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On Compositions of D.C. Functions and Mappings

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A d.c. (delta-convex) function on a normed linear space is a function representable as a difference of two continuous convex functions. We show that an infinite dimensional analogue of Hartman's theorem on stability of d.c. functions under compositions does not hold in general. However, we prove that it holds in some interesting particular cases. Our main results about compositions are proved in the more general context of d.c. mappings between normed linear spaces.

Keywords: D.c. function, composition of d.c. functions, d.c. mapping, delta-convex mapping

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Introduction

Let C be a convex set in a (real) normed linear space X. A function $f\colon C \to \mathbb{R}$ is called *d.c.* or *delta-convex* if it can be represented as a difference of two continuous convex functions on C. We say that f is locally d.c. on C if each $c \in C$ has a convex neighbourhood U such that f is d.c. on $U \cap C$. A mapping $F\colon C \to \mathbb{R}^n$ is a *d.c. mapping* if each of its n components is a d.c. function. There are many articles which work with d.c. functions (see, e.g., the references in [11] and [7]).

In 1959, P. Hartman [10] proved the following interesting now well-known results.

(I) Let $A \subset \mathbb{R}^m$ be a convex set which is either open or closed. Let $f\colon A \to \mathbb{R}$ be locally d.c. on A. Then f is d.c. on A.

(II) Let X be a normed linear space, $A \subset X$ a convex set which is either open or closed, and $B \subset \mathbb{R}^n$ an open convex set. If $F\colon A \to B$ is a d.c. mapping and $g\colon B \to \mathbb{R}$ is a d.c. function, then the function $g \circ F$ is locally d.c. on A.

In fact, Hartman [10] formulated (II) only for the case $X = \mathbb{R}^m$, but he mentioned (see

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the end of p. 707) that his proof clearly works also in more general settings (we could even suppose that X is a topological linear space and A is an arbitrary convex set). For a generalization of (II), proved in a quite different way, see Proposition 3.1.

Hartman also remarked that his proof of (I) does not work for infinite dimensional spaces. A corresponding counterexample was provided by E. Kopecká and J. Malý [14]: given a nonempty open convex set $A \subset \ell_2$, there exists a locally d.c. function on A which is not d.c. on A. (They also remark without proof that a similar example can be constructed in each infinite dimensional normed linear space; we prove this claim in Corollary 5.6.)

The results (I) and (II) immediately imply the following superposition theorem.

Theorem H. *Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be convex sets. Let A be either open or closed, and let B be open. If $F\colon A \to B$ and $g\colon B \to \mathbb{R}$ are d.c., then the function $g \circ F$ is d.c.*

Note that Hartman did not mention Theorem H explicitly, but he formulated its corollary (obtained by putting $F := (f_1, f_2)$ and $g(x, y) := xy$ or $g(x, y) := x/y$):

Corollary H. *Let $A \subset \mathbb{R}^m$ be either an open or a closed convex set. Let f_1, f_2 be d.c. on A. Then the product $f_1 \cdot f_2$ and, if $f_2(x) \neq 0$ for $x \in A$, the quotient f_1/f_2 are d.c. functions on A.*

Note that the case of the product can be proved in a more elementary way (see [11]), but the stability with respect to quotients probably cannot be proved more easily.

Though (I) cannot be used to generalize Theorem H to infinite dimensions, it remained open whether such a generalization is possible. The present paper concerns this question. We show that an infinite dimensional analogue of Theorem H does not hold (see Corollary 5.6):

For each infinite dimensional normed linear space X, there exists a positive d.c. function f on X such that $1/f$ is not d.c.

However, using a modification of Hartman's methods, we prove (Theorem 4.1) the following variant of Theorem H (for other variants see Theorem 4.2), in which the function g is defined on the whole $\mathbb{R}^n$.

Let X be a normed linear space. Let $A \subset X$ be an open convex set, and $F\colon A \to \mathbb{R}^n$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ be d.c. Then the function $g \circ F$ is d.c.

Consequently, if f, h are d.c. on A, then, for instance, $\exp(f)$ and $\frac{fh}{1+f^2+h^2}$ are d.c. on A (see the text after Theorem 4.1).

Another positive result, in which F is a real continuous convex (or concave) function, is Proposition 3.4. It implies (see Remark 3.5(i)) the following:

Let X be a reflexive Banach space and f_1, f_2 be continuous convex functions on X. If the quotient f_1/f_2 is defined on X, then it is d.c.

(Note that the above statement is true only in reflexive spaces, see [12].)

We prove our results in a more general context of d.c. mappings between normed linear spaces. In particular, we prove (see Corollary 3.9) that, in some interesting cases, the inner product (and even a general "product" given by a bilinear mapping) of two d.c. mappings is d.c. as well.

1. Preliminaries

We consider only normed linear spaces over the reals $\mathbb{R}$. If X is a normed linear space, we denote by B_X its closed unit ball. By $B(x,r)$ we denote the open ball with center x and radius r. We say that a Lipschitz mapping F is L-Lipschitz if $\mathrm{Lip}F \leq L$, where $\mathrm{Lip}F$ is the (least) Lipschitz constant of F.

Throughout the paper, all mappings and functions are supposed to be defined on non-empty sets.

Definition 1.1 ([16]). Let X, Y be normed linear spaces, $C \subset X$ be a convex set, and $F\colon C \to Y$ be a continuous mapping. We say that F is *d.c.* (or *delta-convex*) if there exists a continuous (necessarily convex) function $f\colon C \to \mathbb{R}$ such that $y^* \circ F + f$ is convex on C whenever $y^* \in Y^*$, $\|y^*\| \leq 1$. In this case we say that f controls F, or that f is a *control function* for F.

Remark 1.2. The following facts are easy to prove (cf. [16]).

(a) For $Y = \mathbb{R}^n$, the above definition of a d.c. mapping coincides with the one in the beginning of Introduction. Moreover, if $F = (F_1, \dots, F_n)$ and f_i controls F_i, then $f := f_1 + \cdots + f_n$ controls F.

(b) If $g = f_1 - f_2$, where f_1, f_2 are continuous convex functions on a convex subset of a normed linear space, then $f_1 + f_2$ controls g.

(c) The notion of delta-convexity does not depend on the choice of equivalent norms on X and Y.

A theory of d.c. mappings on open convex sets was developed in [16]. Some further results, together with a survey of main results from [16], can be found in [7]. We shall need the following two propositions.

