Journal of Convex Analysis
Volume 16 (2009), No. 1, 165–168

A New Proof of the Maximal Monotonicity of Subdifferentials

S. Simons
*Department of Mathematics, University of California,
Santa Barbara, CA 93106-3080, USA*
simons@math.ucsb.edu

Received: August 9, 2007

We give a new proof based on the recent very elegant argument of Marques Alves and Svaiter that the subdifferential of a proper, convex lower semicontinuous function on a real Banach space is maximally monotone. We also show how the argument can be simplified in the reflexive case.

1. Basic notation and preliminary results

We suppose throughout that E is a nonzero real Banach space. In this note, we give a new proof of *Rockafellar's maximal monotonicity theorem* (first proved in [5]) that the subdifferential of a proper, convex lower semicontinuous function on E is maximally monotone. Our proof is based on the very elegant argument by M. Marques Alves and B. F. Svaiter, which appeared recently in [2]. We refer the reader to Remark 2.2 for comparisons between the argument given here and those of [2] and [6], and to Remark 2.3 for an explanation of how the argument can be shortened when E is reflexive.

We recall that if $f\colon E \to]-\infty,\infty]$ then f is said to be *proper* if there exists $x \in E$ such that $f(x) \in \mathbb{R}$. If $f\colon E \to]-\infty,\infty]$ is proper and E^* is the norm–dual of E then $f^*\colon E^* \to]-\infty,\infty]$ is defined by $f^*(x^*) := \sup_E [x^* - f]$. If $f\colon E \to]-\infty,\infty]$ is proper and convex then the multifunction $\partial f\colon\ E \rightrightarrows E^*$ is defined by: $x^* \in \partial f(x)$ exactly when

$$y \in E \quad\Longrightarrow\quad f(x) + \langle y - x, x^*\rangle \le f(y).$$

This is easily seen to be equivalent to the statement that

$$f(x) + f^*(x^*) = \langle x, x^*\rangle.$$

We write $G(\partial f)$ for the *graph* of ∂f, that is to say

$$G(\partial f) := \big\{(x,x^*) \in E \times E^*\colon\ f(x) + f^*(x^*) = \langle x, x^*\rangle\big\}.$$

To say that ∂f is *maximal monotone* means that if $(t,t^*) \in E \times E^*$ and

$$(s,s^*) \in G(\partial f) \quad\Longrightarrow\quad \langle s - t, s^* - t^*\rangle \ge 0 \tag{1}$$

then

$$(t,t^*) \in G(\partial f). \tag{2}$$

ISSN 0944-6532 / $ 2.50

We now state the three basic results from convex analysis that we will use. Lemma 1.1 follows from the Fenchel–Moreau theorem, which was first proved in [3], Section 5–6, pp. 26–39; Lemma 1.2 follows from Rockafellar, [4], Theorem 3(a), p. 85; Lemma 1.3 is the Brøndsted–Rockafellar theorem, which was first proved in [1], p. 608.

Lemma 1.1. *Let $k\colon E \to]-\infty,\infty]$ be proper, convex and lower semicontinuous. Then k^* is proper.*

Lemma 1.2. *Let $k\colon E \to]-\infty,\infty]$ be proper and convex and $g\colon E \to \mathbb{R}$ be convex and continuous. Then, for all $x \in E$, $\partial(k+g)(x) = \partial k(x) + \partial g(x)$.*

Lemma 1.3. *Let $h\colon E \to]-\infty,\infty]$ be proper, convex and lower semicontinuous, $\alpha,\beta > 0$, $(x,x^*) \in E \times E^*$ and $h(x) + h^*(x^*) \le \langle x,x^*\rangle + \alpha\beta$. Then there exists $(z,z^*) \in G(\partial h)$ such that $\|z-x\| \le \alpha$ and $\|z^*-x^*\| \le \beta$.*

2. The main result

Theorem 2.1. *Let $f\colon E \to]-\infty,\infty]$ be proper, convex and lower semicontinuous. Then $\partial f\colon\ E \rightrightarrows E^*$ is maximally monotone.*

