|b'), e_T(t|b)$ and $e'_T(t|b')$ are bounded, $u.h|x)s'(t|x') + s(t|x)v.h|x') = 0\mu_{b\times b'}$ *a.e.* and: $\int_{b\times b'} \|u.h|x)\ v.h|x')\|\ d\mu_{b\times b'} \leq \int_b \|u.|x)\|\ d\mu_b \int_{b'} \|v.|x')\|\ d\mu_{b'}\ \|h\|^2$. The result follows. □

4.2. Differentiability and contingent derivatives

We follow for the notions of graphical set valued analysis theory the notations given in [1].

Theorem 4.8.

1. *Let $f_J : O \sim> E$ be a non singular covering differentiable at t, then:*

$$Gr(\underline{df_J}(t)) \subset \bigcup_{y\in \underline{f_J}(t)} Gr(D\underline{f_J}(t,y)).$$

   More precisely let $f : O \sim> E$ be a non singular dynamic differentiable at t, for any representation $r($ of f, any $y \in \underline{f}(t)$ and any $x \in r(t|^{-1}y)$, $Gr(r^{<1>}(t).|x)) = \{(\alpha, r^{<1>}(t).\alpha|x),\ \alpha \in T\}$ is both a vector subspace of $T \times E$ and a subset of $Gr(D\underline{f}(t,y))$.

2. *Let $F : O - \circ E$ be a non singular set-valued map differentiable at t:*

$$Gr(dF(t)) \subset \bigcup_{y\in F(t)} Gr(DF(t,y)).$$

Proof. Let's assume that $r(\equiv [b, i, e(, 0] \in f,\ u. = r^{(1)}(t)$.

We have then $y = r(t|x)$ and $x' \to \|h\|\,\varepsilon(h|x') = e(t+h|x') - e(t|x') - u.h|x')$ is continuous for any fixed h $(*)$.

Let be $h_n \to^{>} 0$, $\alpha \in T$, : $r(t + h_n\alpha|x) = r(t|x) + u.h_n\alpha|x) + \|h_n\alpha\|\,\varepsilon(h_n\alpha|x)$, which can be written $y_n = y + h_n\beta_n(x)$ with $t_n = t + h_n\alpha$, $y_n = r(t_n|x) \in \underline{f}(t_n)$ and $\beta_n(x) = u.\alpha|x) + \|\alpha\|\,\varepsilon(h_n\alpha|x)$.

As $\|\varepsilon(h_n\alpha|\|_b \to 0$, if $h_n \to 0$ there is t_ν, extracted, $/\,\varepsilon(h_\nu\alpha| \to 0\,\mu_b$ *a.e.*.

$M = \{x' \in b\,/\,\varepsilon(h_\nu\alpha|x') \to 0\}$ is dense in b and $\forall x \in M$, $\beta_\nu(x') \to u.\alpha|x')$, $y'_\nu = y + h_\nu\beta_\nu(x') \to y$.

Let be $x'_m \in M \to x$ $(**)$: $\|\beta_\nu(x) - \beta_\nu(x'_m)\| \leq \|u.\alpha\|\ \|x - x'_m\| + \|\alpha\|\ \|\varepsilon(h_\nu\alpha|x)$ $-\varepsilon(h_\nu\alpha|x'_m)\|$.

Following $(*)$ and $(**)\,\forall\nu\ /\ h_\nu \leq 1,\ \|h_\nu\| \leq 1,\ \forall\rho > 0,\ \exists m_\nu\ /\ \left\|x - x'_{m_\nu}\right\| < \frac{\rho}{3}$ and $\|h_\nu\alpha\|\ \left\|\varepsilon(h_\nu\alpha|x) - \varepsilon(h_\nu\alpha|x'_{m_\nu})\right\| < \frac{\rho}{3}$ and then: $\left\|h_\nu(\beta_\nu(x) - \beta_\nu(x'_{m_\nu}))\right\| \leq \frac{2\rho}{3}$.

Therefore $\forall\rho > 0, \exists\nu_\rho\ /\forall\nu \geq \nu_\rho,\ \|y - y_\nu\| \leq \|y - y'_\nu\| + h_\nu\left\|\beta_\nu(x) - \beta_\nu(x'_{m_\nu}\right\| \leq \frac{\rho}{3} + \frac{2\rho}{3} = \rho$, that is to say $y_\nu \to y$ and then $(\alpha, u.\alpha|x)) \in T_{Gr(\underline{f}}(t_o, y) = D\underline{f}(t_o, y)$. The result follows. $\square$

Convention. We call unproper cone of a cone Γ the set $\Gamma_{unp} = \{v \in \Gamma / -v \in \Gamma\}$. A cone Γ is then proper if and only if $\Gamma_{unp} \subset \{0\}$.

Theorem 4.9. *Let be K a subset of E and $\mathcal{K} : t \in \mathbb{R} \multimap K$.*

1. *If for any y in $K\, T_K(y)$ is proper, then for any t $d\mathcal{K}(t) = \{0\}$, or equivalently, any differentiable covering of $\mathcal{K}$ is essentially unchanged.*
2. *If K is convex and E a Banach space:*

$$\bigcup_{y\in K} S_K(y)_{unp} \subset d\mathcal{K}(t) \underset{def}{=} dK \subset \bigcup_{y\in K} T_K(y)_{unp} \subset \overline{dK}\,\forall t,$$

dK is the differential of K,

$$\bigcup_{y\in K} S_K(y) \subset d^+\mathcal{K}(t) \underset{def}{=} d^+K \subset \bigcup_{y\in K} T_K(y) \subset \overline{d^+K}\,\forall t,$$

where $d^+\mathcal{K}(t)$ denote the right differential at t of $\mathcal{K}$ and d^+K is said right differential of K.

Proof. For $(t, y) \in Gr(\mathcal{K}) = \mathbb{R} \times K,\ D\mathcal{K}(t, y) = T_{\mathbb{R}\times K}(t, y) = \mathcal{R} \times T_K(y)$.

Let t_o be fixed, $\forall[b, r(] \in f_j \in f_J$ differentiable covering of $\mathcal{K}$ and $(x, y) \in b \times K$ such as $r(t_o|x) = y$, we have: $Gr(r^{(1)}(t_o).|x)) \subset Gr(d\underline{f}(t_o, y))_{unp} \subset Gr(d\mathcal{K}(t_o, y))_{unp} = \mathbb{R} \times T_K(y)_{unp}$.

1. We have then $Gr(r^{(1)}(t_o).|x)) \subset \mathbb{R} \times \{0\}$;then $r^{(1)}(t_o).|x) = 0$, $\forall t_o$, $\forall x$: $t \mapsto e_T(t|$ is a function from $\mathbb{R}$ to $L(b)$ of null derivative for any t_o and then $e_T(= c$ fixed in $L(b)$ and $r($ is essentially unchanged, or in an equivalent way $d\mathcal{K}(t_o) = \{0\}, \forall t_o$.

2. Let $v \in S_K(y)_{unp}$ where $S_K(y) = \cup_{h>0} \frac{K-y}{h}$ [1]: it exists $\rho > 0$ such as $y + tv \in K$, if $|t| \leq \rho$.

Let then $f : t \mapsto \begin{cases} y + (t - t_o)v, & \text{if } |\mathrm{t} - \mathrm{t_o}| \leq \rho, \\ y + \frac{t-t_o}{|t-t_o|}\rho v, & \text{if } |\mathrm{t} - \mathrm{t_o}| \geq \rho; \end{cases}$ f is an internal dynamic of $\mathcal{K}$, with image y and derivative v at t_o: $v \in \underline{df}(t_o) \subset d\mathcal{K}(t_o)$. Therefore: $S_K(y)_{unp} \subset d\mathcal{K}(t_o)$, $\forall y \in K$.

Independently if $t \rightsquigarrow f(t)$ is an internal dynamic of $\mathcal{K}$, differentiable at t_o, the same stands for $t \rightsquigarrow f_1(t) = f(t + t_o - t_1)$ and we have $df_1(t_1) = df(t_o)$. It follows that $d\mathcal{K}(t_1) = d\mathcal{K}(t_o),\ \forall t_o, t_1$.

The result follows then immediately from: $\overline{S_K(y)_{unp}} = \overline{S_K(y)}_{unp} = T_K(y)_{unp}$.

The second inclusion follows formally from a same proof. $\square$

4.3. Product spaces

Theorem 4.10. *A covering* $f_J : O \sim> \times_{k=1,\dots,n} E_k$ *(resp. a set valued map* $F : O \multimap \times_{k=1,\dots,n} E_k$*) is differentiable if and only if every one of its coordinates* $f_J^k : O \sim> E_k$ *(resp.* $F^k : O \multimap E_k$*) is differentiable. Therefore:*

$$(df_J)^k = d(f_J^k), \qquad (dF)^k = d(F^k).$$

Proof. Strictly identical to the one of classical analysis. □

Definition 4.11. If $T = \mathop{\times}\limits_{k=1}^{n} T_k$, we define the k^{th}differentials of the covering f_J and set-valued map F, following the usual way, by:

$$\frac{\partial f_J}{\partial t_k}(a) = d[h \sim> f_J(a_1, .., a_k + h, a_{k+1}, .., a_n)](0, \quad \text{and}$$
$$\frac{\partial F}{\partial t_k}(a) = d[h \multimap F(a_1, .., a_k + h, a_{k+1}, .., a_n)](0).$$

Theorem 4.12.