Proposition 1.3 ([16]). *Let X, Y, Z be normed linear spaces, and let $A \subset X$ and $B \subset Y$ be convex sets. Let $F\colon A \to B$ and $G\colon B \to Z$ be d.c. mappings with control functions $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$, respectively. If G and g are Lipschitz on B, then $G \circ F$ is d.c. on A with a control function $h = g \circ F + (\mathrm{Lip}G + \mathrm{Lip}g)f$.*

Proof. This was proved in [16, Proposition 4.1] assuming that the sets A, B are also open, since this was the context the authors were interested in. However, it is easy to see that the proof does not need this additional assumption. Indeed, the proof is based on the equivalence of (i) and (iii) in [16, Proposition 1.13], whose proof does not use the openness of A. □

Proposition 1.4. *Let X, Y be normed linear spaces, $C \subset X$ a bounded open convex set, and $F\colon C \to Y$ a d.c. mapping with a Lipschitz control function. Then F is Lipschitz.*

Proof. This was stated in [7, Theorem 18(i)] for X and Y Banach spaces, but the proof therein works for normed linear spaces as well. (Note that the question for which open convex sets C the proposition holds was answered in [3].) □

Notation 1.5. Let A, B, A_n, B_n $(n \in \mathbb{N})$ be subsets of a normed linear space X. We shall use the notation:

- $A \subset\subset B$ whenever there exists $\varepsilon > 0$ such that $A + B(0, \varepsilon) \subset B$;
- $A_n \nearrow A$ whenever $A_n \subset A_{n+1}$ for each $n \in \mathbb{N}$, and $\bigcup_{n\in\mathbb{N}} A_n = A$;
- $A_n \nearrow\!\!\nearrow A$ whenever $A_n \subset\subset A_{n+1}$ for each $n \in \mathbb{N}$, and $\bigcup_{n\in\mathbb{N}} A_n = A$.

Fact 1.6. *Let C be a nonempty convex set in a normed linear space X, and $f\colon C \to \mathbb{R}$ a convex function.*

(*a*) *If C is open and bounded, and f is continuous, then f is bounded below on C.*
(*b*) *If f is bounded on C, then f is Lipschitz on each $D \subset\subset C$.*
(*c*) *If f is L-Lipschitz on C, then f admits a convex L-Lipschitz extension to the whole X.*

Proof. (*a*) follows from the fact that f is minorized by a continuous affine function (by the Hahn-Banach theorem).

(*b*) can be proved in the same way as local Lipschitz continuity of continuous convex functions. For the sake of completeness, we give a sketch of proof. Let $|f| \le M$ on C, $r > 0$ be such that $D + B(0, 2r) \subset C$, and $x, y \in D$, $x \ne y$. Then $z := y + \frac{r(y-x)}{\|y-x\|} \in C$, and $y = \frac{r}{\|y-x\|+r}x + \frac{\|y-x\|}{\|y-x\|+r}z$. By convexity, $f(y) \le \frac{r}{\|y-x\|+r}f(x) + \frac{\|y-x\|}{\|y-x\|+r}f(z)$. It easily follows that $f(y) - f(x) \le \frac{\|y-x\|(f(z)-f(y))}{r} \le \frac{2M}{r}\|y-x\|$. The rest follows by interchanging x and y.

(*c*) It is well-known (and easy-to-prove) that the function $\widehat{f}\colon X \to \mathbb{R}$, given by $\widehat{f}(x) = \inf\{f(c) + L\|x - c\| : c \in C\}$, is a convex, L-Lipschitz extension of f (cf. [4]). □

We shall need the following well-known and very easy fact. Let us recall that $\operatorname{dist}(A, B) = \inf\{\|a - b\| : a \in A,\ b \in B\}$ with the usual agreement that $\inf \emptyset = +\infty$.

Fact 1.7. *Let C be a convex set in a normed linear space X, and $r > 0$. Then the sets (called "inner parallel set" and "outer parallel set" of C)*

$$D := \{x \in C : \operatorname{dist}(x, X \setminus C) > r\}, \qquad E := \{x \in X : \operatorname{dist}(x, C) < r\}$$

are convex.

Observation 1.8. Let X, Y be normed linear spaces, $C \subset X$ a convex set, and $F\colon C \to Y$ a d.c. mapping with a bounded above control function f. Then both F and f are Lipschitz on each bounded convex set $B \subset\subset C$.

Proof. By Fact 1.7, there exist open, bounded, convex sets D and E such that $B \subset D \subset\subset E \subset C$. By Fact 1.6(*a*), f is bounded on E. Hence f is Lipschitz on D by Fact 1.6(*b*), and F is Lipschitz on D by Proposition 1.4. □

Definition 1.9. A normed linear space X is said to have *modulus of convexity of power type 2* if there exists $a > 0$ such that $\delta_X(\varepsilon) \ge a\varepsilon^2$ for each $\varepsilon \in (0, 2]$ (where δ_X denotes the classical modulus of convexity of X; see, e.g., [5] for the definition).

Fact 1.10.

(a) *The ℓ_2-direct sum $(X \oplus Y)_{\ell_2}$ has modulus of convexity of power type 2 whenever both X and Y do.*

(b) *All $L_p(\mu)$ spaces with $1 < p \leq 2$ (μ arbitrary nonnegative measure) have modulus of convexity of power type 2 (in their canonical norms).*

Proof. (a) follows immediately from the following result by Bynum [1]: X has modulus of convexity of power type 2 if and only if there exists $b > 0$ such that $2\|x\|^2 + 2\|y\|^2 \geq \|x+y\|^2 + b\|x-y\|^2$ for each $x, y \in X$.

(b) is due to Hanner [9]. $\square$

Let X, Y be normed linear spaces, and $A \subset X$ an open set. Recall that a mapping $F\colon A \to Y$ is said to be $C^{1,1}$ on A if its Fréchet derivative $F'(x)$ exists at each point $x \in A$ and $F'\colon A \to \mathcal{L}(X,Y)$ is Lipschitz.

The next proposition follows from the proof of the implication (i) $\Rightarrow$ (ii) in [7, Theorem 11].

Proposition 1.11. *Let X, Y be normed linear spaces, $A \subset X$ an open convex set, $F\colon A \to Y$ a $C^{1,1}$ mapping. If X admits an equivalent norm $|\cdot|$ with modulus of convexity of power type 2, then F is d.c. on A with a control function of the form $f(x) = c|\cdot|^2$ for some $c > 0$.*

2. A consequence of Hartman's construction

Hartman's construction [10], which gives the proof that locally d.c. functions in $\mathbb{R}^n$ are d.c., has some consequences also in infinite dimensional spaces. It was observed (independently) already in [15] and [14] (cf. Remark 2.6). The main new observation of the present article is that Hartman's construction gives even a characterization of d.c. mappings on open sets (Proposition 2.4) which (together with Proposition 1.3) implies some infinite dimensional versions of Hartman's superposition theorem. First we formulate a lemma which describes Hartman's construction in a general setting.

Lemma 2.1. *Let X, Y be normed linear spaces, $C \subset X$ a nonempty convex set, and $F\colon C \to Y$ a mapping. Let $\emptyset \neq D_n \subset C$ ($n \in \mathbb{N}$) be convex sets such that $D_n \nearrow C$ and, for each n, $\operatorname{dist}(D_n, C \setminus D_{n+1}) > 0$, D_n is relatively open in C, and $F|_{D_n}$ is d.c. with a control function $\gamma_n\colon D_n \to \mathbb{R}$ which is either bounded or Lipschitz. Then F is d.c. on C.*

Proof. First, fix $a \in D_1$, and observe that the bounded sets $\widetilde{D}_n := D_n \cap B(a,n)$ satisfy the same assumptions as the sets D_n. Thus we can (and do) suppose that each D_n is bounded, and hence each γ_n is bounded on D_n. Adding a constant to γ_n if necessary, we can suppose that $0 < \gamma_n(x) < b_n < \infty$ for each $n \in \mathbb{N}$ and $x \in D_n$.

