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An Optimal Control Theory Approach to the Blaschke-Lebesgue Theorem

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According to the Blaschke-Lebesgue theorem, among all plane convex bodies of given constant width the Reuleaux triangle has the least area. The area of a convex set can be written as an integral involving the support function h and the radius of curvature ρ of the set. The support function satisfies a second order ordinary differential equation where the datum is the radius of curvature. The function ρ is non-negative and bounded above, so that the Blaschke-Lebesgue theorem can be formulated as an optimal control problem, where the functional to be minimized is the area. In the same way, the control theory can be used to find the body of minimum volume among all 3-dimensional bodies of revolution having constant width.

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1. Introduction

Let K be a planar convex body (i.e., a compact convex subset of $\mathbb{R}^2$) having the origin O as an interior point, let H_K denote its support function, defined by

$$H_K(\xi) = \sup\{\langle x, \xi\rangle;\ x \in K\}, \quad \xi \in \mathbb{R}^2, \tag{1}$$

and let

$$h_K(\theta) = H_K(\cos\theta, \sin\theta), \quad \theta \in \mathbb{R}. \tag{2}$$

We say that K is a body of constant width w if $h_K(\theta) + h_K(-\theta) = w$ for every θ. This amounts to require that any two distinct parallel lines tangent to the boundary of K are separated by a distance w. The simplest example of a body of constant width is a circle. A less trivial example is the so-called Reuleaux triangle $\mathcal{R}$ of constant width w, that is, the plane convex set obtained by the intersection of three circles of radius w, centered at the vertices of an equilateral triangle of side w (see Figure 1.1). It is easily seen that its area is

$$A(\mathcal{R}) = \frac{\pi - \sqrt{3}}{2} w^2.$$

Other examples are the Reuleaux polygons with an odd number of sides.

The following celebrated result, due to Blaschke and Lebesgue (see [3, 19]), characterizes the bodies of minimal area among all planar convex bodies of given constant width.

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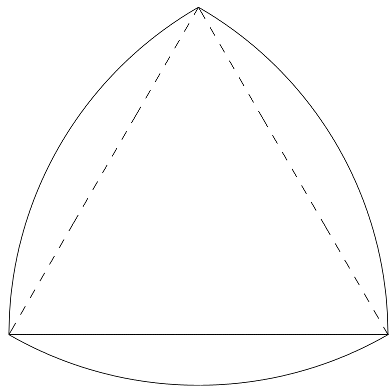

Figure 1.1: The Reuleaux triangle

Theorem 1.1 (Blaschke-Lebesgue). *Among all planar convex bodies of given constant width, the Reuleaux triangle has the least area.*

The major disadvantage of the proof given by Blaschke and Lebesgue, and by many subsequent authors (see [8, 5, 2, 6, 11, 16, 17]), is that it relies on the prior knowledge of the minimizer.

Recently, some analytical proofs of the Blaschke-Lebesgue theorem have been developed. We mention the attempt by Ghandehari [12] and the proof given by Harrell [13].

In this paper we give a proof of Theorem 1.1 following the approach of Ghandehari, which is based on optimal control theory. Unfortunately, in our opinion the proof given in [12] is not entirely correct; actually, some conclusion is not rightly supported by the given arguments:
Considering the cases that one has to analyze to find the maximum of the Hamiltonian function, Ghandehari excludes the cases $\lambda \neq 0$ and $\lambda x_1 + p_2 \neq 0$ together with $\lambda \neq 0$ and $\lambda x_1 + p_2 = 0$, where (λ, p_1, p_2) is the auxiliary vector, as leading to contradictions. While we agree with the second hypothesis (Case 1, p. 397), our analysis leads us to exclude also the case $\lambda = 0$ (the one saved by Ghandehari) and the case $\lambda \neq 0$ and $\lambda x_1 + p_2 \neq 0$ on certain subintervals I_i of $[0, \pi]$, which Ghandehari does not take into consideration at all. We conclude that we have to study the problem just when $\lambda \neq 0$ (that is $\lambda = -1$) and $p_2 + \lambda x_1 \neq 0$, following a quite different path and reasoning also on some geometrical aspect.
Our aim is to give a rigorous proof of the Blaschke-Lebesgue theorem following the optimal control theory approach. More precisely, we rewrite the problem of minimizing the area among the class of planar convex bodies of given constant width as an optimal control problem, where the curvature radius of the convex body appears as a control, and the unknown function h_K satisfies an ordinary differential equation. At this point, we show that the only functions h_K satisfying the necessary condition for optimality given by the Pontryagin Maximum Principle are the support functions of the Reuleaux polygons with an odd number of sides. The conclusion now follows from the fact that, among all these Reuleaux polygons, the Reuleaux triangle attains the least area.
The second part of the paper is devoted to prove the 3-dimensional analogue of the Blaschke-Lebesgue Theorem for rotational bodies, using the Pontryagin Maximum Principle; this proof is our main and new result. Actually, Theorem 4.1 has been approached

only via geometrical techniques and the use of the Optimal Control Theory here is the starting point for its application to 3-dimensional problems in convex geometry.

The plan of the paper is the following.
In the second section, we recall some of the main definitions, properties and formulas in convex geometry that we are going to use in the subsequent pages.
In the third section, we study the smaller class of plane convex bodies of constant width $w = 1$ and we write the problem of minimizing the area A of such bodies using an optimal control theory formulation, where the function ρ, the radius of curvature, is the control parameter.
To gain the solution of the problem, we introduce new variables, depending on h and ρ, then fix the values of the functions h and h' at the points $\theta = 0$ and $\theta = \pi$, and finally we apply the Pontryagin Maximum Principle, looking for the maximum value of the Hamiltonian function of the problem, with respect to the control u.
Due to the exclusion of the bodies having maximum area and of those with empty interior, we can analyse just the case where some of the varying parameters are not completely free.
In order to satisfy the initial conditions for the support function and its regularity, we conclude that the only possible solutions are Reuleaux polygons. Finally, since the area of a Reuleaux polygon is an increasing function with respect to the number of sides, we can conclude that the Reuleaux triangle is the solution to the Blasckhe-Lebesgue theorem.
In the last section, the 3-dimensional case of bodies of revolution having constant width is analyzed as a planar problem, considering the plane figures whose rotations around one of their symmetry axis generate such bodies. The minimum volume is gained when the rotating figure is a Reuleaux triangle, as stated in [5].
The author wishes to express her gratitude to Prof. G. Crasta and to Prof. C. Benassi for many valuable advices.