1. *A covering (resp. a set-valued map) which is differentiable at* t_o *has partial differentials at* t_o *for any* $k = 1, ..., n$
2. *A covering (a set-valued map) is* C^1*-differentiable on* O *if and only if it has* n *continuous partial differentials.*

Proof. *1.* is obvious. *2.* As differentiability and continuity of representations are compatible and independent of the breakdown, it exists a representation of total evolution $e_T($ which has for any k a continuous partial differential $u_{k\cdot}$ on O. It follows immediately that $e_T($ is continuously differentiable on O and of (total) differential at t_o $u_{\cdot} = \sum_{k=1,\dots,n} u_{k\cdot} \circ \pi_k$. □

4.4. High order differentials

Definition 4.13. 1. We define the high order differentials of a covering f_J, following the usual way, by:

$$d^0 f = f, \quad d^{k+1} f = d(d^k f), \; k \geqslant 0.$$

2. We say then, following general rules, that a set-valued map $F : O \multimap E$ is k^{ce}-differentiable if it has a k^{ce}-differentiable covering, and we define $d^k F : O \multimap \mathcal{L}^k(T, E)$ by:

$$d^k F(t) = \bigcup_{f_J \in \mathcal{C}(F)} \underline{d^k f_J(t)} = \bigcup_{f \in \mathcal{I}(F)} \underline{d^k f}(t).$$

We have then obviously the statement:

Theorem 4.14. *If* $f : O \mapsto E$ *is a function, it is equivalent to say:*

- *The function* $f : O \mapsto E$ *is* k^{ce}*-differentiable (resp.* C^k*).*
- *The dynamics* $f : O \sim> E$ *is* k^{ce}*-differentiable (resp.* C^k*).*
- *The set-valued map* $\{f\} : O \multimap E$ *is* k^{ce}*-differentiable (resp.* C^k*).*

Comments. If a set-valued map $F : O \multimap E$ is $(k+1)^{ce}$-differentiable, it is k^{ce}-differentiable, but, a priori, its k^{ce}-differential is not necessary differentiable. Nevertheless, if it is the case:

$$d^{k+1}F \subset d(d^k F).$$

4.5. Differential operators

To extend the differential operator concept to coverings, and then to set-valued maps, we adapt the notion of "Thom's multijet" [2] as follows:

1. *Differential Rewriting of order k*:

$$t, h \in T,\ f_J = \widehat{[b_j, r_j(\,]}_{j\in J} \mapsto \mathcal{D}^{(k)}(f_J)(t).(h)^k$$

$$= \left[b_j, \widehat{\sum_{|(m_1,\ldots,m_n)|=k} \partial^m r_j(t).h^{(m_1,\ldots,m_n)}} \right]_{j\in J},$$

and

$$\mathcal{D}^{k)}F = \underset{f\in\mathcal{I}(F)}{\cup} \underline{\mathcal{D}^{(k)}f}$$

for $T = \overset{n}{\underset{k=1}{\times}} T_k,\ f_J,\ F,\ k^{ce}$-differentiable; $\partial^m = \frac{\partial^{(m_1,\ldots,m_n)}}{\partial t_1{}^{m_1}\ldots\partial t_n{}^{m_n}}$.

2. *Leibnizian of order k*:

$$\mathcal{L}^{(k)} : \left(f_I = \widehat{[b_i, r_i(\,]}_{i\in I}, g_J = \widehat{[b'_j, r'_j(\,]}_{j\in J} \right)$$

$$\mapsto \left[b_i \times b'_j, \widehat{\sum_{0\leqslant m\leqslant k} \binom{k}{m} r_i^{(m)}(\otimes r'_j{}^{(k-m)}(} \right]_{(i,j)\in I\times J}$$

and

$$\mathcal{L}^{(k)}(F,G) = \underset{f\in\mathcal{I}(F),g\in\mathcal{I}(G)}{\cup} \underline{\mathcal{L}^{(k)}(f,g)}$$

for f_I, g_J, F, G simple, non singular, C^k-differentiable.

3. *Taylorian of order k*:

$$t, h \in T,\ f_I = \widehat{[b_i, r_i(\,]}_{i\in I} \mapsto \mathcal{T}^{(k)}(f_J)(t).h = \left[b_i, \widehat{\sum_{0\leqslant m\leqslant k} \frac{1}{m!} r_i^{(m)}(t).(h)^m} \right]_{i\in I}$$

and

$$\mathcal{T}^{(k)}(F) \underset{f\in\mathcal{I}(F)}{\cup} \underline{\mathcal{T}^{(k)}(f)}$$

for $f_J,\ F,\ k^{ce}$-differentiable.

Comments. If is obviously possible to extend in this way to coverings and set valued maps every classical differential operator such as gradian, Laplacian, rotational, etc...

Theorem 4.15.

1. *Let's assume that f_J, and F are k^{ce}-differentiable:*

$$\mathcal{D}^{(k)} f_J = d^k f_J \quad \textit{and} \quad \mathcal{D}^{(k)}(k)F = d^k F.$$

2. *Let's assume that f_I, g_J, F, G are simple, non singular, C^k-differentiable:*

$$\mathcal{L}^{(k)}(f_I, g_J) = d^k(f_I\ g_J) \quad \textit{and} \quad \mathcal{L}^{(k)}(F,G) \subset d^k(FG).$$

3. *Let's assume that:*

- *f_J is $(k-1)$-differentiable on O and k^{ce}-differentiable at t, then:*

$$f_J(t+h) = \mathcal{T}^{(k)}(f_J)(t).h + ||h||^k \epsilon_J(h|, \quad \textit{with } \lim_{h\mapsto 0} \epsilon_J(h| = 0$$

  in this way that if $\widehat{[b_j, r_j(\,]}_{j\in J} = f_J$, $\exists [b_j, \epsilon_j(\,]_{j\in J}$: $\forall j$, $r_j(t+h| = \underset{m=0,..,k-1}{\Sigma} \frac{1}{m!} r_j^{(m)}(t).(h)^m + \|h\|^k \epsilon_j(h|$, with $\lim_{h\mapsto 0} \|\epsilon_j(h|\|_{b_j} = 0$.

- *f_J is non singular, C^k-differentiable on O and such as: $\sup_{u.\in \underline{d^k f_j(s)}} \|u.\|_{\mathcal{L}} \leq M_j, /, \forall j \in J, s \in [t; t+h] \subset O$, then:*

$$f_J(t+h) = \mathcal{T}^{(k-1)}(f_J) + \int_0^1 \frac{(1-s)^{(k-1)}}{(k-1)!} d^k f_J(t+sh).(h)^k ds$$

  in the sense of operators, that is to say: $\forall \widehat{[b_j, r_j(\,]}_{j\in J} = f_J$, $x \in b_j, j \in J$,

$$r_j(t+h|x) = \underset{m=0,..,k-1}{\Sigma} \frac{1}{m!} r_j^{(m)}(t).(h)^m|x) + \int_0^1 \frac{(1-s)^{(k-1)}}{(k-1)!} r_j^{(k)}(t+sh).(h)^k|x) ds.$$

Therefore if f_J is subjacent to F and such as: $\sup_{j\in J)} M_j \leq M$ we have:

$$F(t+h) \subset \underline{\mathcal{T}^{(k-1)}(f_J)}(t).h + M\|h\|^k \subset \mathcal{T}^{(k-1)}(F)(t).h + M\|h\|^k.$$

Proof. *1.* follows directly from the defintions of differentiability and partial differentiability.

As for representations there is identity between differential and derivative of $e_T(: O \mapsto L^1(b)$, *2.* and the first result of *3.* are for dynamics simple translations of the classical one-one analysis corresponding results, [5].

For the second point of *3.* the one-one anlysis give similarly for $t \mapsto e_T(t| = e(t| \,/\, [b, i, e(\,] \in f$: $e(t+h| = \underset{m=0,..,k-1}{\Sigma} \frac{1}{m!} e^{(m)}(t).(h)^m| + \int_0^1 \frac{(1-s)^{(k-1)}}{(k-1)!} e^{(k)}(t+sh).(h)^k ds|$ in $L^1(b)$.

As $L^1\left(]0;1[, L^1(b, d\mu), ds\right) = L^1\left(]0;1[\times b, ds \otimes d\mu\right)$,[7], we can write following Fubini's theorem: $e(t+h|x) = \underset{m=0,..,k-1}{\Sigma} \frac{1}{m!} e^{(m)}(t).(h)^m|x) + \int_0^1 \frac{(1-s)^{(k-1)}}{(k-1)!} e^{(k)}(t+sh).(h)^k|x) ds$, μ a.e. $x \in b$.

As $\|e^{(k)}(t+sh).(h)^k|x)\| \leq M\|h\|^k$ on $[0;1] \times b$, the x-continuity of the integral term follows from Lebesgue's theorem and then, by continuity of the two members, the egality can be written for any x in b. Therefore the last result follows then immediately.