For each $n \in \mathbb{N}$, choose $0 < d_n < \operatorname{dist}(D_n, C \setminus D_{n+1})$, and consider the Lipschitz convex functions $\varphi_n(x) := \frac{b_{n+1}}{d_n} \operatorname{dist}(x, D_n)$ on C. Define

$$h_n(x) := \max\{\gamma_{n+1}(x), \varphi_n(x)\},\ x \in D_{n+1}, \quad \text{and} \quad h_n(x) := \varphi_n(x),\ x \in C \setminus D_{n+1}.$$

If $z \in D_{n+1}$, then there exists $\varepsilon > 0$ such that $h_n(x) = \max\{\gamma_{n+1}(x), \varphi_n(x)\}$ for $x \in C \cap B(z,\varepsilon)$, since D_{n+1} is open in C. If $z \in C \setminus D_{n+1}$, then $\operatorname{dist}(z, D_n) > d_n$ and therefore there exists $\varepsilon > 0$ such that $\varphi_n(x) > b_{n+1}$, and thus $h_n(x) = \varphi_n(x)$, for each $x \in C \cap B(z, \varepsilon)$. Therefore, h_n is continuous and convex on C. Moreover, clearly

- $h_n \geq 0$, and h_n is bounded on each bounded subset of C.

Since $\varphi_n(x) = 0$ for $x \in D_n$, we see that $h_n(x) = \gamma_{n+1}(x)$ for $x \in D_n$. So,

- h_n is a control function for F on D_n.

Let us define, by induction, a sequence $\{f_n\}$ of continuous convex functions on C such that:

(a) f_n is bounded on bounded subsets of C,
(b) $f_n \geq 0$,
(c) f_n controls F on D_{n+1}, and
(d) $f_{n+1} = f_n$ on D_n.

Put $f_1 := h_2$. Suppose we already have $f_1, \ldots, f_n$. Set

$$s := \sup f_n(D_{n+2}), \qquad \sigma := \sup h_{n+2}(D_n) \quad \text{and}$$

$$g_n(x) := h_{n+2}(x) - \sigma + \frac{\sigma + s + 1}{d_n} \operatorname{dist}(x, D_n) \ \text{ for } x \in C.$$

Then clearly g_n is continuous and convex on C, and it controls F on D_{n+2}. Define $f_{n+1} = \max\{f_n, g_n\}$. Clearly f_{n+1} is continuous convex, $f_{n+1} \geq f_n \geq 0$, and f_{n+1} is bounded on bounded subsets of C. If $x \in D_n$, then $g_n(x) \leq 0 \leq f_n(x)$, consequently $f_{n+1} = f_n$ on D_n.

Let us show that f_{n+1} controls F on D_{n+2}; i.e., that the function $\varphi_{y^*} := y^* \circ F + f_{n+1} = \max\{y^* \circ F + f_n, y^* \circ F + g_n\}$ is continuous and convex on D_{n+2} for each $y^* \in B_{X^*}$. To this end, fix $y^* \in B_{X^*}$ and $z \in D_{n+2}$. If $z \in D_{n+1}$, then there is $\varepsilon > 0$ such that φ_{y^*} is continuous and convex on $B(z, \varepsilon) \cap C$ (since D_{n+1} is open in C and both f_n and g_n control F on D_{n+1}). If $z \in D_{n+2} \setminus D_{n+1}$, then $\operatorname{dist}(z, D_n) \geq d_n$, and consequently $g_n(z) \geq 0 - \sigma + (\sigma + s + 1) > f_n(z)$. Therefore there exists $\varepsilon > 0$ such that $U := B(z, \varepsilon) \cap C \subset D_{n+2}$ and φ_{y^*} equals to the continuous convex function $y^* \circ F + g_n$ on U. Hence we can conclude that φ_{y^*} is continuous and convex on D_{n+2}.

Now, for each $x \in C$, the sequence $\{f_n(x)\}$ is constant for large n's, hence $f(x) := \lim_{n\to\infty} f_n(x)$ is well defined on C. Since $f = f_n$ on D_n, (c) easily implies that f is a continuous convex function which controls F on C. □

Remark 2.2. The assumptions of Lemma 2.1 allow the possibility that $D_n = D_{n+1} = \cdots = C$ for some n.

Lemma 2.3. *Let X be a normed linear space and let $C \subset X$ be nonempty, open and convex. Let $\{C_n\}$ be a sequence of convex sets with nonempty interior, such that $C_n \nearrow C$. Then there exists a sequence $\{D_n\}$ of nonempty bounded open convex sets such that $D_n \nearrow\!\!\!\nearrow C$, and $D_n \subset\subset C_n$ for each n.*

Proof. We can (and do) suppose that each C_n is bounded. (If this is not the case, replace, for each n, the set C_n with the set $C_n \cap B(x_0, n)$ where x_0 is an arbitrary interior point of C_1.)
First we claim that $C = \bigcup_n \operatorname{int} C_n$. Indeed, let $x \in C$ be any point. Then $x \in C_n$ for

some n. If $x \notin \operatorname{int} C_n$, choose any $y \in \operatorname{int} C_n$. There exists $z \in C$ such that $x \in (y, z)$ (i.e., x is a relative interior point of the segment $[y, z]$). There exists $k > n$ such that $z \in C_k$. Then $x \in \operatorname{int} C_k$, since $y \in \operatorname{int} C_k$.
Now, fix $\delta > 0$ such that C_1 contains an open ball of radius 2δ, and define

$$D_n := \{x \in C_n : \operatorname{dist}(x, X \setminus C_n) > \delta/n\}.$$

Obviously $D_n \subset\subset C_n$ for each n, and the sets D_n are nonempty, open and (by Fact 1.7) convex. Moreover

$$D_n \subset\subset \{x \in C_n : \operatorname{dist}(x, X \setminus C_n) > \delta/(n+1)\} \subset D_{n+1}.$$

To finish the proof, fix $x \in C$. Then $x \in \operatorname{int} C_n$ for some n. Fix $k > n$ such that $\operatorname{dist}(x, X \setminus C_n) > \delta/k$. Then $\operatorname{dist}(x, X \setminus C_k) \geq \operatorname{dist}(x, X \setminus C_n) > \delta/k$ which means that x belongs to D_k. $\square$

Now, we are ready to state the main result of this section.

Proposition 2.4. *Let X, Y be normed linear spaces, $C \subset X$ a nonempty open convex set, and $F \colon C \to Y$ a mapping. Then the following assertions are equivalent:*

(*i*) *F is d.c. on C;*
(*ii*) *there exists a sequence $\{C_n\}$ of convex sets with nonempty interior such that $C_n \nearrow C$ and, for each n, $F|_{C_n}$ is d.c. with a control function that is bounded above on C_n;*
(*iii*) *there exists a sequence $\{D_n\}$ of bounded open convex sets such that $D_n \nearrow\!\!\nearrow C$ and, for each n, $F|_{D_n}$ is Lipschitz and d.c. with a Lipschitz control function on D_n.*

Proof. (*i*) $\Rightarrow$ (*ii*). Let $f \colon C \to \mathbb{R}$ be a control function for F. Fix $x_0 \in C$ and consider the sets $C_n = \{x \in C : f(x) < f(x_0) + n\}$ ($n \in \mathbb{N}$). They are nonempty, open and convex, and they obviously satisfy (*ii*).