2. Preliminaries

A planar *convex body* K is a compact, convex subset of $\mathbb{R}^2$ with nonempty interior. It is not restrictive to assume that the origin is an interior point of K.
The *support function* and the *support function in the direction* $\overrightarrow{u_\theta} = (\cos\theta, \sin\theta)$ of K are defined respectively in (1) and (2). The *width* of K in the direction $\overrightarrow{u_\theta}$, given by

$$w_K(\theta) = h_K(\theta) + h_K(\theta + \pi), \tag{3}$$

measures the distance from distinct parallel lines tangent to the boundary ∂K of K, and perpendicular to the direction $\overrightarrow{u_\theta}$.

We say that K is a body of constant width w if $w_K(\theta) = w$ for every θ. It is well known that every convex body of constant width K is strictly convex. As a consequence, for every θ there exists a unique point $\Gamma(\theta) \in \partial K$ with supporting line perpendicular to $\overrightarrow{u_\theta}$, and such that $h_K(\theta) = \langle \overrightarrow{u_\theta}, \Gamma(\theta)\rangle$.

Let $\kappa(\theta)$ be the principal curvature of ∂K at $\Gamma(\theta)$, and let

$$\rho_K(\theta) = 1/\kappa(\theta)$$

denote the *curvature radius* of ∂K at $\Gamma(\theta)$. It is well known that the functions h_K and ρ_K satisfy the ordinary differential equation

$$h''_K(\theta) + h_K(\theta) = \rho_K(\theta) \tag{4}$$

(see, for example, [22], p. 3 or [23], p. 110). In order to use (4) correctly, *we work with functions h_K which are differentiable a.e. two times.* Furthermore, we recall that if K has constant width, then h_K is continuously differentiable (see [23], §1.7).

It is not difficult to see that the area of K is given by

$$A(K) = \frac{1}{2}\int_0^{2\pi} h_K(\theta)\rho_K(\theta)\, d\theta \tag{5}$$

(see [22], §1).

3. The optimal control theory formulation

As a first reduction, in order to prove the Blaschke-Lebesgue theorem, it is not restrictive to consider only the class $\mathbb{K}_1$ of convex bodies of constant width $w = 1$.

Let us consider a convex body $K \in \mathbb{K}_1$. From the relation (3) of width, we have that

$$h_K(\theta) + h_K(\theta + \pi) = 1, \tag{6}$$

hence $h''_K(\theta) + h''_K(\theta + \pi) = 0$. It follows, according to (4), that

$$\rho_K(\theta) + \rho_K(\theta + \pi) = h_K(\theta) + h''_K(\theta) + h_K(\theta + \pi) + h''_K(\theta + \pi) = 1, \tag{7}$$

hence the radius of curvature ρ_K is bounded and satisfies the condition

$$0 \le \rho_K(\theta) \le 1 \quad \forall\theta. \tag{8}$$

From an Optimal Control Theory point of view, we can consider the problem of minimizing the area A, given by (5), among all functions $h = h_K$ satisfying the differential equation (4), where the control $\rho = \rho_K$ takes values in the control set $[0, 1]$ (see (8)). Moreover, h and ρ satisfy the additional symmetry conditions (6) and (7).

In conclusion, the problem can be summarized in the following way:

$$\min \frac{1}{2}\int_0^{2\pi} h(\theta)\rho(\theta)\, d\theta \tag{9}$$

subject to

$$h(\theta) + h''(\theta) = \rho(\theta), \tag{10}$$

$$h(\theta) + h(\theta + \pi) = 1, \tag{11}$$

$$\rho(\theta) + \rho(\theta + \pi) = 1, \tag{12}$$

$$0 \le \rho \le 1. \tag{13}$$

It is worth to remark that, thanks to the symmetry conditions (11) and (12), the mean values of h and ρ in $[0, 2\pi]$ are fixed. Namely

$$\begin{aligned}\int_0^{2\pi} h(\theta)d\theta &= \int_0^{\pi} h(\theta)d\theta + \int_\pi^{2\pi} h(\theta)d\theta \\ &= \int_0^{\pi} h(\theta)d\theta + \int_0^{\pi} h(\theta+\pi)d\theta = \int_0^{\pi} 1 d\theta = \pi,\end{aligned} \tag{13'}$$

which is just the perimeter of the body having support function equal to h. Moreover, using (12) in place of (11) in (13'), one obtains

$$\begin{aligned}\int_0^{2\pi} \rho(\theta)d\theta &= \int_0^{2\pi} (h(\theta) + h''(\theta))d\theta \\ &= \int_0^{2\pi} h(\theta)d\theta + [h']_0^{2\pi} = \int_0^{2\pi} h(\theta)d\theta = \pi.\end{aligned} \tag{14}$$

We can also fix the values of h and h' at the points $\theta = 0$ and $\theta = \pi$, that means that we prevent the figure from translating in any direction. Indeed, if we consider any diameter of K, two support lines to K pass through its ends and are perpendicular to such a chord. Therefore, we can take one end of that diameter as the origin and fix the following values:

$$\begin{aligned} h(0) = 1, &\quad h(\pi) = 0, \\ h'(0) = 0, &\quad h'(\pi) = 0. \end{aligned}$$

Using (6) and (7), the area functional can be written as an integral from 0 to π. Namely,

$$\begin{aligned}\int_0^{2\pi} h(\theta)\rho(\theta)\, d\theta &= \int_0^{\pi} h(\theta)\rho(\theta)\, d\theta + \int_0^{\pi} h(\theta+\pi)\rho(\theta+\pi)\, d\theta \\ &= \int_0^{\pi} (h(\theta)\rho(\theta) + (1 - h(\theta))(1 - \rho(\theta)))\, d\theta \\ &= \int_0^{\pi} (2h(\theta)\rho(\theta) + 1 - \rho(\theta) - h(\theta))\, d\theta.\end{aligned}$$