For coverings and set valued maps the results follow from the general rules of transfer as usually. $\square$

4.6. Chains

Theorem 4.16. *Let be $f_I : O \sim O'$ and $g_J : O' \sim E$, of subjacent set valued maps $F : O \multimap O'$ and $G : O' \multimap E$.*

1. *Let's assume that:*
   - *f_I is a "differentiable functions stream" in the way that it has a system of representations $[b_i, r_i(\,]_{i\in I}$ which verifies the strongly compatible property: $h \mapsto r_i(t+h|x)$ is differentiable at $h=0$ for any $i \in I, x \in b_i$,*
   - *g_J is differentiable on $F(t)$,*

   then $G \circ F$ is differentiable at t.
2. *Let's assume the following $\mathcal{H}$ set properties:*
   - *f_I is differentiable at t and has a system of representations $[b_i, r_i(\,]_{i\in I}$ which verifies the strongly compatible property:*

   $$\int_{b_i} \|r_i(t+h| - r_i(t|\,\| d\mu_{b_i} \leq K_i \|h\|^{\alpha_i},\ \alpha_i > 0, \forall i \text{ locally at } h = 0;$$

   - *g_J and dg_J are C-chainable with f_I,*
   - *O' is convex and g_J 2^{ce}-differentiable on O' such as $\underline{d^2 g_J} \circ \underline{f_I}$ is locally C-bounded at t,*

   then $g_J \circ f_I$ is differentiable at t. We have then in the sense of differential operators:

   $$d(g_J \circ f_I)(t) = (dg_J \circ f_I)(t).df_I(t).$$

   Therefore, $G \circ F$ is differentiable at t.

Proof. *1.* Let $f_I = f$, $g_J = g$, and $[b', r'(\,] \in g$. For $x \in b$ fixed, $r^{(1)}(t) = u$, $r'^{(1)}(r(t|x)) = v$, we have then for any $x' \in b'$:

$$\begin{aligned}
& (*)\, r'(r(t+h|x)|x') \\
= \;& r'(r(t|x) + u.h|x) + \|h\|\epsilon(h|x)|x') \\
= \;& r'(r(t|x)|x') + v.\left(u.h|x) + \|h\|\epsilon(h|x)\right)|x') \\
& \quad + \|u.h|x) + \|h\|\epsilon(h|x)\|\epsilon'\left(r(t+h|x) - r(t|x)|x'\right) \\
= \;& r'(r(t|x)|x') + v.u.h|x)|x') \\
& \quad + \|h\| \left(v.\epsilon(h|x)|x') + \|u.\frac{h}{\|h\|}|x) + \epsilon(h|x)\|\epsilon'\left(r(t+h|x) - r(t|x)|x'\right) \right).
\end{aligned}$$

$x' \mapsto [h \mapsto v.u.h|x)|x')] \in \mathcal{E}^k(b', \mathcal{L}(T,E))$ if $r'($ is k-regular differentiable and, independently: $\|v.\epsilon(h|x)|\,\|_{b'} = \int_{b'} \|v.\epsilon(h|x)|x')\| d\mu_{b'}(x') \leq \|v.\|_{\mathcal{L},b'} \|\epsilon(h|x)\| \to 0$ if $h \to 0$ because $\zeta \mapsto r(\zeta|x)$ differentiable at t.

In the same way: $\|u.\frac{h}{\|h\|}|x) + \epsilon(h|x)\| \leq \|u.|x)\|_{\mathcal{L}} + \|\epsilon(h|x)\|$ is locally bounded at $h=0$ and $\|\epsilon'(r(t+h|x) - r(t|x)|\,\|_{b'} = \int_{b'} \|\epsilon'(r(t+h|x) - r(t|x)|x')\| d\mu_{b'} \to 0$ if $h \to 0$ because $\zeta \mapsto r(\zeta|x)$ is continuous at t.

Therefore $\|v.\epsilon(h|x)|\, + \|u.\frac{h}{\|h\|}|x) + \epsilon(h|x)\|\epsilon'(r(t+h|x) - r(t|x)|\,\|_{b'} \to 0$ if $h \to 0$ and the result follows.

2. We have now:

$$\begin{aligned}(*)\, r' \odot r(t+h|x,x') &= r' \odot r(t|x,x') \\ &= r'^{(1)} \odot r(t|x,x').u.h|x) + \|h\|(r'^{(1)} \odot r(t|x,x').\epsilon(h|x)|x') \\ &\qquad + \frac{1}{\|h\|}\|r(t+h|x) - r(t|x)\|\epsilon'(r(t+h|x) - r(t|x)|x'));\end{aligned}$$

and then:

$$\begin{aligned}\|r'^{(1)} \odot r(t|x,x').\epsilon(h|\,\|_{b\times b'} &\leq \int_b \int_{b'} \|r'^{(1)}(r(t|x)).\|_{\mathcal{L}} d\mu_{b'} \|\epsilon(h|\,\| d\mu_b \\ &\leq \int_b \|r'^{(1)}(r(t|x)).\|_{\mathcal{L},b'} \|\epsilon(h|\,\| d\mu_b.\end{aligned}$$

Therefore: $\|r'^{(1)} \odot r(t|x,x').\epsilon(h|\,\|_{b\times b'} \leq M\|\epsilon(h|\,\|_b \to 0$ if $h \to 0$, because $\zeta \mapsto r'^{(1)}(\zeta)$ is continuous on the compact $r(t|b)$ and then bounded.

We have by Theorem 4.15 *3.*:

$$\begin{aligned}&\frac{1}{\|h\|}\|r(t+h| - r(t|\,\|\epsilon'(r(t+h| - r(t|\,|\,\|_{b\times b'} \\ =\ & \frac{1}{2}\int_{b\times b'} \|r'^{(2)}(r(t+h|x) + \theta(h,x)r(t|x)).(r(t+h|x) - r(t|x))|x')\| d\mu_{b\times b'} \\ \leq\ & \frac{M}{2}\int_b \|r(t+h|x) - r(t|x)\| d\mu_b \leq \frac{M}{2}K\|h\|^{\alpha} \to 0 \ \text{ if h} \to 0\end{aligned}$$

and the result follows. □

Theorem 4.17.

1. *If $g = u. + c$ is a continuous affine function and F a differentiable set valued map of covering f_I, then $g \circ F$ is differentiable. Therefore:*

$$d(g \circ f_I) = u.df_I \quad \text{and} \quad u.dF \subset d(g \circ F)$$
$$\text{(resp. } u.dF = d(g \circ F) \text{ if } g \text{ is a diffeomorphism).}$$

2. *If a function f and a set valued map G, of covering g_J, are differentiable, then $G \circ f$ is differentiable. Therefore:*

$$(dg_J \circ f).df = d(g_J \circ f) \quad \text{and} \quad (dG \circ f).df \subset d(G \circ f).$$

Proof. The two results follow immediately from the definitions of differentiability of functions and coverings. □

5. Integrability

The extension process is here applied to the concepts of measurability and integrability in a very simple framework. This choice has been done only for a global maximal simplicity and homogeneity of the presentation of my work. It is obviously possible to consider the more general context used for this questions (see [1] for example).

As usually I begin to enlist the basics and the elementary properties of the extension, then I present the fundamental theorems of comparison between Aumann-integral and integral (in the sense of coverings). I enlist then some results, formally classical, on integrability and "primitive" existence and I end this part with the main theorems of regularization and approximation by convolution.

General conventions. ν a Radon positive measure on T, $dim(T) < \infty$; $M \subset O$ a measurable set. To lighten the account, we say measurable and integrable for ν-measurable and ν-integrable; furthermore, we will denote $\overset{\mathcal{A}}{\int} d\nu$ the Aumann integral [1], [3].

5.1. Measurability

Theorem 5.1.

1. *The measurability property: "the representation $[b, r(\,]$ has a breakdown $[b, i, e_T(\,]$ such has $e_T\, O \mapsto L(b)$ is Borel measurable" is strongly compatible.*
2. *The μ-simplicity property: "the representation $[b, r(\,]$ has a breakdown $[b, i, e_T(\,]$ of μ_b-measurable invariant" is strongly compatible.*

Proof. *1.* Let be $t_o \in O$ $[b, i, e_T(\,]$, $[b, i', e'_T(\,]$ two breakdowns of $r($, then: $e_T(t| = r(t| - r(t_o| + c = e'_T(t| + c'$ with c, c' in $L^1(b)$. Therefore, if the property is compatible, this compatibilty is strong.

Let be $r'(\sim r(: r'_{b'_1}(t| = \tau^{\#}._{b_1}r(t| \forall t \Rightarrow r'_{b'_1}(t| - r'_{b'_1}(t_o| = \tau^{\#}._{b_1}r(t| - {}_{b_1}r(t_o| = (\pi_b \circ \tau)^{\#}.(r(t| - r(t_o|)$ where π_b is the canonical projector from $b_1 \times b$ on b.

From $(\pi_b \circ \tau) \in Tr(b, b' \times b'_1$ it follows that $(\pi_b \circ \tau)^{\#}.$ is continuous and then $t \mapsto r'_{b'_1}(t| - r'_{b'_1}(t_o|$ is Borel measurable.

Let then: $r'_{b'_1}($ be measurable from O to $L(b' \times b'_1, E)$, i.e. $e'_{T,b'_1}($ Borel measurable.

Let be A a closed set in $L(b', E)$ and $\widetilde{A} = \{u \circ \pi_{b'} /\, u \in A\}$. We have $\|u \circ \pi_{b'}\|_{b' \times b'_1} = \|u\|_{b'}$, then $\widetilde{A}$ is closed in $L^1(b' \times b'_1, E)$ and therefore $e'_{T,b'_1}(^{-1}\widetilde{A}$ is Borel measurable.

As $e'_{T,b'_1}(t| = u \circ \pi_{b'} \Leftrightarrow e'_T(t| \circ \pi_{b'} = u \circ \pi_{b'} \Leftrightarrow e'_T(t| = u$, then $e'_T(^{-1}A = e'_{T,b'_1}(^{-1}\widetilde{A}$, i.e. $r'($ is measurable.