(*ii*) $\Rightarrow$ (*iii*). Let $\{C_n\}$ be as in (*ii*). Let D_n ($n \in \mathbb{N}$) be the bounded, open, convex sets constructed in Lemma 2.3 from the sets C_n. Then (*iii*) follows immediately from Observation 1.8.

(*iii*) $\Rightarrow$ (*i*) follows from Lemma 2.1. $\square$

Proposition 2.4 easily implies the following generalization of Hartman's result (I) from Introduction, which was stated (for open A) already in [16, Theorem 1.20] with only a hint for the proof.

Corollary 2.5. *Let $A \subset \mathbb{R}^d$ be a convex set which is either open or closed, and let Y be a normed linear space. Then each locally d.c. mapping $F \colon A \to Y$ is d.c. on A.*

Proof. First we will show that F is d.c. on each compact convex set $C \subset A$. Using compactness of C and [10, Lemma 1], we easily see that there exist continuous convex functions f_i on A, $x_i \in C$, and $r_i > 0$, $i = 1, \ldots, k$, such that $C \subset \bigcup_{i=1}^{k} B(x_i, r_i)$ and f_i controls F on $C \cap B(x_i, r_i)$. Consequently, $f_C = f_1 + \cdots + f_k$ controls F on C.

Now, distinguish two cases. First suppose that A is open. Then choose compact convex sets C_n with nonempty interior such that $C_n \nearrow A$. Since f_{C_n} is bounded on C_n, Proposition 2.4 implies that F is d.c.

If A is closed, choose $z \in A$ and put $D_n := A \cap B(z,n)$. Since $\overline{D_n} \subset A$ is compact and convex, F is d.c. on D_n (with a bounded control function), and we can apply Lemma 2.1. □

Remark 2.6. It is known (see [2]) that, on each infinite dimensional Banach space, there exists a continuous convex function which is unbounded on a ball. This implies (via Fact 1.6(b) and Proposition 1.4) that the implication (ii) ⇒ (i) in Proposition 2.4 is a *strict* generalization of both [15, Theorem 2.3] and [14, Corollary 18], where delta-convexity of F was proved under the following stronger assumption: F is d.c. on each bounded closed convex $B \subset C$ with a Lipschitz ([15]) or bounded ([14]) control function on B.

3. Global delta-convexity of composed mappings

Let us start with the following generalization of (II) (see Introduction) which is essentially proved in [16, Theorem 4.2].

Proposition 3.1. *Let X, Y, Z be normed linear spaces, $A \subset X$ a convex set, and $B \subset Y$ an open set. Let $F\colon A \to B$ and $G\colon B \to Z$ be locally d.c. mappings. Then $G \circ F$ is locally d.c.*

Proof. Fix $a \in A$. Since G is locally d.c. and each d.c. mapping on an open convex subset of Y is locally Lipschitz (see [16, Proposition 1.10]), there exists an open convex neighborhood $B_0 \subset B$ of $F(a)$ on which G is Lipschitz and d.c. with a Lipschitz control function. Find $\delta > 0$ such that, for $A_0 := B(a,\delta) \cap A$, we have that $F(A_0) \subset B_0$ and $F|_{A_0}$ is d.c. Then $G \circ F|_{A_0} = (G|_{B_0}) \circ (F|_{A_0})$ is d.c. by Proposition 1.3. □

Our results on global delta-convexity of composed mappings will follow from the next basic lemma.

Lemma 3.2. *Let X, Y, Z be normed linear spaces, let $A \subset X$ and $B \subset Y$ be convex sets, and let $F\colon A \to B$ and $G\colon B \to Z$ be mappings. Suppose there exist sequences of convex sets $A_n \subset A$, $B_n \subset B$ such that $F(A_n) \subset B_n$, $G|_{B_n}$ is Lipschitz and d.c. with a Lipschitz control function, and at least one of the following conditions holds:*

- (i) *A_n is relatively open in A, $A_n \nearrow A$, $\operatorname{dist}(A_n, A \setminus A_{n+1}) > 0$, $F|_{A_n}$ is either bounded or Lipschitz and it is d.c. with a control function which is either bounded or Lipschitz.*
- (ii) *A is open, F is d.c., $\operatorname{int} A_n \neq \emptyset$, and $A_n \nearrow A$.*

Then $G \circ F$ is d.c. on A.

Proof. Let (i) hold. As in the proof of Lemma 2.1, we can (and do) suppose that the sets A_n are bounded. Then, on each A_n, F is bounded and admits a bounded control function. Proposition 1.3 implies that the mapping $G \circ F|_{A_n} = (G|_{B_n}) \circ (F|_{A_n})$ is d.c. with a bounded control function. By Lemma 2.1, $G \circ F$ is d.c.

Now, suppose that (ii) holds. By Lemma 2.3, we can (and do) suppose that $A_n \nearrow\!\!\nearrow A$ and each A_n is open. By Proposition 2.4, there exists a sequence $\{D_n\}$ of bounded, open, convex sets such that $D_n \nearrow\!\!\nearrow A$ and, for each n, $F|_{D_n}$ is Lipschitz and d.c. with a Lipschitz control function. Then the sets $\widetilde{A}_n := A_n \cap D_n$ are open and convex,

$F(\widetilde{A}_n) \subset B_n$, and $\widetilde{A}_n \nearrow\!\!\nearrow A$. Thus the condition (i) holds with A_n replaced by $\widetilde{A}_n$. So $G \circ F$ is d.c. by the first part of the proof. $\square$

As a simpler but still rather general consequence we obtain:

Proposition 3.3. *Let X, Y, Z be normed linear spaces, let $A \subset X$ and $B \subset Y$ be convex sets, and let $F\colon A \to B$ and $G\colon B \to Z$ be mappings. Suppose that the restriction of G to each bounded convex subset of B is Lipschitz and d.c. with a Lipschitz control function, and at least one of the following conditions holds.*

(i) *The restriction of F to each bounded convex subset of A is bounded and d.c. with a bounded control function.*

(ii) *A is open and F is d.c.*

Then $G \circ F$ is d.c.

Proof. To prove (i), choose an arbitrary $a \in A$ and, for each $n \in \mathbb{N}$, set $A_n := B(a, n) \cap A$, $B_n := \operatorname{conv} F(A_n)$. It is easy to see that $\operatorname{dist}(A_n, A \setminus A_{n+1}) > 0$ and $B_n \subset B$ is bounded for each n. Thus $G \circ F$ is d.c. by Lemma 3.2.

To prove (ii), use Proposition 2.4 to choose a sequence $\{A_n\}$ of bounded open convex sets such that $A_n \nearrow\!\!\nearrow A$ and, for each n, $F|_{A_n}$ is Lipschitz and d.c. with a Lipschitz control function. Then $B_n := \operatorname{conv} F(A_n)$ is clearly bounded and convex, and thus $G|_{B_n}$ is Lipschitz and d.c. with a Lipschitz control function. Apply Lemma 3.2. $\square$

Most of the next results are corollaries of Proposition 3.3. One of the exceptions is the following interesting proposition.