Let us introduce the new variables

$$\begin{cases} x_1(\theta) = 2h(\theta) - 1, \\ x_2(\theta) = x_1'(\theta) = 2h'(\theta), \\ u(\theta) = 2\rho(\theta) - 1. \end{cases}$$

Since the area is given by $\frac{1}{4}\int_0^{\pi}(1 + x_1(\theta)u(\theta))\, d\theta$, the optimal control problem can be formulated in the following new way:

$$\min \int_0^{\pi} x_1(\theta)u(\theta)\, d\theta, \tag{15}$$

subject to

$$x_1' = x_2, \tag{16}$$

$$x_2' = u - x_1, \tag{17}$$

$$x_1(0) = 1, \quad x_1(\pi) = -1, \tag{18}$$

$$x_2(0) = 0, \quad x_2(\pi) = 0, \tag{19}$$

$$|u| \leq 1. \tag{20}$$

The Blaschke Selection Theorem (see [23], p. 50) guarantees the existence of a minimizer even to such a problem. Using the Pontryagin Maximum Principle for the analysis of the problem (15)–(20), we will show that the solution to the Blaschke-Lebesgue Theorem is to be found among the regular Reuleaux polygons. We recall that a Reuleaux polygon is a convex figure which is obtained drawing arcs of circumference on the sides of a regular polygon, in such a way that each arc connects two consecutive vertices of the polygon and is centered in the vertex opposite to the corresponding side. This implicitly means that we can construct Reuleaux polygons only with an odd number of sides.

Let us write the Pontryagin Maximum Principle for the problem (15)–(20) (see, for example, [20, 21]). The Hamiltonian function of the problem is given by

$$H(x, p, u) = u(\lambda x_1 + p_2) + p_1 x_2 - p_2 x_1,$$

where λ is a non-positive constant. The state variables $x = (x_1, x_2)$ satisfy (16)–(17) with boundary conditions (18)–(19), whereas the adjoint vector $p = (p_1, p_2)$ satisfies the adjoint system of differential equations

$$\begin{cases} p_1' = -\dfrac{\partial H}{\partial x_1} = p_2 - \lambda u\,, \\[2ex] p_2' = -\dfrac{\partial H}{\partial x_2} = -p_1\,. \end{cases} \tag{21}$$

According to the Pontryagin Maximum Principle, if (x^*, u^*), $x^* = (x_1^*, x_2^*)$, is an optimal solution for the problem (15)–(20), then there exist a constant $\lambda \leq 0$ and an absolutely continuous adjoint vector $p^* = (p_1^*, p_2^*)$ such that

$$\max_{|w|\leq 1} H(x^*(\theta), p^*(\theta), w) = H(x^*(\theta), p^*(\theta), u^*(\theta)), \quad \text{a.e. } \theta \in [0, \pi].$$

Due to the linearity of the equations, we can always assume that $\lambda = 0$ or $\lambda = -1$. Since the function H is linear with respect to u, the maximum value in the left hand side is reached when the control u satisfies

$$u = \begin{cases} 1, & \text{if } \lambda x_1 + p_2 > 0, \\ -1, & \text{if } \lambda x_1 + p_2 < 0, \end{cases} \tag{22}$$

whereas u can take any value in $[-1, 1]$ when $\lambda x_1 + p_2 = 0$.

Let us consider all the possible cases.

Case 1: $\lambda = -1$ and $\lambda x_1 + p_2 \equiv 0$ in the whole interval $[0, \pi]$.
In this case, $p_2 = -\lambda x_1$, therefore $p_2' = -\lambda x_1' = -p_1 = -\lambda x_2$, and $p_1' = p_2 - \lambda u = \lambda x_2' = \lambda(u - x_1)$, where we use the differential equations for p_1', p_2' and (16), (17).
Then, $-\lambda x_1 - \lambda u = p_2 + u = -\lambda x_1 + \lambda u$, which implies $u = 0$. This means $\rho(\theta) = 1/2$, so we have found a disc with radius 1/2, which has maximum area among the convex bodies with constant width 1. Clearly, this case has to be excluded.

Case 2: $\lambda = 0$. The maximality condition is satisfied if

$$u = \begin{cases} 1 & \text{if } p_2 > 0, \\ -1 & \text{if } p_2 < 0. \end{cases}$$

The differential equations for p_1 and p_2 become

$$\begin{cases} p_1' = p_2, \\ p_2' = -p_1, \end{cases}$$

that is $p_2'' = -p_2$, from which we get $p_2(\theta) = B\sin(\theta - \gamma)$, for some constants B and γ. Hence we find $x_1(\theta) = A\sin(\theta - \alpha) \pm 1$, so that $x_2(\theta) = A\cos(\theta - \alpha)$, with $x_1(0) = 1$, $x_1(\pi) = -1$, $x_2(0) = 0 = x_2(\pi)$. If we use these conditions on x_1 and x_2, we find

$$\begin{cases} x_1(0) = -A\sin\alpha \pm 1 = 1, \\ x_2(0) = A\cos\alpha = 0. \end{cases}$$

The second equation means $\alpha = \pi/2$, therefore the first is true only if $p_2(0) < 0$ and $A = -2$. Hence, in a right neighborhood of $\theta = 0$, we have $x_1(\theta) = 2\cos\theta - 1$.
The function p_2 satisfies $p_2'' + p_2 = 0$, therefore, if $\gamma > 0$, it is negative in a right neighborhood $[0, \tau_1)$ of 0, it is positive in $(\tau_1, \tau_1 + \pi)$ and again negative in $(\tau_1 + \pi, 2\pi)$. Consequently, we have

$$u = \begin{cases} -1, & \text{in } [0, \tau_1), \\ 1, & \text{in } (\tau_1, \pi + \tau_1), \\ -1, & \text{in } (\pi + \tau_1, 2\pi), \end{cases}$$

and then

$$\rho = \begin{cases} 0, & \text{in } [0, \tau_1), \\ 1, & \text{in } (\tau_1, \pi + \tau_1), \\ 0, & \text{in } (\pi + \tau_1, 2\pi), \end{cases}$$

We conclude that $\rho = 0$ in an interval of length π. Such a case corresponds to a degenerate convex body (i.e. a convex body with empty interior). Then, this case also must be excluded.