2. is obvious. □

Definition 5.2. A representation $[b, r(\,]$ is said to be measurable (resp. μ-simple) if it has the measurable (resp. μ-simple) property.

The definitions of measurability and μ-simplicity of dynamics, coverings, and set valued maps, follow then the general rules.

Theorem 5.3 (Case of functions). *Let $f : O \mapsto E$ be a function; it is equivalent to say:*

- *The function $f : O \mapsto E$ is Borel measurable.*
- *The dynamic $f : O \sim> E$ is measurable.*
- *The set-valued map $\{f\} : O \multimap E$ is measurable.*

Proof. To say that $\sum 1_{O_j} c_j$ is an approximation sequence of simple(*) measurable functions for $f : O \mapsto E$ is equivalent to say that $1_b(\sum 1_{O_j} c_j)$ is an approximation sequence of simple(*) measurable functions for $[b, 1_b f()] \in f$, because: $\forall t, \|f(t) - \sum 1_{O_j}(t) c_j\| = \|1_b f(t) - 1_b(\sum 1_{O_j} c_j)\|_b$. The result follows.

(*)simple is used here in the sense of classical functions measurability theory, [4]. □

Theorem 5.4.

1. *Any continuous covering (resp. set valued map) is measurable.*
2. *Measurability and continuity have the same algebraic and boolean stability properties.*

Proof. Immediate for *1.* and strictly identical to the corresponding continuity proofs for *2.*. □

Theorem 5.5.

1. *Any measurable μ-simple set-valued map of order at most countable is Borel measurable.*
2. *For any Borel measurable set-valued map F with closed images, it exists a measurable set-valued map of order at most countable F_o dense in F, that is to say such as $\forall t \in O$, $\overline{F_o(t)} = F(t)$.*

Proof.

Lemma. *For any measurable μ-simple dynamic f, if $[b, r(\,]$ is a representation of f, then the function $(t, x) \mapsto r(t|x)$ is $\nu \otimes \mu_b$ Borel-measurable.*

Proof. Let f be a measurable μ-simple dynamic and $[b, r(\,] = [b, i, e_T(\,] \in f$; $e_T(: O \mapsto L(b)$ measurable.

$e_T(= \lim_n \sum_j 1_{O_j^n} l_j^n$ with $l_j^n \in L(b), \forall j, n$ and then: $\forall t, \lim_n \int_b \| \sum_j 1_{O_j^n}(t) l_j^n(x) - e_T(t|x)\| d\mu_b(x) = 0$. It follows that there is an extracted sequence $(\alpha)/ \sum_j 1_{O_j^\alpha}(t) l_j^\alpha(x) \mapsto e_T(t|x)\, \mu_b$ a.e. Independently i is μ_b Borel-measurable. Then $(t, x) \mapsto i(x) + \sum_j 1_{O_j^\alpha}(t) l_j^\alpha(x)$ is $\nu \otimes \mu_b$ Borel-measurable and converges $\nu \otimes \mu_b$ a.e. to $\widetilde{r}(t, x) = r(t|x)$. The result follows. □

1. Let then F be measurable and such as $\omega(F) \le card(\mathbb{N})$

- $\omega(F) = 1$, $F = \underline{f}$: for all open set U in E, $\widetilde{r}^{-1}(U)$ is $\nu \otimes \mu_b$-measurable and then $F^{-1}(U) = \pi_T(\widetilde{r}^{-1}(U))$ is measurable (with π_T canonical projector on T) and F is therefore Borel-measurable.
- $\omega(F) \le card(\mathbb{N})$: let f_J be a measurable covering of F, $\forall j \in J, \underline{f_j}$ is then Borel-measurable and, for all open set U in E, $\underline{f_j}^{-1}(U)$ is measurable and therefore $F^{-1}(U) = \cup_{j \in J} \underline{f_j}^{-1}(U)$ is measurable because $|J| \le |\mathbb{N}|$; F is Borel-measurable.

2. Following the characterization Castaing's theorem, [1], [3], for any Borel-measurable set valued map $F : O \multimap E$, with closed images, there is a countable dense family of Borel-measurable selections f_n of F such as $\forall t \in O$, $F(t) = \overline{\cup_n f_n(t)}$. The set valued map F_0 subjacent to the covering $f_{\mathbb{N}} = (f_n)_{n \in \mathbb{N}}$ is then solution. □

5.2. Integrability of representations

Definition 5.6. A representation $[b, r(]$ is said integrable on M if it is measurable and has an integrable breakdown $[b, i, e_T(]$, i.e. a breakdown such as:

- $\forall x \in b,\ i(x)1_M$ is integrable,
- its total evolution $e_T(: O \mapsto L(b)$ is Lebesgue integrable on M.

Comments. Let $[b, r(]$ be an integrable representation. If $\nu(M) = +\infty$, there is only one integrable breakdown: $i = 0$, $e_T(= r($. If $\nu(M) < \infty$ any breakdown is integrable because if $[b, i', e'(] = [b, i, e(]$, we have: $e'_T = (i - i') + e($, i.e. $i - i' = ct \in L(b)$ and therefore ν integrability of $e($ and $e'($ are equivalent.

We have then the preliminary statement:

Theorem 5.7. *If $[b, i, e_T(]$ is an integrable breakdown of $r($ on M, μ_b a.e. $x \in b$, $t \mapsto e_T(t|x)$ is Lebesgue integrable on M, and*

$$\left[x \mapsto \left(\int_M e_T(t|\, d\nu\right)(x)\right] =_{L(b)} \left[x \mapsto \int_M e_T(t|x) d\nu\right].$$

Therefore the function $x \mapsto \int_M r(t|x)d\nu(t) = i(x)\nu(M) + \int_M e_T(t|x)d\nu(t)$ (in particular its domain b_M) is independent of the integrable breackdown of $r($ which is used.

Proof. Let $[b, r(] = [b, i, e_T(]$ be integrable on M.

Following lemma of Theorem 5.5, $\widetilde{e_T} : (t, x) \mapsto e_T(t|x)$ is $\nu\otimes\mu_b$ Borel-measurable. Therefore by hypothesis $t \mapsto e_T(t| \in L(M, L(b))$: $\int_{M\times b} \|e_T(t|x)\| d(\nu\otimes\mu_b) = \int_M (\int_b \|e_T(t|\,\| d\mu_b) d\nu$ $= \int_M \|e_T(t|\,\|_b d\nu < \infty$. Then $\widetilde{e_T} \in L(M \times b, \nu \otimes \mu_b)$ and for μ_b a.e. $x\, t \mapsto \int_M e_T(t|x)d\nu \in L^1(b, E)$.

Let g_n be an approximation sequence of $\widetilde{e_T}$:

$$\begin{aligned} \|e_T(t| - g_n(t,)\|_{L^1(M,L(b),\nu)} &= \int_M \|e_T(t| - g_n(t,\ )\|_b d\nu \\ &= \int_{M\times b} \|e_T(t|x) - g_n(t, x)\| d(\nu \otimes \mu_b \\ &= \|\widetilde{e_T} - g_n\|_{L^1(M\times b,E,d(\nu\otimes\mu_b))} \to 0, \end{aligned}$$

that is to say: $t \mapsto g_n(t,)$ is an approximation sequence of $t \mapsto e_T(t|$ and the results follow. □

Definition 5.8. Let $[b, r(] = [b, i, e(, s(]$ be an integrable on M representation.

1. We call integral of r (on M the subset of E: $\underline{\int_M} r(t|\, d\nu = \{i(x)\nu(M) + \int_M e_T(t|x)d\nu(t),\ x \in b_M\}$

2. $r($ is said totally integrable on M if furthermore $b_M = b$ and $[b, \int_M e(t|\, d\nu]$ is at least 0-regular. The non singular representation $[b, t \mapsto \int_M r(t|\, d\nu]$ is then said total integral of $r($ on M.

Theorem 5.9. *The integrability and total integrability properties are compatible. Therefore if $r(\sim r'($, we have $\underline{\int_M} r(t|\, d\nu = \underline{\int_M} r'(t|\, d\nu$ and $\int_M r(t|\, d\nu \sim \int_M r'(t|\, d\nu$.*

Proof. Let be $[b, r(\,] = [b, i, e_T(\,] \sim [b', r'(\,] = [b', i', e'_T(\,]$ and let's assume that: $\forall t,\ r'_{b'_1}(t| = \tau * \cdot_{b_1} r(t|$.

If $\nu(M) = +\infty$, then: $i = i' = 0$. If $\nu(M) < +\infty$, we can always consider the breackdown of $r($ initialized at an arbitrary $t_o \in O$. Then $r(t| = i + e_T(t| = r(t_o| + (r(t| - r(t_o|))$ (similarly for $r'(\,)$. We have then $e_T(t_o| = 0, e'_T(t_o| = 0$ and from $r'_{b'_1}(t_o| = \tau * \cdot_{b_1} r(t_o|$, in both cases we deduce $i'_{b'_1} = \tau * \cdot_{b_1} i$ and then $\forall t,\ e'_{T,b'_1}(t| = \tau *_{b_1} e_T(t|$. Therefore:

$$\begin{aligned}
\int_M \|e'_T(t|\,\|_{b'} d\nu &= \int_M \int_{b'} \|e'_T(t|x')\| d\mu_{b'} d\nu \\
&= \int_M \int_{b'\times b'_1} \|e'_{T,b'_1}(t|x', x'_1)\| d\mu_{b'\times b'_1}\, d\nu \\
&= \int_M \int_{b'\times b'_1} \|\tau * \cdot_{b_1} e_T(t|x', x'_1)\| d\mu_{b'\times b'_1}\, d\nu \\
&= \int_M \int_{b_1\times b} \|_{b_1} e_T(t|x_1, x)\| d\mu_{b_1\times b}\, d\nu \\
&= \int_M \|e_T(t|\,\|_b d\nu < \infty.
\end{aligned}$$

Then by continuous linearity: $\left(\int_M e'_T(t|\,d\nu\right)_{b'_1} = \int_M e'_{T,b'_1}(t|\,d\nu = \int_M \tau * \cdot_{b_1} e_T(t|\,d\nu = \int_M {}_{b_1}e_T(t|\,d\nu = \tau * \cdot_{b_1} \left(\int_M e_T(t|\,d\nu\right)$.