Proposition 3.4. *Let C be an open convex subset of a reflexive Banach space X, and $f\colon C \to \mathbb{R}$ be a continuous convex function. Let $I \subset \mathbb{R}$ be an open interval containing $f(C)$. Then, for every normed linear space Z and every d.c. mapping $G\colon I \to Z$, the composed map $G \circ f$ is d.c. on C.*

Proof. Let $\{b_n\} \subset \big(\inf f(C), \sup I\big)$ be an increasing sequence tending to $\sup I$. Then clearly the sets $C_n := \{x \in C : f(x) < b_n\}$ are nonempty, open and convex, and $C_n \nearrow C$. By Lemma 2.3, there exist nonempty bounded open convex sets $D_n \subset\subset C_n$ ($n \in \mathbb{N}$) with $D_n \nearrow\!\!\nearrow C$. Since f attains its infimum on the weakly compact set $\overline{D_n}$ (see e.g. [6, Theorem 25.1(b)]), we have $a_n := \min f\big(\overline{D_n}\big) > \inf I$ and hence $f(D_n) \subset [a_n, b_n] \subset I$ ($n \in \mathbb{N}$). Since G and its control function are locally Lipschitz on I (cf. [16, Proposition 1.10]), they are Lipschitz on each $[a_n, b_n]$. Apply Lemma 3.2 with $A := C$, $A_n := D_n$, and $B_n := [a_n, b_n]$. $\square$

Remark 3.5. (i) Proposition 3.4 implies that $1/f$ is d.c. whenever f is a positive continuous convex function on an open convex subset of a reflexive Banach space.

(ii) It is easy to see that Proposition 3.4 holds for concave (instead of convex) f as well. However it is not true for *all* d.c. functions f (see Corollary 5.6).

(iii) Proposition 3.4 fails in any nonreflexive Banach space X: by [12], a Banach space X is reflexive if and only if $1/f$ is d.c. for each positive continuous convex function f on X.

Theorem 3.6. *Let X, Y, Z be normed linear spaces, let $A \subset X$ and $B \subset Y$ be open convex sets, and let $F\colon A \to B$ and $G\colon B \to Z$ be d.c. mappings. Then $G \circ F$ is d.c. on A, provided at least one of the following conditions is satisfied:*

(*a*) *$B = Y$ and G admits a control function g that is bounded on bounded sets;*
(*b*) *Y is finite-dimensional and $\overline{F(A)} \subset B$;*
(*c*) *Y admits a renorming with modulus of convexity of power type 2, and G is $C^{1,1}$ on bounded open subsets of B.*

Proof. Let (a) hold. Let $E \subset Y$ be an arbitrary bounded convex set. Choose a bounded convex set C such that $E \subset\subset C$. Since g is bounded on C, Observation 1.8 implies that both G and g are Lipschitz on E. Thus $G \circ F$ is d.c. by Proposition 3.3 .

Now, suppose (b) holds. By Proposition 2.4, there exists a sequence $\{A_n\}$ of nonempty bounded open convex sets such that $A_n \nearrow A$ and F is Lipschitz on each A_n. Since each $\overline{F(A_n)}$ is a compact subset of B (Y is finite-dimensional!), $B_n := \operatorname{conv} \overline{F(A_n)} \subset\subset B$ is a compact convex subset of B. Let $\widetilde{g}$ be a control function of G. We can clearly find $\varepsilon > 0$ such that $\widetilde{g}$ is bounded on $C := B_n + B(0, \varepsilon) \subset B$. Observation 1.8 implies that both G and $\widetilde{g}$ are Lipschitz on B_n. Now, Lemma 3.2 shows that $G \circ F$ is d.c.

Finally, let (c) hold. For each bounded convex set $E \subset B$, let $B_0 \subset B$ be a bounded convex open set containing E. Since G is $C^{1,1}$ on B_0, it is also Lipschitz on B_0. Moreover, Proposition 1.11 easily implies that G admits a Lipschitz control function on B_0, and hence also on E. Thus, we can apply Proposition 3.3. □

Let X, Y be vector spaces. Recall that a mapping $Q\colon X \to Y$ is *quadratic* if there exists a bilinear mapping $B\colon X \times X \to Y$ such that $Q(x) = B(x, x)$ for each $x \in X$. In this case, we say that Q *is generated* by B.

Definition 3.7 ([13]). A normed linear space X is said to have the *property* (D) if every continuous quadratic form on X can be represented as a difference of two nonnegative continuous quadratic forms.

Proposition 3.8. *Let X, Y, Z be normed linear spaces, $C \subset X$ an open convex set, $F\colon C \to Y$ a d.c. mapping, and $Q\colon Y \to Z$ a continuous quadratic mapping. Then $Q \circ F$ is d.c. on C, provided at least one of the following conditions is satisfied:*

(*a*) *Y admits a renorming with modulus of convexity of power type 2;*
(*b*) *Z is finite-dimensional and Y has the property (D).*

Proof. The case (a) follows immediately from Theorem 3.6(c), since each continuous quadratic mapping is $C^{1,1}$.

Suppose (b) holds. We can suppose that $Z = \mathbb{R}^d$ for some $d \in \mathbb{N}$. Then the components Q_j $(j = 1, \ldots, d)$ of the quadratic mapping Q are continuous quadratic forms. Since Y has (D), we can write $Q_j = p_j - q_j$ where p_j, q_j are nonnegative continuous quadratic forms, in particular, they are convex continuous functions that are bounded on bounded sets. By Remark 1.2(a) and (b), Q is d.c. with a control function which is bounded on bounded subsets of Y. Apply Theorem 3.6(a). □

The following Corollary 3.9 improves [16, Corollary 4.3.] which states only that $B\circ(F,G)$ is locally d.c. whenever Y and V are Hilbert spaces.

Corollary 3.9. *Let X,Y,V,Z be normed linear spaces, $C\subset X$ an open convex set, $F\colon C\to Y$ and $G\colon C\to V$ d.c. mappings, and $B\colon Y\times V\to Z$ a continuous bilinear mapping. Then the mapping $B\circ(F,G)\colon x\mapsto B\big(F(x),G(x)\big)$ is d.c. on C, provided at least one of the following conditions is satisfied:*

(a) *both Y and V admit renormings with modulus of convexity of power type 2;*
(b) *Z is finite-dimensional and $Y\times V$ has the property (D).*

Proof. Observe that B is also a quadratic mapping on $Y\times V$; indeed, it is generated by the bilinear mapping $\widetilde{B}\big((y,v),(y',v')\big)=B(y,v')$ on $(Y\times V)\times(Y\times V)$. Moreover, by [16, Lemma 1.7], the mapping $x\mapsto\big(F(x),G(x)\big)$ is d.c. on C. Apply Fact 1.10(a) and Proposition 3.8. □

Remark 3.10. (a) By Fact 1.10(b), the assumptions in Proposition 3.8(a) and Corollary 3.9(a) are satisfied, for instance, if each of Y,V is isomorphic to a subspace of some $L_p(\mu)$ with $1<p\le 2$ (not necessarily with the same p and μ).

(b) By [13, Theorem 1.6 and Observation 3.13], the assumptions in Proposition 3.8(b) and Corollary 3.9(b) are satisfied, for instance, if each of Y,V is isomorphic to one (not necessarily the same) of the spaces $C(K)$, $c_0(\Gamma)$, $L_p(\mu)$ with $2\le p\le\infty$.