Case 3: $\lambda = -1$ and $\lambda x_1 + p_2 \equiv 0$ on certain subintervals I_j of $[0, \pi]$.
Suppose that $p_2 - x_1 \neq 0$ on I_i and $p_2 - x_1 = 0$ on $I_{i+1} = [\tau_1, \tau_2]$ or viceversa. Since the differential equations for p_2 and x_1 can be written as $p_2'' + p_2 = -u$ and $x_1'' + x_1 = u$ respectively, we have that $p_2(\theta) = B_1 \sin\theta + B_2 \cos\theta - u$ and $x_1(\theta) = A_1 \sin\theta + A_2 \cos\theta + u$.
This means that $p_2 = x_1 \;\Rightarrow\; (A_1 - B_1)\sin\theta + (A_2 - B_2)\cos\theta + 2u \equiv 0$ for all $\theta \in I_{i+1}$.

In case $u = constant$, $u = \frac{A_1 - B_1}{2} \sin\theta + \frac{A_2 - B_2}{2} \cos\theta$ in the whole interval I_{i+1} if and only if $u \equiv 0$ and (4) gives us $\rho \equiv \frac{1}{2} \, \forall\, \theta \in I_{i+1}$.
Due to conditions (11) and (12), our extremal figure is delimited by two circular arcs of radius $\frac{1}{2}$ centered in the origin, one facing the other.
But one can draw another body of smaller area and constant width one via suitable cuts on its boundary, as it is shown in the following picture:

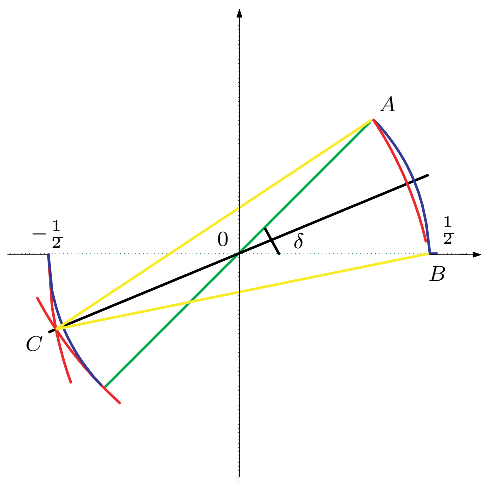

Figure 3.1:

Actually, we consider an arc of radius 1 connecting A and B instead of the arc AB of radius $\frac{1}{2}$, while the facing arc is substituted by two arcs of radius 1 centered in A and B respectively. If δ is the measure of the interval we are working with, the missing area between the arc AB of radius 1 and the arc AB of radius $\frac{1}{2}$, written as

$$A_1 = \frac{\delta}{8} - \arcsin\left(\frac{1}{2}\sin\left(\frac{\delta}{2}\right)\right) + \frac{1}{2}\sin\left(\frac{\delta}{2}\right)\left[\sqrt{\left(1 - \frac{1}{4}\left(\sin\left(\frac{\delta}{2}\right)\right)^2\right)} - \frac{1}{2}\cos\left(\frac{\delta}{2}\right)\right],$$

is bigger than the opposite added area which can be calculated as

$$A_2 = \frac{3\delta}{8} - \arcsin\left(\frac{1}{2}\sin\left(\frac{\delta}{2}\right)\right) - \frac{1}{2}\sin\left(\frac{\delta}{2}\right)\left[\sqrt{\left(1 - \frac{1}{4}\left(\sin\left(\frac{\delta}{2}\right)\right)^2\right)} - \frac{1}{2}\cos\left(\frac{\delta}{2}\right)\right],$$

for all $\delta \in [0, \pi]$.
On the other hand, the segments intercepted by the parallel lines to the bisecting line of the δ-angle, on the figure delimited by the arc AB of radius 1 and the arc AB of radius $\frac{1}{2}$, are longer than the corresponding segments on the figure delimited by the two arcs of radius 1 centered in A and B, except for the ones on the bisecting line itself.

In case $u \neq constant$ in I_{i+1}, we study the behaviour of the function $z = p_2 - x_1$:
Actually,

$$z = \begin{cases} (B_1 - A_1)\sin\theta + (B_2 - A_2)\cos\theta - 2, & \text{if } u = 1, \\ (B_1 - A_1)\sin\theta + (B_2 - A_2)\cos\theta + 2, & \text{if } u = -1, \end{cases}$$

in I_i. So it can take the value 0 in τ_1, having also horizontal tangent line in that point, if it does not change sign in I_i; in any case we need $\mid A \mid = 2$.
The Hamiltonian function reaches its maximum value with

$$u = \begin{cases} 1 & \text{if } z > 0 \\ -1 & \text{if } z < 0, \end{cases}$$

but we get

$$z = \begin{cases} (B_1 - A_1)\sin\theta + (B_2 - A_2)\cos\theta - 2 \leq 0, & \text{if } u = 1, \\ (B_1 - A_1)\sin\theta + (B_2 - A_2)\cos\theta + 2 \geq 0, & \text{if } u = -1, \end{cases}$$

since $\mid A \mid = 2$.
We exclude these solutions.

Case 4: $\lambda = -1$ and $\lambda x_1 + p_2 \not\equiv 0$.
In this case, since $x_1'' = x_2' = u - x_1$, we have $x_1(\theta) = A_u \sin(\theta - \alpha_u) + u$, for some constants A_u and α_u, in each interval where u is constant.