The results follow by symmetry and macrogerm image. □

5.3. Integrability of coverings and set-valued maps

Definition 5.10. The definitions of integrability and total integrability follow the general rules. Therefore:

1. For integrable dynamics:

$$\text{if } r(\in f,\ \underline{\int_M} f\, d\nu = \underline{\int_M} r(t|\, d\nu\ \subset E$$

and for total integrability:

$$\int_M f\, d\nu = \int_M r(t|\, d\nu \quad \text{and} \quad \underline{\int_M f\, d\nu} = \underline{\int_M} f\, d\nu.$$

2. For integrable coverings:

$$\underline{\int_M} f_J\, d\nu = \bigcup_{j\in J} \underline{\int_M} f_j\, d\nu$$

and, for total integrability:

$$\int_M f_J d\nu = \left(\int_M f_j\, d\nu\right)_{j\in J} \quad \text{then } \underline{\int_M f_J\, d\nu} = \underline{\int_M} f_J\, d\nu.$$

3. For integrable set-valued maps:

$$\int_M F\, d\nu = \bigcup_{f_J \in \mathcal{C}(F)} \underline{\int_M} f_J\, d\nu = \bigcup_{f\in \mathcal{I}(F)} \underline{\int_M} f\, d\nu.$$

Comments. The definition relative to $\mathcal{C}(F)$ is, as for differentiability, only necessary if the structure property $\omega = 1$ is included in the reference properties set $\mathcal{H}$.

5.4. Elementary properties

Theorem 5.11 (Case of functions). *Let $f : O \mapsto E$ be a function, it is equivalent to say:*

- *The function $f : O \mapsto E$ is Lebesgue integrable on M.*
- *The dynamic $f : O \sim> E$ is integrable on M.*
- *The set-valued map $\{f\} : O \multimap E$ is integrable on M.*

Therefore: $\int_M f\,d\nu = \underline{\int_M f\,d\nu} = \int_M \{f\}\,d\nu$.

Proof. Immediate. ☐

Theorem 5.12.

1. *The spaces of integrable (resp. totally integrable) on M dynamics, coverings, set-valued maps, are stable by linear combination and union. Therefore:*
   - *The operators $f_J \mapsto \underline{\int_M} f_J\,d\nu$ and $f_J \mapsto \int_M f_J\,d\nu$ are linear and boolean.*
   - *The operator $F \mapsto \int_M F\,d\nu$ is sublinear, subboolean, and increasing.*
2. *The integrability (resp. total integrability) is stable by relaxation and:*
   - $\underline{\int_M} co(f_J)\,d\nu = co(\underline{\int_M} f_J\,d\nu)$ *and* $\int_M co(f_J)\,d\nu = co(\int_M f_J\,d\nu)$.
   - $co(\int_M F\,d\nu) \subset \int_M co(F)\,d\nu$. *Therefore if F has convex images, $\int_M F\,d\nu$ is convex.*

Proof. Immediate consequences of the definitions and algebra results. ☐

5.5. Integrability and Aumann integrability

Theorem 5.13. *For any set-valued map $F : O \multimap E$ which is both measurable and Borel-measurable it is equivalent to say that F is Aumann integrable on M and that F is integrable on M. Therefore:*

$$^{\mathcal{A}}\!\!\int_M F\,d\nu = \int_M F\,d\nu.$$

Proof. Any Lebesgue integrable selection of F defines an internal dynamic of F. Reciprocally for any internal dynamic $\widehat{[b, r(\,]}$of F and any $x \in b_o$ the function $t \mapsto r(t|x)$ defines a Lebesgue integral selection of F. The result follows then immediately from the definitions of the integral and Aumann integral. ☐

Definition 5.14. We call "Aumann kernel" of a set-valued map F the set-valued map subjacent to the stream, possibly empty, of its locally Lebesgue integrable selections. We denotes it $\mathcal{A}(F)$.

Theorem 5.15.

1. - *For any set-valued map $F : O \multimap E$, $\mathcal{A}(F) : O \multimap E$, if it is non empty, it is the maximal locally integrable sub set-valued map of F. Therefore $\mathcal{A}(F)$ is always simple and non singular.*

- *The operator $F \mapsto \mathcal{A}(F)$ is sublinear, subboolean, and increasing.*
- *If F, with closed images, is locally integrably bounded* [1] *$\mathcal{A}(F)$ is dense in F.*

2. *For any locally Aumann integrable set-valued map $F : O \multimap E$ and any compact $K \subset O$:*

$$^{\mathcal{A}}\int_K F\, d\nu = \int_K \mathcal{A}(F)\, d\nu.$$

Proof. The results follow immediately from the definitions. The last point of *1.* follows immediately from Castaing's theorem. □

5.6. Local integrability rules

Theorem 5.16.

1. *Any continuous covering (resp. set-valued map F) is locally integrable.*

2. *Any measurable covering f_J of integrably bounded evolution, in the sense that: $\forall K \subset O$ compact, $\forall j \in J$, $\exists [b_j, i_j, e_{j,T}(\,] \in f_j$, $\exists g_j \in L^1_{loc}(O, E)\,/$, $\sup_{y \in e_{j,T}(t|b_j)} \|y\| \leq g_j(t), \forall t \in K$ is locally totally integrable.*

3. *Any measurable locally integrably bounded covering f_J (resp. measurable C-locally integrably bounded set-valued map F) in the sense that: $\forall K \subset O$ compact, $\forall j \in J$, $\exists g_j \in L^1_{loc}(O, E)\,/\, \sup_{y \in \underline{f_j(t)}} \|y\| \leq g_j(t), \forall t \in K$ is locally totally integrable.*

Proof. *1.* Let be f a continuous dynamic, K a compact subset of O. Any representation $[b, i, e_T(\,]$ of f is then integrable on K because $\nu(K) < \infty$ and $t \mapsto e_T(t|$ is continuous in $L^1(b)$.

2. We have then for a suitable $[b_j, i_j, e_{j,T}(\,] \in f_j$: $\|e_{j,T}(t|x)\| \leq g_j(t)$, $\forall x \in b_j$. The statement follows from Lebesgue's integrably bounded theorem.

3. Let be an arbitrary t_0 and a suitable $[b_j, r_j(\,] \in f_j$. Because $e_j(t| = e_{j,T}($ *μ a.e.,* $x \mapsto e_j(t_0|x)$ is continuous and K is compact, we have: $\|e_{j,T}(t|x)\| \leq \|r_j(t|x) - r_j(t_o|x)\| + \|e_j(t_0|x)\| \leq g_j(t) + g_j(t_o) + M$, $\forall t \in K$, μ_b *a.e.* $x \in b$.

The result follows then from *2.* □

5.7. Primitives

Theorem 5.17. *Let be $O =]a, b[\subset \mathbb{R}$, ν the Lebesgue measure on $\mathbb{R}$.*

1. *If $f_J = \widehat{[b_j, e_j(, s_j(\,]}_{j \in J} :]a, b[\sim> E$ is a simple locally totally integrable covering, for any initial condition to $\in O$, $t \sim> \int_{t_o}^t f_J\, d\tau$ is a simple covering which is differentiable at any t where f_J is continuous. Therefore:*

$$\int_{t_o}^t f_J\, d\tau = \left[b_j, \widehat{\int_{t_o}^t e_j(\tau|\, d\tau, \int_{t_o}^t} s_j(\tau|\, d\tau\right]_{j \in J} \quad \text{and} \quad d\int_{t_o}^t f_J\, d\tau = f_J(t).$$

2. *If $F :]a, b[\multimap E$ is a simple locally totally integrable set-valued map, for any initial condition to $\in O$, $t \multimap \int_{t_o}^t F\, d\tau$ is a simple set-valued map which is differentiable* *at any t where F is continuous. Therefore:*

$$F(t) \subset d \int_{t_o}^{t} F \, d\tau.$$

Proof. *1.* $\forall \zeta$, $s_j(\zeta|x) = 0$ μ_j *a.e.* $x \in b_j$, then: $\int_{b_j} \int_{t_o}^{t} \|s_j(\zeta|x)\| \; d\zeta \; d\mu_j(x) = \int_{t_o}^{t} \int_{b_j} \|s_j(\zeta|x)\| \, d\mu_j(x) \, d\zeta = 0$ and then: $\int_{t_o}^{t} s(\zeta|x) \, d\zeta = 0 \, \mu_j$ *a.e.* $x \in b_j$.