4. Global delta-convexity of composed functions

Here we present positive results which are formulated without using the notion of d.c. operators, i.e., those which directly concern Hartman's results. Probably most interesting is the following immediate consequence of Theorem 3.6(a). Note that it is important that the function g is defined on the whole $\mathbb{R}^n$ (see Corollary 5.6(a)).

Theorem 4.1. *Let X be a normed linear space. Let $A\subset X$ be an open convex set, and $F\colon A\to\mathbb{R}^n$ and $g:\mathbb{R}^n\to\mathbb{R}$ be d.c. Then the composed function $g\circ F$ is d.c.*

Since each C^2 function $g:\mathbb{R}^n\to\mathbb{R}$ is d.c by Proposition 1.11 and (I) from Introduction, applying Theorem 4.1 to $F=(f,h)$ and $g(x,y)=xy$, we obtain that $f\cdot h$ is d.c. on A, whenever f and h are real d.c. functions on A. However, this fact is well-known (cf. [11]) and can be proved in a quite elementary way. But the fact that, for instance, $\exp(f)$ and $\frac{fh}{1+f^2+h^2}$ are d.c. on A seems to be new. (Hartman's results only imply that these functions are locally d.c.)

For compositions of special d.c. functions, we obtain the following.

Theorem 4.2. *Let X be a normed linear space and $A\subset X$, $B\subset\mathbb{R}^n$ convex sets. Let $F=(F_1,\dots,F_n)\colon A\to B$ be a d.c. mapping and $g\colon B\to\mathbb{R}$ a d.c. function. Then $g\circ F$ is d.c. on A, provided at least one of the following conditions is satisfied:*

(a) *A is open, F is d.c., and g is a difference of two Lipschitz convex functions;*
(b) *each F_i is a difference of two continuous convex functions which are bounded on bounded subsets of A, and the restriction of g to each bounded convex subset of B is a difference of two Lipschitz convex functions;*

(c) *$X = \mathbb{R}^k$, A is open or closed, F is d.c., and, for each $a \in A$, there exists $\varepsilon > 0$ such that g is a difference of two Lipschitz convex functions on $B \cap B(F(a), \varepsilon)$.*

Proof. To prove (a), observe that, by Fact 1.6(c), we can suppose that g is a difference of two Lipschitz convex functions on the whole $\mathbb{R}^n$. Hence $g \circ F$ is d.c. on A by Theorem 4.1.

The part (b) follows from Remark 1.2(a) and (b), and Proposition 3.3.

Let (c) hold. By Corollary 2.5 (or (I)), it is sufficient to show that $g \circ F$ is locally d.c. on A. To this end, choose an $a \in A$ and find $\varepsilon > 0$ such g is a difference of two Lipschitz convex functions on $B \cap B(F(a), \varepsilon)$. Since F is continuous, we can find $\delta > 0$ such that $F(B(a, \delta) \cap A) \subset B \cap B(F(a), \varepsilon)$. Using Proposition 1.3 (and Remark 1.2(b)), we obtain that $g \circ F$ is d.c. on $B(a, \delta) \cap A$. $\square$

Note that the case (c) follows also from proofs in [10]. However, a claim of P. Hartman (see [10], p. 708, lines 12–17), which would imply (via (I) from Introduction) that, in (c), it is sufficient to write "g is d.c. and Lipschitz" instead of "g is a difference of two Lipschitz convex functions", is false (presumably due to a misprint). This is shown by the following example.

Example 4.3. Let $d\colon \mathbb{R} \to \mathbb{R}$ be the characteristic function of the set $S := \bigcup_{n \in \mathbb{N}} [-2^{-2n+2}, -2^{-2n+1})$ and put $g(x) := \int_{-1}^{x} d$ for $x \in [-1, 0]$. First we will show that g is a Lipschitz d.c. function which is not a difference of two Lipschitz convex functions on $[-1/2, 0]$. Since d is bounded, g is clearly Lipschitz. Clearly $g'_+(x) = d(x)$, $x \in [-1, 0)$, since d is right continuous. For $x \in [-1, 0)$, let $v(x)$ be the total variation of d on the interval $[-1, x]$. It is easy to check that $v(x) = n - 1$ for $x \in [-2^{1-n}, -2^{-n})$, and consequently $\int_{-1}^{0} v = \sum_{n=1}^{\infty} (n-1) 2^{-n} < \infty$. Thus both v and $w := v - d$ are nondecreasing and (Lebesgue) integrable on $[-1, 0)$. So, $c_1(x) := \int_{-1}^{x} v$ and $c_2(x) := \int_{-1}^{x} w$ are continuous convex functions on $[-1, 0]$, and

$$g(x) = \int_{-1}^{x} d = \int_{-1}^{x} (v - w) = c_1(x) - c_2(x), \quad x \in [-1, 0].$$

Therefore, g is d.c. on $[-1, 0]$.

Now, suppose to the contrary that $g = p - q$ on $[-1/2, 0]$, where p, q are convex Lipschitz functions on $[-1/2, 0]$. It is well-known that then the right derivatives p'_+, q'_+ are finite, bounded and nondecreasing functions on $[-1/2, 0]$. Further $d = g'_+ = p'_+ - q'_+$ on $[-1/2, 0)$. Let $V_a^b \varphi$ denote the total variation of φ on $[a, b]$. Then, for $x \in [-1/2, 0)$,

$$\begin{aligned} v(x) - v(-1/2) &= V_{-1/2}^{x} (p'_+ - q'_+) \le V_{-1/2}^{x}\, p'_+ + V_{-1/2}^{x}\, q'_+ \\ &= \big(p'_+(x) - p'_+(-1/2)\big) + \big(q'_+(x) - q'_+(-1/2)\big) =: z(x), \end{aligned}$$

which is a contradiction, since $\lim_{x \to 0-} v(x) = \infty$ and z is a bounded function.

Now, set $F(x) := -|x|$ for $x \in [-1, 1]$. Then $g \circ F$ is not d.c. even on $(-1, 1)$. Indeed, otherwise $g \circ F$ would be a difference of two Lipschitz convex functions on $[-1/2, 0]$, which is not true, since $g \circ F = g$ on $[-1/2, 0]$.

5. The main counterexample

The main result of this section (Theorem 5.5) provides a general construction of non-d.c. composed mappings. Its proof uses some ideas from [14].

The following lemma, implicitly contained in [14], is useful for showing that certain functions or mappings are not d.c.