Summarizing what we said until now, we have to analyze the case $\lambda = -1,\ p_2 - x_1 \neq 0$. With these conditions, the problem can be written as

(1) $H = p_1 x_2 + p_2(u - x_1) - x_1 u$;

(2) $\begin{cases} p_1' = p_2 + u \\ p_2' = -p_1, \end{cases}$ therefore $p_2'' = -p_2 - u$ and $p_2(\theta) = B\sin(\theta - \gamma) - u$;

(3) H reaches its maximum value if $u = \begin{cases} 1 & \text{if } p_2 - x_1 > 0 \\ -1 & \text{if } p_2 - x_1 < 0; \end{cases}$

(4) $x_1'' = u - x_1$, therefore $x_1(\theta) = A\sin(\theta - \alpha) + u$ and $x_2(\theta) = A\cos(\theta - \alpha)$.

The mean values for h and u in $[0, 2\pi]$ are fixed; indeed,

$$\begin{aligned} \int_0^{2\pi} h(\theta)d\theta &= \int_0^{\pi} h(\theta)d\theta + \int_{\pi}^{2\pi} h(\theta)d\theta \\ &= \int_0^{\pi} h(\theta)d\theta + \int_0^{\pi} h(\theta + \pi)d\theta = \int_0^{\pi} 1 d\theta = \pi, \end{aligned}$$

which is just the perimeter of the body having support function equal to h.
Moreover, we use (14) and get

$$\begin{aligned} \int_0^{2\pi} u(\theta)d\theta &= \int_0^{2\pi} (2\rho(\theta) - 1)d\theta = \int_0^{2\pi} 2(h(\theta) + h''(\theta))d\theta - 2\pi \\ &= 2\int_0^{2\pi} h(\theta)d\theta + 2[h']_0^{2\pi} - 2\pi = 2\int_0^{2\pi} h(\theta)d\theta - 2\pi = 0. \end{aligned}$$

Given $z = p_2 - x_1$, we get

$$z'' = p_2'' - x_1'' = -p_2 - u - u + x_1 = -z - 2u$$

and $u = \operatorname{sign} z$.
Therefore, $z(\theta) = C\sin(\theta - \delta) - 2\operatorname{sign} z$ and $z' = C\cos(\theta - \delta)$, where z and z' belong to

the class C^1.
Moreover, since $x_1(0) = 1 = -A\sin\alpha \pm 1$ and $x_2(0) = 0 = A\cos\alpha$, we get $\alpha = \frac{\pi}{2}, A = -2, z(0) < 0$; this implies $C\sin\delta - 2 > 0$, that is $|C| > 2$.
Let τ be the first point after which z changes from negative to positive; if $z(\theta) = \overline{C}\sin(\theta - \varphi) - 2$ for some θ bigger than τ and $z(\tau) = 0$, the following conditions imply the connection of the curves representing z and z' for $\theta < \tau$, with the same curves for $\theta > \tau$:

$$\begin{cases} C\sin(\tau - \delta) + 2 = \overline{C}\sin(\tau - \varphi) - 2 = 0 \\ C\cos(\tau - \delta) = \overline{C}\cos(\tau - \varphi). \end{cases}$$

From the first equation we get

$$C = -\frac{2}{\sin(\tau - \delta)}, \qquad \overline{C} = \frac{2}{\sin(\tau - \varphi)};$$

substituting in the second equation, we have $\tan(\tau - \delta) = -\tan(\tau - \varphi)$, that is,

$$\tau - \delta = \varphi - \tau \quad \text{or} \quad \tau - \delta = \pi + \varphi - \tau.$$

Then,

$$\overline{C} = \frac{2}{-\sin(\tau - \delta)} \quad \text{or} \quad \overline{C} = \frac{2}{\sin(\tau - \delta)},$$

that is $|\overline{C}| = |C|$.
Now we conclude the proof showing that the interval $[0, \pi]$ can be divided in an odd number of intervals; in each of them, z can be written as

$$z(\theta) = G\sin(\theta - \nu) \pm 2.$$

The condition $|\overline{C}| = |C|$ tells us that the bodies corresponding to such functions z are regular Reuleaux polygons.
Indeed, the graph of the functions z can be drawn in the following way:

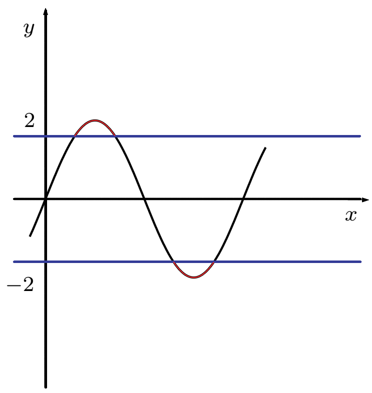

Figure 3.2:

we start with the function $y(\theta) = C\sin\theta$ and consider the piece of graph outside the strip $|y| \leq 2$. In this way, we obtain several arcs of the curve, which are defined in particular subintervals of $[0, 2\pi]$ and that are suitably connected.
To be able to do this, the length of the subintervals must be always the same.

Moreover, since $z(0) = z(2\pi)$, $u = \operatorname{sign} z$, and $\int_0^{2\pi} u(\theta)d\theta = 0$, the number of intervals of the subdivision of $[0, 2\pi]$ is integer and even.
This implies that $[0, \pi]$ is split in M subintervals, where M is integer, in which u is 1 or -1 in turn.
To be even more precise, M is necessarily odd, otherwise the corresponding bodies don't have constant width. We already said, indeed, that you can construct Reuleaux polygons only with an odd number of sides.
Leaving the degenerate case M = 1, we can conclude that the Reuleaux triangle is the solution to the Blaschke-Lebesgue problem, since the area is an increasing function with respect to M.
This propriety can be directly verified and the explicit calculus can be found in [10].
So, the minimum is given by the Reuleaux triangle with width 1, whose function ρ is 1 or 0 in turn, on intervals of length $\frac{\pi}{3}$.