Independently, $x \in b_j \mapsto \int_{t_o}^{t} e_j(\tau|x) \, d\tau$ is following the hypothesies continuous and integrable on b_j. The first result follows. Therefore:

$$\begin{aligned} \int_{t_o}^{t+h} e_j(\tau|x) \, d\tau - \int_{t_o}^{t} e_j(\tau|x) \, d\tau &= \int_{t}^{t+h} e_j(\tau|x) \, d\tau \\ = \; \int_{t}^{t+h} e_j(t|x) \, d\tau &+ \int_{t}^{t+h} (e_j(\tau|x) - e_j(t|x)) \, d\tau \\ = \; e_j(t|x) \, h + |h| &\left(\frac{1}{|h|} \int_{t}^{t+h} (e_j(\tau|x) - e_j(t|x)) \, d\tau \right) \end{aligned}$$

and we have, if $\tau \mapsto e_j(\tau|$ is continuous, for arbitrary $\varepsilon > 0$, $|h| < \delta$, δ suitable:

$$\begin{aligned} \left\| \frac{1}{|h|} \int_{t}^{t+h} (e_j(\tau| - e_j(t| \, d\tau_{b_j} \right\|_{b_j} &\leq \frac{1}{|h|} \int_{\alpha=\min(t,t+h)}^{\beta=\max(t,t+h)} \|e_j(\tau| - e_j(t| \, \|_{b_j} \, d\tau \\ &\leq \frac{\varepsilon}{|h|} \int_{\alpha}^{\beta} d\tau = \varepsilon. \end{aligned}$$

2. Because following general rules $\mathcal{C}(F)$ and $\mathcal{I}(F)$ are only made respectively of coverings and internal dynamics which verify the same hypotheses than F, for any covering, (resp. internal dynamic) of F its primitive is a differentiable covering (resp. internal dynamic) of $t \multimap \int_{t_o}^{t} F \, d\tau$. The result follows. □

5.8. Regularization by convolution

$T = \mathbb{R}^n$, $d\nu = dt$, $\mathcal{D}(O)$ the set of C^∞ functions $\theta : O \mapsto \mathbb{R}$ with compact support.

Theorem 5.18.

1. *Let $f : O \sim> E$ be a locally totally integrable dynamic.*
The dynamic $t \in O \sim> f * \theta(t) = \int f(t-\tau)\theta(\tau) \, d\tau = \int f(\tau)\theta(t-\tau) \, d\tau$, said "convolution of f by θ", is C^∞.
If $f = \widehat{[b, r(]}$ with $[b, r(] = [b, i, e(, s(]$ has for representation $[b, r * \theta(,] = [b, (\int \theta \, d\tau) i, e * \theta(, s * \theta(]$ with $r * \theta(t| = \int r(t-\tau| \, \theta(\tau) \, d\tau = \int r(\tau| \, \theta(t-\tau) \, d\tau$ ($e * \theta($ and $s * \theta($ similarly defined) [7].
Therefore for any $m = (m_1, .., m_n) \in \mathbb{N}^n$:

$$\partial^m (f * \theta) = f * \partial^m \theta \;\; \textit{and if } f \textit{ is } C^k, \, |m| \leq k, \; (\partial^m f) * \theta = f * (\partial^m \theta).$$

2. *Let $F : O \multimap E$ be a locally totally integrable set-valued map of covering f_J.*

The set-valued map $F * \theta : t \in O \multimap \int \theta(\tau)\,(F(t-\tau)d\tau) \overset{def}{=} \underset{f\in\mathcal{I}(F)}{\cup} \underline{f * \theta}(t)$ *is* C^∞, *of covering* $f_J * \theta$.
Therefore for any $m = (m_1, .., m_n) \in \mathbb{N}^n$*:*

$$F * \partial^m \theta \subset \partial^m(F * \theta) \quad \text{and if } F \text{ is } C^k,\ |m| \leq k,\ F * (\partial^m \theta) \subset (\partial^m F) * \theta).$$

Proof. *1.* $\int r(t-\tau|x)\theta(\tau)\,d\tau = i(x)\int \theta(\tau)d\tau + \int e(t-\tau|x)\theta(\tau)\,d\tau + \int s(t-\tau|x)\theta(\tau)\,d\tau$.
Therefore: $\int_b \| \int e(t-\tau|x)\theta(\tau)\,d\tau\|\,d\mu_b = \int_b \|\int e(\tau|x)\theta(t-\tau)\,d\tau\|\,d\mu_b \leq \sup|\theta| \int_{supp(\theta)} \|e_T(\tau|,\|_b\,d\tau < \infty$ and we have: $\|\int e(\tau|x)\theta(t-\tau)d\tau - \int e(\tau|x_o)\theta(t-\tau)d\tau\| \leq \sup|\theta| \int_{supp(\theta)} \|e(\tau|x) - e(\tau|x_o)\|\,d\tau$, the continuity follows: $[t \mapsto \int e(t-\tau|x)\theta(\tau)\,d\tau] \in \mathcal{E}(b)$.

$\int_b \int \|s(\tau|x)\|\theta(t-\tau)\,d\tau\,d\mu_b = \int (\int_b \|s(\tau|x)\|\,d\mu_b)\theta(t-\tau)d\tau = 0$ because $s(\tau|x) = 0\,\mu_b$ *a.e.* x, then $\forall t\ \int s(\tau|x)\theta(t-\tau)\,d\tau = 0\,\mu_b$ *a.e.* x and $r * \theta$ is a representation of $f * \theta$.

Let's put $\frac{\partial^k}{\partial t_1^k} = \partial^k$ and $h = (h_1, 0..)$. We have: $\theta(t+h-\tau) = \theta(t-\tau)\partial^1\theta(t-\tau).h + \frac{h_1^2}{2}\int_0^1 (1-u)\partial^2\theta(t-\tau+uh)du$ and then:

$$\begin{aligned} &\int e_T(\tau|x)\theta(t+h-\tau)\,d\tau \\ = &\int e_T(\tau|x)\theta(\tau)\,d\tau + h_1 \int e(\tau|x)\partial\theta(t-\tau)\,d\tau + h_1 \int s(\tau|x)\partial\theta(t-\tau)\,d\tau \\ &\quad + \frac{h_1^2}{2}\int e_T(\tau|x)\int_0^1 (1-u)\partial^2\theta(t-\tau+uh)\,du\,d\tau, \end{aligned}$$

or again: $r * \theta(t+h) = r * \theta(t) + h_1 \int e(\tau|x)\partial\theta(t-\tau)\,d\tau + |h_1|\varepsilon(h|x)$ with: $\varepsilon(h|x) = \frac{|h_1|}{h_1}\int s(\tau|x)\partial\theta(t-\tau)\,d\tau + \frac{h_1}{2}\int e_T(\tau|x)\int_0^1 (1-u)\partial^2\theta(t-\tau+uh)\,du\,d\tau$.

Similarly to $x \mapsto e * \theta(t|x)$ and $x \mapsto s * \theta(t|x)$, we have $[x \mapsto \int e(\tau|x)\partial\theta(t-\tau)\,d\tau] \in \mathcal{E}(b)$ and $\int s(\tau|x)\partial\theta(t-\tau)\,d\tau = 0\,\mu_b$ *a.e.* x. Then:

$$\begin{aligned} \|\varepsilon(h|x)\|_b &\leq \|\int_b (\frac{h_1}{2}\int e_T(\tau|x)\int_0^1 (1-u)\partial^2\theta(t-\tau+uh)\,du\,d\tau)\,d\mu_b\| \\ &\leq \frac{|h_1|}{2}\sup|\partial^2\theta| \underbrace{\int_{supp(\theta)} \|e_T(\tau|\,\|_b d\tau}_{<\infty} \to 0 \quad \text{if } \mathrm{h}_1 \to 0. \end{aligned}$$

The differentiability follows and the last statements follow of general rules on representations and coverings.

2. Immediate application of *1.* to set-valued maps. ☐

Theorem 5.19. *Let* $(\theta_k)_{k\in\mathbb{N}}$ *be a "regularization sequence":* $(\theta_k)_{k\in\mathbb{N}} \subset \mathcal{D}(O)$ *such as:* $\theta_k \geq 0$, $\int \theta_k d\tau = 1$, $\forall k$, $supp(\theta_k) \subset B(0, \varepsilon_k)$ *with* $\varepsilon_k \to 0$.