Lemma 5.1. *Let X, Y be normed linear spaces, let $A \subset X$ be an open convex set with $0 \in A$, and let $F\colon A \to Y$ be a mapping. Suppose there exist $\lambda \in (0,1)$ and a sequence of balls $B(x_n, \delta_n) \subset A$ such that $\{x_n\} \subset \lambda A$, $\delta_n \to 0$ and F is unbounded on each $B(x_n, \delta_n)$. Then F is not d.c. on A.*

Proof. Suppose the contrary. Let f be a control function for F on A. We can suppose $f \geq 0$ (otherwise choose an affine function g such that $g \leq f$ on A, and consider $f - g$ instead of f). For each n, let $z_n \in A$ be such that $x_n = \lambda z_n$. Observe that $\|h\| < \delta_n$ implies $x_n + h = \lambda z_n + (1-\lambda)\frac{h}{1-\lambda}$ and $\|\frac{h}{1-\lambda}\| < \frac{\delta_n}{1-\lambda}$. Now, fix $m \in \mathbb{N}$ so large that $B(0, \frac{\delta_m}{1-\lambda}) \subset A$ and both F and f are bounded on $B(0, \frac{\delta_m}{1-\lambda})$. Then, using [16, Proposition 1.13] or [7, Theorem 1], we get

$$\left\|\lambda F(z_m) + (1-\lambda)F\left(\frac{h}{1-\lambda}\right) - F(x_m+h)\right\| \tag{1}$$

$$\leq \lambda f(z_m) + (1-\lambda)f\left(\frac{h}{1-\lambda}\right) - f(x_m+h)$$

$$\leq \lambda f(z_m) + (1-\lambda)f\left(\frac{h}{1-\lambda}\right) \tag{2}$$

whenever $\|h\| < \delta_m$. But this is a contradiction since the expression (2) is bounded on $\{h : \|h\| < \delta_m\}$ while (1) is not (because F is unbounded on $B(x_m, \delta_m)$). □

Lemma 5.2. *Let X be a normed linear space. Let $e \in S_X$, $e^* \in S_{X^*}$ and $c > 0$ be such that $e^*(e) = 1$ and the implication*

$$e^*(u) > 1 - \varepsilon \quad \textit{and} \quad \|u\| \leq 1 \quad \Rightarrow \quad \|u - e\| \leq c\varepsilon \tag{3}$$

holds for $u \in X$ and $\varepsilon > 0$. Then the following implication holds for $x \in X$ and $0 < \delta < \frac{1}{2}$:

$$\frac{1}{2}\|x\|^2 < \frac{1}{2}\|e\|^2 + e^*(x-e) + \delta \quad \Rightarrow \quad \|x - e\| < (1+2c)\sqrt{2\delta}\,. \tag{4}$$

Proof. Let $x \in X$ and $0 < \delta < \frac{1}{2}$ satisfy the left-hand side of (4). Then

$$\frac{1}{2}\|x\|^2 < e^*(x) - \frac{1}{2} + \delta \leq \|x\| - \frac{1}{2} + \delta$$

which implies $\frac{1}{2}(1 - \|x\|)^2 < \delta$. Thus $0 < 1 - \sqrt{2\delta} < \|x\| < 1 + \sqrt{2\delta}$.

If $\|x\| \leq 1$, then $e^*(x) > \frac{1}{2}\|x\|^2 + \frac{1}{2} - \delta > \frac{1}{2}(1-\sqrt{2\delta})^2 + \frac{1}{2} - \delta = 1 - \sqrt{2\delta}$. By the assumption (3), $\|x - e\| \leq c\sqrt{2\delta} < (1+2c)\sqrt{2\delta}$.

If $\|x\| > 1$, then (as above) $e^*(\frac{x}{\|x\|}) > \frac{1}{\|x\|}\left(\frac{1}{2}\|x\|^2 + \frac{1}{2} - \delta\right) > \frac{1-\sqrt{2\delta}}{\|x\|} > \frac{1-\sqrt{2\delta}}{1+\sqrt{2\delta}} = 1 - \frac{2\sqrt{2\delta}}{1+\sqrt{2\delta}}$. By (3), we have $\|\frac{x}{\|x\|} - e\| \le c\,\frac{2\sqrt{2\delta}}{1+\sqrt{2\delta}}$. Consequently, $\|x-e\| \le \|x - \frac{x}{\|x\|}\| + \|\frac{x}{\|x\|} - e\| \le (\|x\|-1) + \frac{2c\sqrt{2\delta}}{1+\sqrt{2\delta}} < \sqrt{2\delta}\left(1 + \frac{2c}{1+\sqrt{2\delta}}\right) < (1+2c)\sqrt{2\delta}$. □

Lemma 5.3. *For each infinite dimensional normed linear space, there exists a countable biorthogonal system $\{e_n, e_n^*\} \subset X \times X^*$ such that:*

$$\|e_n\| = 1\ (n \in \mathbb{N}), \qquad R := \sup_n \|e_n^*\| < \infty, \qquad r := \inf_{m\neq n} \|e_m - e_n\| > 0.$$

Proof. The completion of X contains a normalized basic sequence $\{e_n\}$ (see [8, Theorem 6.14]). By the "small perturbation lemma" [8, Theorem 6.18], we may assume that $\{e_n\} \subset X$. Let e_n^* ($n \in \mathbb{N}$) be Hahn-Banach extensions of the corresponding coefficient functionals; it is well-known that they are equi-bounded (cf. [8, p. 164]). Moreover, for $m \neq n$, we have $\|e_n - e_m\| \ge 1/R$, since $1 = e_n^*(e_n - e_m) \le R\,\|e_n - e_m\|$. □

Lemma 5.4. *Let X, Y be normed linear spaces, X infinite dimensional. Then, for each bounded sequence $\{y_n\} \subset Y$, there exists a d.c. mapping $\Phi\colon X \to Y$ such that:*

(*a*) $\Phi = 0$ *outside* B_X*;*
(*b*) Φ *admits a control function that is Lipschitz on bounded sets;*
(*c*) $\{y_n\} \subset \Phi(B_X)$ *and* $\Phi(X) \subset \operatorname{conv}\big[\{0\} \cup \{y_n\}_{n\in\mathbb{N}}\big]$.

Proof. Let $\{e_n\}$, $\{e_n^*\}$, R and r be as in Lemma 5.3. Observe that $R \ge 1$ since $e_1^*(e_1) = 1$. Fix an arbitrary $\rho \in (0, \frac{1}{R})$. The symmetric closed convex set

$$C := \overline{\operatorname{conv}}\big(\rho B_X \cup \{\pm e_n\}_{n\in\mathbb{N}}\big)$$

is the unit ball of an equivalent norm $|\!|\!|\cdot|\!|\!|$ on X since $\rho B_X \subset C \subset B_X$.

Fix an arbitrary $n \in \mathbb{N}$. It is easy to see that $|\!|\!|e_n^*|\!|\!| = \max e_n^*(C) = e_n^*(e_n) = 1$, which implies that also $|\!|\!|e_n|\!|\!| = 1$. Let $\varepsilon > 0$ and $u \in C$ be such that

$$e_n^*(u) > 1 - \varepsilon.$$

Observe that $C = \operatorname{conv}(\{e_n\} \cup C_n)$ where

$$C_n = \overline{\operatorname{conv}}\big(\rho B_X \cup \{-e_k\}_{k\in\mathbb{N}} \cup \{e_k\}_{k\in\mathbb{N}\setminus\{n\}}\big).$$

Thus we can write $u = (1-\lambda)e_n + \lambda v$ where $v \in C_n$ and $0 \le \lambda \le 1$. Since

$$1 - \varepsilon < e_n^*(u) \le 1 - \lambda + \lambda \sup e_n^*(C_n) \le 1 - \lambda + \lambda R\rho,$$

we easily get $\lambda < \frac{\varepsilon}{1-R\rho}$. Consequently,

$$|\!|\!|u - e_n|\!|\!| = \lambda|\!|\!|v - e_n|\!|\!| \le \frac{2\varepsilon}{1-R\rho}.$$