4. The Blaschke-Lebesgue Theorem in the 3-dimensional case for bodies of revolution

In this section, we apply the Pontryagin Maximum Principle [20, 21] to the Blaschke-Lebesgue Theorem in the three-dimensional case, as it can be written for bodies of revolution.
The Theorem we refer to is the following [5]:

Theorem 4.1 (Blaschke-Lebesgue for bodies of revolution). *Among all 3-dimensional bodies of revolution having constant width, the body obtained by rotating a Reuleaux triangle around one of its axis of symmetry has the least volume.*

If the rotating figure of the extremal body is placed in a coordinate system, such as to be symmetric with respect to the y axis, and $h(\theta)$ is its support function, to find the minimum volume means to find the minimum of the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (h(\theta)\cos\theta - h'(\theta)\sin\theta)\rho(\theta)\, d\theta.$$

The plane figure we are studying has to be of constant width 1, so that we work under the following conditions:
$\rho(\theta) = h(\theta) + h''(\theta)$, $h(\theta) + h(-\theta) = 1$, $\rho(\theta) + \rho(-\theta) = 1$; the integral above then becomes

$$\begin{aligned}
&\int_{-\frac{\pi}{2}}^{0} (h(\theta)\cos\theta - h'(\theta)\sin\theta)\rho(\theta)d\theta \;+ \int_{0}^{\frac{\pi}{2}} (h(\theta)\cos\theta - h'(\theta)\sin\theta)\rho(\theta)d\theta \\
&= \int_0^{\frac{\pi}{2}} (h(-\theta)\cos(-\theta) - h'(-\theta)\sin(-\theta))\rho(-\theta)d\theta + \int_0^{\frac{\pi}{2}} (h(\theta)\cos\theta - h'(\theta)\sin\theta)\rho(\theta)d\theta \\
&= \int_0^{\frac{\pi}{2}} ((1-h(\theta))\cos\theta + h'(\theta)\sin\theta)(1-\rho(\theta))d\theta \;+ \int_0^{\frac{\pi}{2}} (h(\theta)\cos\theta - h'(\theta)\sin\theta)\rho(\theta)d\theta \\
&= \int_0^{\frac{\pi}{2}} (\cos\theta - h\cos\theta + h'\sin\theta + \rho(2h\cos\theta - 2h'\sin\theta - \cos\theta))d\theta \\
&= \int_0^{\frac{\pi}{2}} (2\rho(h\cos\theta - h'\sin\theta) - 2h\cos\theta)d\theta + 1 + h'(0),
\end{aligned}$$

where $\int_0^{\frac{\pi}{2}} \rho\cos\theta\, d\theta = \int_0^{\frac{\pi}{2}} (h\cos\theta + h''\cos\theta)d\theta = \int_0^{\frac{\pi}{2}} h\cos\theta\, d\theta + [h'\cos\theta]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} h'\sin\theta\, d\theta$.

Since $\int_0^{\frac{\pi}{2}} \rho(\theta)\, h'(\theta)\, \sin\theta\, d\theta = [\frac{1}{2}(h^2 + h'^2)(\theta)\sin\theta]_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos\theta(h^2 + h'^2)(\theta)d\theta$ and integrating by parts $\int h'^2(\theta)\, \cos\theta\, d\theta$, we get

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \big(2\rho(\theta)\, h(\theta)\cos\theta - 2h(\theta)\cos\theta + h'^2(\theta)\cos\theta + h^2(\theta)\, \cos\theta\big)\, d\theta + constant\\
=& \int_0^{\frac{\pi}{2}} \bigg(3h^2(\theta)\cos\theta + 2h(\theta)h''(\theta)\cos\theta - 2h(\theta)\cos\theta\\
&\qquad -h(\theta)\, h''(\theta)\cos\theta - \frac{h^2(\theta)}{2}\cos\theta\bigg)\, d\theta + constant\\
=& \int_0^{\frac{\pi}{2}} h(\theta)\cos\theta\left(\rho(\theta) + \frac{3h(\theta)}{2} - 2\right) d\theta + 1 + h'(0)\\
&\qquad - h^2\left(\frac{\pi}{2}\right) - h'^2\left(\frac{\pi}{2}\right) + \frac{h^2}{2}\left(\frac{\pi}{2}\right) - h(0)h'(0).
\end{aligned}$$

Let us now introduce the new variables x_1 and x_2 and the control u such that

$$\begin{cases}
\rho(\theta) = u(\theta), & 0 \le u \le 1,\\
x_1(\theta) = h(\theta), &\\
x_2(\theta) = x_1'(\theta) = h'(\theta). &
\end{cases}$$

Due to the regularity of the figure as concerns symmetry and width, we have, up to vertical translations, $h(0) = \frac{1}{2}$; moreover, since we are working with convex bodies, $h'(\frac{\pi}{2}) = 0$. The only uncertain boundary condition is $h'(0)$, but we can say $h'(0) = 0$ and consider that $h(\frac{\pi}{2})$ belongs to the interval $(0, 1)$. Summarizing we have to solve

$$\begin{cases}
\min \displaystyle\int_0^{\frac{\pi}{2}} x_1\cos\theta\left(u + \frac{3x_1}{2} - 2\right) d\theta & \text{such that}\\
\rho(\theta) = u(\theta) & 0 \le u \le 1\\
x_1(\theta) = h(\theta) &\\
x_2(\theta) = x_1'(\theta) = h'(\theta) &\\
x_1(0) = \frac{1}{2}, & x_2(0) = 0\\
x_1(\frac{\pi}{2}) = a \in (0,\, 1), & x_2(\frac{\pi}{2}) = 0.
\end{cases}$$

The differential equation for x_1 can be written as $x_2' = x_1'' = u - x_1 \;\Rightarrow\; x_1'' + x_1 = u \;\Rightarrow\; x_1 = A\sin(\theta - \alpha) + u$, in those intervals where u is constant.
The Hamiltonian function is

$$\begin{aligned}
H &= \lambda\, x_1\, \cos\theta\left(u + \frac{3x_1}{2} - 2\right) + p_2\,(u - x_1) + p_1\, x_2\\
&= u(p_2 + \lambda\, x_1\, \cos\theta) + \lambda\, x_1\, \cos\theta\left(\frac{3x_1}{2} - 2\right) - p_2\, x_1 + p_1\, x_2,
\end{aligned}$$

which is linear with respect to the control u.
Therefore, $p_1' = -\lambda\cos\theta(u-2+3x_1)+p_2$, $p_2' = -p_1$ and

$$p_2'' + p_2 = \lambda\cos\theta(u-2+3x_1).$$

The function H reaches its maximum value if

$$u = \begin{cases} 1, & p_2 + \lambda x_1 \cos\theta > 0 \\ 0, & p_2 + \lambda x_1 \cos\theta < 0. \end{cases}$$