1. *Let* $f_J : O \sim> E$ *be a simple non singular locally totally integrable covering. Let's assume that* f_J *is continuous at a fixed* t *then:*

$$\underline{f_J}(t) \subset \limsup_k \underline{f_J * \theta_k}(t) \overset{def}{=} \underset{j\in J}{\cup} \limsup_k \underline{f_j * \theta_k}(t).$$

Furthermore if f_J is locally uniformly b-Lipschitz, i.e.: $\exists\widehat{[b_j, r_j(\,]}_{j\in J} = f_J$: $\forall j, \exists U_j$ neighbourhood of t, $\exists \rho_j > 0 : \|r_j(\tau|x_1) - r_j(\tau|x_2)\| \leq \rho_j \|x_1 - x_2\|$, $\forall x_1, x_2 \in b_j\; \tau \in U_j$, then:

$$\underline{f_J}(t) = \limsup_k \underline{f_J * \theta_k}(t).$$

2. *Let $F : O \multimap E$ be a simple non singular locally totally integrable set-valued map. Let's assume that F is continuous at a fixed t then:*

$$F(t) \subset \limsup_k{}^{cov} F * \theta_k(t) \overset{def}{=} \bigcup_{f\in\mathcal{I}(F)} \limsup_k \underline{f * \theta_k}(t).$$

Furthermore if F is locally uniformly b-Lipschitz then:

$$F(t) = \limsup_k{}^{cov} F * \theta_k(t) = \limsup_k \underline{f_J * \theta_k}(t) \quad \textit{for an arbitrary } f_J \in \mathcal{C}(F).$$

Proof. *1.* It is only necessary to prove the statement for $f_J = f = \widehat{[b, r(\,]}$. We have, as f is continuous at t: $\forall \varepsilon > 0\, \exists \gamma > 0 \,/\, \|\tau\| \leq \gamma \Rightarrow \|e_T(t-\tau| - e_T(\tau|\,\|_b \leq \varepsilon$,

$$\begin{aligned}
\|r * \theta_k(t| - r(t|\,\|_b &\leq \int \|r(t-\tau| - r(t|\,\|_b \theta_k(\tau)\, d\tau \\
&\leq \int \|e_T(t-\tau| - e_T(t|\,\|_b \theta_k(\tau)\, d\tau \\
&\leq \int_{B(0,\gamma)} \|e_T(t-\tau| - e_T(t|\,\|_b \theta_k(\tau)\, d\tau, \\
&\leq \varepsilon \int \theta d\tau = \varepsilon,\ \forall k \geq k(\gamma).
\end{aligned}$$

Hence, for any t, $r * \theta_k(t| \to_{L(b)} r(t|$ and then: $\exists b_t \subset b \,/\, \mu_b(b - b_t) = 0$ and $r * \theta_k(t|x) \to r(t|x)$, $\forall x \in b_t$. It follows that: $r(t|b_t) \subset \limsup_k r * \theta_k(t|b_t) = \limsup_k r * \theta_k(t|b)$.

Because as $r($ is simple non singular so is $r * \theta_k($. Then $r(t|b_t)$ and $r * \theta_k(t|b_t)$ are respectively dense in $r(t|b) = \underline{f}(t)$ and $r * \theta_k(t|b) = \underline{f * \theta_k}(t)$. The result follows.

If f verify the Lipschitz condition, by strong compatibility of the continuity, we can assume the chosen continuous representation $r($ has the Lipschitz property. Let be then $x \in b$, as b_t is dense in b: $x = \lim x_\beta$, $x_\beta \in b_t$ and $\|r*\theta_k(t|x) - r(t|x)\| \leq \|r*\theta_k(t|x) - r*\theta_k(t|x_\beta)\| + \|r*\theta_k(t|x_\beta) - r(t|x_\beta)\| + \|r(t|x_\beta) - r(t|x)\|$, but $\|r*\theta_k(t|x) - r*\theta_k(t|x_\beta)\| \leq \int \|r(t-\tau|x) - r(t-\tau|x_\beta)\|\theta_k(\tau)\, d\tau \leq \rho\|x - x_\beta\|$ for any $k \geq k_U$, $t + supp(\theta_k) \subset U$ and $\|r(t|x) - r(t|x_\beta)\| \leq \rho\|x - x_\beta\|$. It follows immediately: $\lim_\beta (r * \theta_\beta(t|x) - r(t|x)) = 0$ and $b_t = b$.

Let be $y \in \limsup_k \underline{r * \theta_k}(t) :\ y = \lim_\nu r * \theta_\nu(t|x_\nu)$, $x_\nu \in b$. As b is compact, there is an extracted x_α, $x_\alpha \to x \in b$. Then:

$$\begin{aligned}
&\|y - r(t|x)\| \\
\leq\ &\|y - r*\theta_\alpha(t|x_\alpha)\| + \|r*\theta_\alpha(t|x_\alpha) - r*\theta_\alpha(t|x)\| + \|r*\theta_\alpha(t|x) - r(t|x)\| \\
\leq\ &\|y - r*\theta_\alpha(t|x_\alpha)\| + \underbrace{\rho\|x_\alpha - x\|}_{\alpha\geq\alpha_U} + \|r*\theta_\alpha(t|x) - r(t|x)\| \underset{\alpha}{\to} 0.
\end{aligned}$$

Then $y = r(t|x)$ and $\limsup_k \underline{r * \theta_k}(t) \subset \underline{f}(t)$. Hence, the equality follows from *1.*.

2. Immediate application of *1.* to set-valued maps. □

6. Introduction to polydistributions

I end this exposition of my work by a short introduction to polydistributions, extension to set-valued maps of the distributions theory. One of the main difficulties of set-valued analysis is in the bad compatibility between set algebra and functional analysis necessities. In the framework of polydistributions this antagonism totally desappears and leads to an extension which find again the formal simplicity of the initial theory.

I give here just basic definitions of polydistribution, distributive covering and polydistribution associated with a set-valued map. I illustrate each of these concepts with their reciprocal relations and some elementary results on derivatives and Fourier transform.

Conventions.

- O an open set in $(\mathbb{R}^n, dt)$.
- $\mathcal{D}(O)$ the space of C^∞ complex functions on O with compact support in O. $\mathcal{D}'(O)$ its dual space, space of distributions on O.
- $\mathcal{S}(\mathbb{R}^n)$ the space of fast decreasing C^∞ complex functions on $\mathbb{R}^n$. $\mathcal{S}'(\mathbb{R}^n)$ its dual, space of tempered distributions on $\mathcal{S}(\mathbb{R}^n)$, [7].

6.1. General notions on polydistributions

Definition 6.1. We call polydistribution on O any set-valued map $\mathcal{T} : \mathcal{D}(O) \multimap \mathbb{C}$ such as:

$$\exists T_I = \{T_i; i \in I\} \subset \mathcal{D}'(O) \ : < \mathcal{T}, \varphi >= \mathcal{T}(\varphi) = \{< T_i, \varphi >; i \in I\}.$$

We denotes $\mathcal{PD}'(O)$ their set.

We have then the following immediate extrapolations and results:

6.1.1. Algebraic and boolean operations

$$\underbrace{\mathcal{T} + \mathcal{S}}_{set-valued\ maps\ algebra} = \underbrace{T_I + S_J}_{sets\ algebra}$$

and similarly:

$$k\mathcal{T} = kT_I; \qquad \mathcal{T} \cup \mathcal{S} = T_I \cup S_J; \qquad \mathcal{T} \cap \mathcal{S} = T_I \cap S_J \text{ (if it is defined)}.$$

Hence, we identify polydistribution on O and non empty subsets of $\mathcal{D}'(O)$.

6.1.2. Derivative

We define the *"derivative of order* $m = (m_1, ..., m_n)$*"* of a polydistribution $\mathcal{T}$, as in the classical case, by:

$$< D^{(m)}\mathcal{T}, \varphi >= (-1)^{|m|} < \mathcal{T}, \varphi^{(m)} >$$

that is equivalent to:

$$D^{(m)}T_I = \{D^{(m)}T_i; i \in I\}.$$

We have then immediately:

$\mathcal{T} \mapsto D^{(m)}\mathcal{T}$ is a linear and boolean operator from $\mathcal{PD}'(O)$ to itself (see Theorem 5.12)

6.1.3. Fourier analysis

A polydistribution $\mathcal{T}$ on $\mathbb{R}^n$ is said to be *tempered* if it can be identified with a non empty subset of $\mathcal{S}'(\mathbb{R}^n)$ Their set, denoted $\mathcal{PS}'(\mathbb{R}^n)$, is stable by algebraic and boolean operations.

We define the *Fourier transform* $\widehat{\mathcal{T}}$ of $\mathcal{T}$, as in the classical case, by:

$$< \widehat{\mathcal{T}}, \varphi >=< \mathcal{T}, \widehat{\varphi} >$$

that is equivalent to:

$$\widehat{T_I} = \{\widehat{T_i}, i \in I\}.$$

The *co-Fourier transform* $\overline{\mathcal{T}}$ is defined similarly.

We have immediately: The Fourier transform and co-Fourier transform are linear and boolean inverse operators of $\mathcal{PS}'(\mathbb{R}^n)$ on itself:

$$\widehat{\overline{\mathcal{T}}} = \overline{\widehat{\mathcal{T}}} = \mathcal{T}.$$

6.2. Distributive coverings

Definition 6.2. A covering $\tau_I : \mathcal{D}(O) \sim> \mathbb{C}$ is said to be "distributive" if it has a representations system $[b_i, r_i(\ \ ]_{i\in I}$ such as:

$$\forall i \in I, \forall x \in b_i,\ [r_{i,x}\ : \varphi \mapsto r_i(\varphi|x)] \in \mathcal{D}'(O).$$

We extend the classical notation $< \tau_I, \varphi >$ for $\tau_I(\varphi)$.

A distributive covering is always simple and is said tempered ($O = \mathbb{R}^n$) if, for any $i \in I$ and any $x \in b_i$, $r_{i,x}$ belongs to $\mathcal{S}'(\mathbb{R}^n)$ The spaces obtained are stable by linear and boolean operations.