Denote $g(x) = \frac{1}{2}|\!|\!|x|\!|\!|^2$. By Lemma 5.2, for $n \in \mathbb{N}$, $x \in X$ and $0 < \delta < \frac{1}{2}$ the following implication holds:

$$g(x) < g(e_n) + e_n^*(x - e_n) + \delta \quad\Rightarrow\quad \|x - e_n\| \le |\!|\!|x - e_n|\!|\!| < \left(1 + \frac{4}{1-R\rho}\right)\sqrt{2\delta}.$$

Since the sequence $\{e_n\}$ is uniformly discrete, it is possible to fix a $\delta \in (0, \frac{1}{2})$ so small that the open convex sets

$$D_n = \{x \in X : g(x) < g(e_n) + e_n^*(x - e_n) + \delta\}$$

satisfy $\operatorname{dist}_{\|\cdot\|}(D_m, D_n) > \delta$ whenever $m \neq n$. We have $e_n \in D_n$ for each n.

Define $H\colon X \to Y$ by

$$H(x) = \begin{cases} \frac{1}{\delta}\,[g(e_n) + e_n^*(x - e_n) + \delta - g(x)]\, y_n & \text{if } x \in D_n; \\ 0 & \text{for } x \notin \bigcup_{n\in\mathbb{N}} D_n. \end{cases}$$

It is easy to see that H is continuous since we have

$$H(x) = \frac{1}{\delta}\,[\max\{g(x), g(e_n) + e_n^*(x - e_n) + \delta\} - g(x)]\, y_n\,, \quad x \in D_n + \delta B_X. \tag{5}$$

Put $s := \sup_{n\in\mathbb{N}} \|y_n\|$. We claim that the formula

$$h(x) = \frac{s}{\delta}\, \sup_{n\in\mathbb{N}} \bigl(\max\{g(x), g(e_n) + e_n^*(x - e_n) + \delta\}\bigr) \; + \frac{s}{\delta}\, g(x) \tag{6}$$

defines a control function for H, which is Lipschitz on bounded sets. First, observe that $h(0) = \frac{s}{\delta}\max\{0, \frac{1}{2} - 1 + \delta\} = 0$. Moreover, since g is Lipschitz on bounded sets and the functionals e_n^* ($n \in \mathbb{N}$) are equi-Lipschitz, (6) defines a real convex function that is Lipschitz on bounded sets. Fix $y^* \in B_{Y^*}$. To prove that the function $\psi := y^* \circ H + h$ is convex, it is sufficient to show that it is locally convex. For $x \notin \bigcup_n \overline{D_n} = \overline{\bigcup_n D_n}$, we have $\psi(x) = h(x)$. For $x \in D_n + \delta B_X$, we have $g(x) \geq g(e_k) + e_k^*(x - e_k) + \delta$ whenever $k \neq n$, and hence

$$h(x) = \frac{s}{\delta}\, \max\{g(x), g(e_n) + e_n^*(x - e_n) + \delta\} \; + \frac{s}{\delta}\, g(x)\,, \quad x \in D_n + \delta B_X.$$

Consequently, (5) implies that, on the set $D_n + \delta B_X$, the function

$$\psi(x) = \frac{s + y^*(y_n)}{\delta}\, \max\{g(x), g(e_n) + e_n^*(x - e_n) + \delta\} \; + \frac{s - y^*(y_n)}{\delta}\, g(x)$$

is convex (since it is a sum of convex functions).

Observe that $H(e_n) = y_n$. Moreover, for each $x \in D_n$,

$$\begin{aligned} 0 < g(e_n) + e_n^*(x - e_n) + \delta - g(x) &\leq \frac{1}{2} + \|x\| - 1 + \delta - \frac{1}{2}\|x\|^2 \\ &= \delta - \frac{1}{2}\bigl(\|x\| - 1\bigr)^2 \leq \delta\,. \end{aligned}$$

Thus, for each n, the image $H(D_n)$ is contained in the segment $[0, y_n]$. Since the support of H is contained in $2B_X$, the mapping $\Phi(x) := H(2x)$ has all the required properties (note that $\varphi(x) := h(2x)$ clearly controls Φ, cf. [16, Lemma 1.5]). $\square$

Theorem 5.5. *Let X, Y, Z be normed linear spaces, X infinite dimensional. Let $A \subset X$ be an open convex set, let $B \subset Y$ be a convex set, and let $G\colon B \to Z$ be a mapping which is unbounded on a bounded subset of B. Then there exists a d.c. mapping $F\colon A \to B$ such that $G \circ F$ is not d.c. on A.*

Proof. We can (and do) suppose that $0 \in A$. Fix $r \in (0,1)$ such that $B(0,2r) \subset A$. By [2], there exists a continuous convex function h on X such that $h(0) = 0$ and $\sup_{x \in B(0,r)} h(x) = \infty$. For $k \in \mathbb{N}$, set

$$A_k := \{x \in A : \ h(x) < k, \ \|x\| < k\}.$$

Clearly each A_k contains 0, is open and convex; moreover, $A_k \nearrow A$. It is easy to see that, for each $k \in \mathbb{N}$, we can choose $v_k \in B(0,r)$ and $0 < \delta_k < 1/k$ such that $B(v_k, 2\delta_k) \subset A_{k+1} \setminus A_k$.

We can (and do) suppose that $0 \in B$. Let $\{y_n\} \subset B$ be a bounded sequence such that $\|G(y_n)\| \to \infty$, and let Φ be the corresponding mapping from Lemma 5.4. For each $k \in \mathbb{N}$, define $F_k\colon X \to Y$ by

$$F_k(x) = \Phi\left(\frac{x - v_k}{\delta_k}\right).$$

Since the supports of these mappings are pairwise disjoint and each A_k intersects only finitely many of them, the mapping

$$F\colon A \to Y\,, \qquad F(x) := \sum_{k \in \mathbb{N}} F_k(x)$$

is well-defined and continuous. Observing that $\varphi_k(x) := \varphi(\frac{x - v_k}{\delta_k})$ controls F_k if φ controls Φ (cf. [16, Lemma 1.5]), we obtain that F is d.c. on each A_k with a Lipschitz (hence bounded) control function. By Proposition 2.4, F is d.c. on A. Moreover, $F(A) \subset \bigcup_k F_k(X) \subset B$ by Lemma 5.4(*c*). Since $G \circ F$ is unbounded on each $B(v_k, \delta_k)$ and $v_k \in \frac{1}{2}A$, Lemma 5.1 implies that $G \circ F$ is not d.c. on A. □

Corollary 5.6. *Let X be an infinite dimensional normed linear space, and $A \subset X$ a nonempty open convex set.*

(*a*) *There exists a positive d.c. function f on A such that $1/f$ is not d.c.*
(*b*) *There exists a locally d.c. function g on A, which is not d.c.*

Proof. Applying Theorem 5.5 with $B = (0, \infty)$ and $G(y) = 1/y$, we obtain (*a*). Now, (*b*) follows from (*a*), since $g := 1/f$ is locally d.c. by Proposition 3.1 (or (II) in Introduction). □

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