Case $\lambda = 0$: $x_1 = A\sin(\theta-\alpha)+u$, $x_2 = A\cos(\theta-\alpha)$, $p_2 = B\sin(\theta-\gamma)$, for some constant A, α, B, γ.
Using the boundary conditions, we get

$$x_2(0) = A\cos\alpha = 0,$$

$$x_1(0) = -A\sin\alpha + 1 = \frac{1}{2} \vee x_1(0) = -A\sin\alpha + 0 = \frac{1}{2},$$

and, consequently,

$$x_1(\theta) = \begin{cases} 1 - \dfrac{1}{2}\cos\theta & u(0) = 1, \\ \dfrac{1}{2}\cos\theta & u(0) = 0, \end{cases}$$

in a right neighbourhood of $\theta = 0$.
As we said in the plane case, the function p_2 satisfies $p_2'' + p_2 = 0$, therefore, if $\gamma > 0$, it is negative in a right neighborhood $[0,\tau_1)$ of 0, it is positive in $(\tau_1, \tau_1+\pi)$ and again negative in $(\tau_1+\pi, 2\pi)$. Consequently, we have

$$u = \begin{cases} -1, & \text{in } [0,\tau_1), \\ 1, & \text{in } (\tau_1, \pi+\tau_1), \\ -1, & \text{in } (\pi+\tau_1, 2\pi), \end{cases}$$

and then

$$\rho = \begin{cases} 0, & \text{in } [0,\tau_1), \\ 1, & \text{in } (\tau_1, \pi+\tau_1), \\ 0, & \text{in } (\pi+\tau_1, 2\pi), \end{cases}$$

The symmetry conditions of our bodies further restrict the possible choices for τ_1, but, in any case, we conclude that $\rho = 0$ in an interval of length π. Such a case corresponds to a degenerate convex body (i.e. a convex body with empty interior). Then, this case must be excluded.

Case $\lambda = -1$ *and* $p_2 + \lambda x_1 \cos\theta \equiv 0$: The auxiliary vector p satisfies $p_2 = x_1\cos\theta$, $p_2' = x_1'\cos\theta - x_1\sin\theta = x_2\cos\theta - x_1\sin\theta = -p_1$, $p_1' = p_2 + \cos\theta(u-2+3x_1) = 2x_1\cos\theta + 2x_2\sin\theta - u\cos\theta$. Then, $x_1\cos\theta + (u-2)\cos\theta + 3x_1\cos\theta = 2x_1\cos\theta + 2x_2\sin\theta - u\cos\theta$ if and only if $(u-1+x_1)\cos\theta = x_2\sin\theta$. Replacing u with $x_1''+x_1$, this leads us to solve the differential equation $h'' = h'\tan\theta - 2h + 1$ for the support function h. Given the boundary conditions $h(0) = \frac{1}{2}$ and $h'(0) = 0$, since the coefficients are continuous, the equation above has a unique solution in $[0, \frac{\pi}{2})$ which is the constant function $h = \frac{1}{2}$. That means that our solution is the circle of radius $\frac{1}{2}$. This case also has to be excluded.

Case $\lambda = -1$ *and* $p_2 - x_1 \cos\theta = z \equiv 0$ *on certain subintervals* I_j *of* $[0, \frac{\pi}{2}]$: Suppose that $p_2 \neq x_1 \cos\theta$ on the interval I_i and $p_2 = x_1 \cos\theta$ on the interval I_{i+1} or viceversa. As we said in the previous case, the Hamiltonian system in I_{i+1} gives us the condition $(u - 1 + x_1)\cos\theta = x_2 \sin\theta$, that is, $u = x_2 \tan\theta + 1 - h = h' \tan\theta + 1 - h$, $\theta \neq \frac{\pi}{2}$. The differential equation $h(\theta) + h''(\theta) = \rho(\theta)$ becomes

$$h'' + 2h = h' \tan\theta + 1. \tag{23}$$

The function $h(\theta) = \frac{1}{2} + A \sin\theta$ is the unique solution of equation (23), for all $A \in \mathbb{R}$. This implies $\rho(\theta) \equiv \frac{1}{2}$, and we find the same configuration as in the plane case of a figure bounded by two circular arcs of radius $\frac{1}{2}$ centered in the origin and facing one another. An analogous argument as in the planar case, leads us to the conclusion that the volume of the bodies obtained by those figures is bigger than the volume of the bodies we get under rotation of the figure of smaller area we can construct cutting the boundary of the previous ones (see Figure 3.1).
Actually, not only the area of the rotating bodies decreases, but also the center of mass moves on average towards the origin.

In conclusion, the only acceptable case is $\lambda = -1$ and $z = p_2 - x_1 \cos\theta \neq 0$.