Theorem 6.3. *A set-valued map $\mathcal{T} : \mathcal{D}(O) \multimap \mathbb{C}$ is a (tempered) polydistribution if and only if it is sujacent to a (tempered) distributive covering.*

Proof. Let $\mathcal{T} : \mathcal{D} \multimap \mathbb{C}$ be a polydistribution. On the first hand, if $\mathcal{T} = \{T_i;\ i \in I\}$, τ_I, defined by $\tau_i = T_i$, $\forall i \in I$, is a distributive covering of $\mathcal{T}$. On the other hand, if τ_I is a distributive covering of $\mathcal{T}$, we have : $\mathcal{T} = \{T_{i,x} = r_{i,x};\ i \in I, x \in b_i\} \subset \mathcal{D}'(O)$. The result follows. □

6.2.1. Derivative

We define the *derivative of order* $m = (m_1, ..., m_n)$ of a distributive covering τ_I by:

$$< D^{(m)}\tau_I, \varphi >= (-1)^{|m|} < \tau_I, \varphi^{(m)} > .$$

We have immediately: $\tau_I \mapsto D^{(m)}\tau_I$ is a linear and boolean operator of the distributive coverings space in itself.

6.2.2. Fourier analysis

We defined the *Fourier transform* of a tempered distributive covering by:

$$< \widehat{\tau}_I, \varphi >=< \tau_I, \widehat{\varphi} > .$$

that is equivalent to: if $[b_i, r_i(\ ]_{i\in I}$ is a representations system of τ_I, $[b_i, \widehat{r_i}(\ ]_{i\in I}$ is a representations system of $\widehat{\tau_I}$ with:

$$\widehat{r_i}(\varphi| = r_i(\widehat{\varphi}|.$$

The *co-Fourier transform* $\tau_I \mapsto \overline{\tau_I}$ is defined similarly.

Then, immediately: The Fourier transform an co-Fourier transform are linear and boolean inverse operators of the tempered distributive coverings space on itself.

6.2.3. Fundamental relation

Theorem 6.4. *Let be $f_I : O \sim> \mathbb{C}$ a simple locally totally integrable covering and $[b_i, e_i(, s_i(\ ]_{i\in I}$ a suitable representations system of f_I. For any $\varphi \in \mathcal{D}(O)$, we define:*

$$< \tau_{f_I}, \varphi >= \int f_I \varphi dt.$$

Then:

1. τ_{f_I} *is a distributive covering on O of representations system*

$$[b_i, \varphi \mapsto \int \varphi e_i(t|\, dt, \varphi \mapsto \int \varphi s_i(t|\, dt]_{i\in I}.$$

$f_I \mapsto \tau_{f_I}$ *is a linear and boolean operator such as, for coverings of at most countable order:*

$$\tau_{f_I} = \tau_{g_J} \Leftrightarrow f_I = g_J \ a.e.\ t \in O.$$

Therefore if f_I is C^k on O: $\forall m \in \mathbb{N}^n$, $/\, |m| \leq k$, $D^{(m)}\tau_{f_I} = \tau_{\partial^m f_I}$.

2. *If we define the Fourier transform of a $\mathbb{R}^n$-totally integrable f_I as the locally totally integrable covering:*

$$t \sim> \widehat{f_I}(t) = \int_{\mathbb{R}^n} e^{2i\pi t.\zeta} f_I(\zeta) d\zeta,$$

then $\tau_{\widehat{f_I}}$ is a tempered distributive covering and $\tau_{\widehat{f_I}} = \widehat{(\tau_{f_I})}$.
(similarly for the co-transform)

Proof. *1.* The proof of the first point is formally the same as the proof given in the regularization by convolution Theorem 5.18.

The calculus rules are obvious and the bijectivity property follows immediately from:

- $f_I = g_J$ naturally mean $\exists i \in I \mapsto j(i) \in J$ bijective such as: $\forall i \in I, g_{j(i)} = f_i$.
- For two dynamics f and g , of representations $[b, e_{f,T}(\ ]$ and $[b, e_{g,T}(\ ]$, after mutual extension of their initial bases, the equality means equality in the space of vector distributions $\mathcal{D}(O, L(b))$. It follows the equality, for almost every t, of functions $e_{f,T}$ and $e_{g,T}$ and hence equality of dynamics f and g.

For the last point, let be $n = 1$, $k = 1$, $f = \widehat{[b, e_T(\ ]}$: $df_I = \widehat{[b, e_T^{(1)}(\ ]}$. $t \mapsto e_T($ is a C^1 function from O in $L(b)$ with derivative $e_T^{(1)}($. The vector distribution with values in $L(b)$ defined in this way is then such as $\frac{d}{dt} e_T(= e_T^{(1)}($. The statement follows immediately.

2. The immediate checks and identification are formally identical to these used in *1b*. $\square$

6.3. Polydistribution associated with a set-valued map

Definition 6.5. We define the polydistribution associated with a locally integrable set-valued map $F : O \multimap \mathbb{C}$ by:

$$\mathcal{T}_F : \varphi \in \mathcal{D}(O) \multimap < \mathcal{T}_F, \varphi >$$
$$= \int \varphi(t)\,(F(t)dt) \overset{def}{=} \{< f, \varphi >;\ f \text{ locally integrable selection of } F\}.$$

Theorem 6.6. *Let be $F, G : O \multimap \mathbb{C}$ and $f_I : O \sim> \mathbb{C}$ locally integrable:*

1. $\forall \varphi \in \mathcal{D}(O)$, $\int \varphi\, Fdt) \subset \int \varphi\, Fdt \subset \overline{\int \varphi\,(Fdt)}$ *and in particular, if F is integrably bounded and has closed images:* $\int \varphi F\, dt = \int \varphi\,(Fdt)$.
2. $\mathcal{T}_F = \mathcal{T}_G \Leftrightarrow \mathcal{A}(F) = \mathcal{A}(G)$,
3. $F \mapsto \mathcal{T}_F$ *is a sublinear and subboolean operator.*
4. $\underline{\tau_{f_I}} \subset \mathcal{T}_{\underline{f_I}}$.
5. *Therefore if $F : \mathbb{R}^n \multimap \mathbb{C}$ is integrable and if we defines the "Fourier transform" of F as the locally integrable set-valued map:*

$$t \multimap \widehat{F}(t) = \int_{\mathcal{R}^n} e^{2i\pi\, t.\zeta} F(\zeta)\, d\zeta,$$

then: $\mathcal{T}_{\widehat{F}} = \widehat{\mathcal{T}_F}$.

Proof. *1.* Only the second inclusion is not obvious. Let be $z \in \int \varphi F dt$ and h an integrable selection of φF such as $\int hdt = z$. We have: $supp(h) \subset supp(\varphi) = K$, K compact such as $\overline{(K^o)} = K$.

Let $K_m \subset K^o$ be an increasing sequence of compacts such as $\cup_m K_m = K^o$. Then $\forall m, i_m = inf_{K_m} |\varphi| > 0$ and $h_m = \frac{h_{|m}}{\varphi}$ is then defined and measurable on K_m. Therefore: $\int_{K_m} |h_m|\, dt \leq \frac{1}{i_m} \int_{K_m} |h_{|K_m}|\, dt \leq \frac{1}{i_m} \int |h|\, dt < \infty$, h_m is integrable on K_m.

Let be then f an locally integrable selection of F and $f_m = h_m 1_{K_m} + f\,(1 - 1_{K_m})$. f_m is a locally integrable selection of F and $\int \varphi f_m\, dt \in \int \varphi(Fdt)$. Therefore: $\int_K |\varphi f_m - h|\, dt \leq$
$$sup|\varphi| \underbrace{\int_{\overline{K-K_m}} |f|\, dt}_{\to 0} + \underbrace{\int_{K_m} |\varphi f_{K_m} - h|\, dt}_{=0} + \underbrace{\int_{\overline{K-K_m}} |h|\, dt}_{\to 0}.$$

Hence: $\int \varphi f_m\, dt \to \int h\, dt$ and then $z \in \overline{\int \varphi(Fdt)}$.

The special case is an immediate consequence of the theorem of integral compacity, [1], [3].

2. and *3.* are immediate.

4. Let f be a locally totally integrable dynamic.

$\underline{\tau_f}(\varphi) = \underline{< \tau_f, \varphi >} = \{\int r(t|x)\varphi(t)\, dt;\ x \in b\}$, where $[b, r(\,] \in f$, and $\forall x \in b$, $[t \mapsto r(t|x)] \in \mathcal{A}(\underline{f})$. Then $\int r(t|x)\varphi(t)\, dt \in < \underline{f}, \varphi >$. Hence, $\underline{\tau_f}(\varphi) \subset < \underline{f}, \varphi >$ and then $\underline{\tau_f} \subset \mathcal{T}_{\underline{f}}$.

5. We have first obviously: $\widehat{F}(t) = \{\widehat{f};\ f$ integrable selection of $F\}$ and the locally integrability of $\widehat{F}$ follows. On the other hand:

$$\begin{aligned} < \widehat{\mathcal{T}_F}, \varphi > = < \mathcal{T}_F, \widehat{\varphi} > = \int \widehat{\varphi}(F dt) &= \left\{ \int \widehat{\varphi} f \, dt;\ f \text{ integrable selection of } F \right\} \\ &= \left\{ \int \varphi \widehat{f} \, dt;\ f \text{ integrable selection of } F \right\} \\ &= \int \varphi(\widehat{F} dt) = < \mathcal{T}_{\widehat{F}}, \varphi > . \end{aligned}$$

□

Conclusion

I have here only explained some fundamentals of coverings theory I developed. In the natural framework of the set-valued analysis many points would be of real interest to be studied in more depth or developed, such as the relations between differentiability and the different cones of derivatives, the reciprocal investment of Aumann integral and integral in each other, the theory of polydistributions, as well as some other analytical properties as analycity or periodicity. Similarly, it would be interesting to search and develop applications for the differential inclusions and viability theories, but also, possibly, applications to others domains less directly connected with set-valued analysis.

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