The differential equation for the function z which determines the change in the control u is $z'' + z = 2[(1 - u - x_1)\cos\theta + x_2 \sin\theta]$. Let us analyze the different situations occurring when the number of switching points varies.
First of all, we observe that the case $u = 0$ is similar to the case $u = 1$ in a right neighbourhood of $\theta = 0$, since they both produce the same rotating figure up to symmetries with respect to the horizontal axes. *Case 1:* one switching point δ

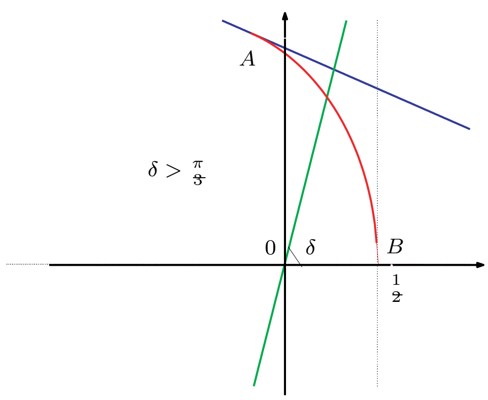

Figure 4.1: One switching point $\delta > \frac{\pi}{3}$

As is shown in Figures 4.1 and 4.2, if we want to preserve the symmetry of the figure with respect to the θ axes and to avoid new vertices, it is necessary that the arc on the boundary drawn in the interval $\theta \in [0, \delta]$ ends on the vertical axes; therefore $\delta = \frac{\pi}{3}$ and,

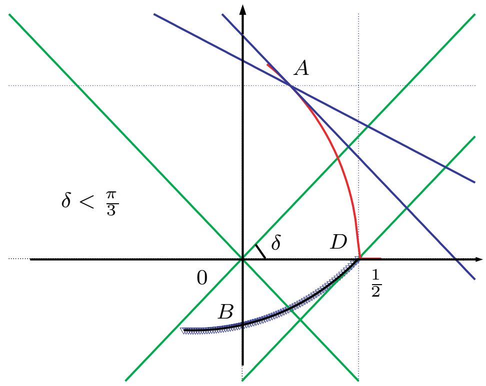

Figure 4.2: One switching point $\delta < \frac{\pi}{3}$

in case $u(0) = 1$,

$$h(\theta) = \begin{cases} 1 - \dfrac{1}{2}\cos\theta & 0 \le \theta \le \frac{\pi}{3} \\ \dfrac{\sqrt{3}}{2}\sin\theta & \frac{\pi}{3} \le \theta \le \frac{\pi}{2}. \end{cases}$$

The rotating figure we get is a Reuleaux triangle.

Case 2: two switching points δ_1 and δ_2

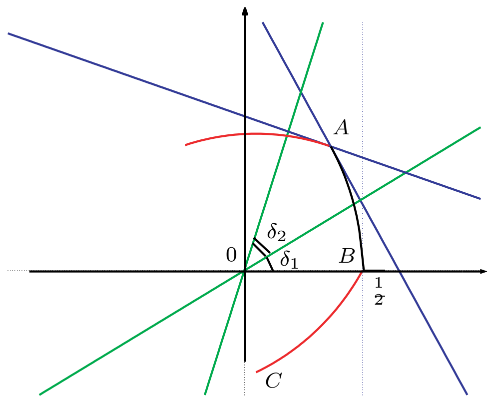

Figure 4.3: Two switching points δ_1 and δ_2

Again, as one can see from Figure 4.3, if we want to preserve the symmetry of the figure with respect to the θ axes and to avoid new vertices, it is necessary that the arc BC ends on the vertical axes and the center of the arc AB is on the same vertical axes.
The configuration we obtain is that of a regular star with sides of length 1, connecting the vertices, while $\delta_1 = \delta_2 = \frac{\pi}{5}$ (see Figure 4.4)

The rotating figure of the extremal body is a regular Reuleaux pentagon.

Case 3 on: three or more switching points Again, the optimal configuration is that of a regular Reuleaux polygon with n sides and switching points $\delta_i = \frac{\pi}{n}$.
Now, both the area of a regular Reuleaux polygon (as we already said) and the x-coordinate of the center of mass x_B of half a Reuleaux polygon (which is our rotating

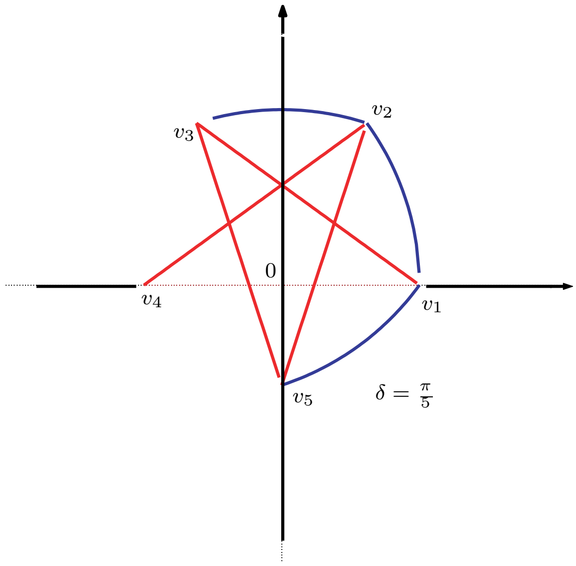

Figure 4.4: Two switching points: final configuration

body) is an increasing function with respect to the number of sides (see footnote).[1]

The conclusion is that the volume of the revolution bodies obtained by the rotation of regular Reuleaux polygons around one of their axes of symmetries is also an increasing function with respect to the number of sides and the solution to the initial problem is the body generated by a Reuleaux triangle.

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[1] $x_B = \frac{1}{3n}\frac{(\frac{\pi}{n}+\tan\frac{\pi}{2n})\frac{\pi}{2n}(\sin\frac{\pi}{4n})^2-\frac{1}{2}\tan\frac{\pi}{2n}\sin\frac{\pi}{2n}}{\frac{\pi}{n}-\tan\frac{\pi}{2n}} + \frac{2}{n}\sum_{k=1}^{[\frac{n}{2}]}[\sin(\frac{\pi}{n}(2k-1)) \cdot$
$\cdot (\frac{2\tan\frac{\pi}{n}(\frac{1}{3}(\cos\frac{\pi}{2n})^3-\frac{1}{4}\cos\frac{\pi}{2n}+\frac{1}{48(\cos\frac{\pi}{2n})^3})+\frac{2}{3}(\sin\frac{\pi}{2n})^3}{\frac{\pi}{2n}-\frac{1}{2}\sin\frac{\pi}{n}+\frac{1}{2}\cos\frac{\pi}{n}\tan\frac{\pi}{2n}} - \frac{1}{2\cos\frac{\pi}{2n}})] \to \frac{2}{3\pi}$ *when* $n \to +\infty$

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