Proof. Let $(t,t^*) \in E \times E^*$ and (1) be satisfied. Let $k := f(\cdot + t)$, $g := \frac{1}{2}\|\cdot\|^2$, and $h := k+g$. It is well known that $g^* = \frac{1}{2}\|\cdot\|^2$ (on E^*). From Lemma 1.2, for all $x \in E$,

$$\partial h(x) = \partial k(x) + \partial g(x) = \partial f(x+t) + Jx, \tag{3}$$

where $J\colon\ E \rightrightarrows E^*$ is the *duality map*. The properties of J that we will need (which are easy to check by direct computation) are that

$$J0 = \{0\} \quad \text{and} \quad x^* \in Jx \Longrightarrow \langle x,x^*\rangle = \|x\|^2. \tag{4}$$

From Lemma 1.1, there exists $y^* \in E^*$ such that $k^*(y^*) \in \mathbb{R}$. It is easily seen that, for all $x \in E$,

$$\left.\begin{aligned} \langle x,t^*\rangle - h(x) &= \langle x,t^*\rangle - k(x) - g(x)\\ &\le \langle x,t^*\rangle - \langle x,y^*\rangle + k^*(y^*) - g(x) = k^*(y^*) + \langle x,t^*-y^*\rangle - g(x)\\ &\le k^*(y^*) + g^*(t^*-y^*) = k^*(y^*) + \tfrac{1}{2}\|t^*-y^*\|^2 < \infty, \end{aligned}\right\} \tag{5}$$

and so $h^*(t^*) < \infty$. Consequently, for all $n \ge 1$, there exists $x_n \in E$ such that

$$\langle x_n,t^*\rangle - h(x_n) \ge h^*(t^*) - 1/n^2. \tag{6}$$

Lemma 1.3 now gives $(z_n,z_n^*) \in G(\partial h)$ such that

$$\|z_n - x_n\| \le 1/n \quad \text{and} \quad \|z_n^* - t^*\| \le 1/n, \tag{7}$$

and (3) gives $y_n^* \in Jz_n$ such that $(z_n+t, z_n^*-y_n^*) \in G(\partial f)$. From (1),

$$\langle z_n, z_n^* - y_n^* - t^*\rangle \ge 0,$$

and so $\langle z_n,y_n^*\rangle \le \langle z_n,z_n^*-t^*\rangle$. From (4), $\langle z_n,y_n^*\rangle = \|z_n\|^2$, thus (7) implies that $\|z_n\|^2 \le \|z_n\|/n$, from which $\|z_n\| \le 1/n$ and so, using (7) again, $\|x_n\| \le 2/n$, thus $x_n \to 0$ as $n \to \infty$. Passing to the limit in (6) and using the lower semicontinuity of h, $h(0) + h^*(t^*) \le 0$, from which $t^* \in \partial h(0)$. Using (3) and (4) again,

$$t^* \in \partial f(t) + J0 = \partial f(t).$$

This completes the proof of (2) and, consequently, also that of Theorem 2.1. $\square$

Remark 2.2. We now compare the argument that we have given above in Theorem 2.1 with those of [6], Chapter VII, pp. 111–139 and [2]. The argument of Theorem 2.1 is completely analytic, while that of [6], Theorem 29.4, pp. 116–118 is much harder and has a much more geometric feel. On the other hand, this latter argument leads to the more general results in the later parts of [6], Chapter VII. We now know that all these more general results can be established using the results on maximal monotone multifunctions of type (ED) established in [7]. This being the case, there is obviously an incentive to give the simplest possible proof of Theorem 2.1. The proof given here is based on the very elegant one found recently by M. Marques Alves and B. F. Svaiter in [2], but is structurally simpler, and exploits the properties of subdifferentials and the duality map to avoid some of the computations in [2].

Remark 2.3. (5) implies that, for all $x \in E$,

$$\langle x, t^* \rangle - h(x) \geq h^*(t^*) - 1 \quad \Longrightarrow \quad \tfrac{1}{2}\|x\|^2 - \|x\|\|t^* - y^*\| \leq 1 + k^*(y^*) - h^*(t^*).$$

Thus

$$\big\{x \in E\colon\ \langle x, t^* \rangle - h(x) \geq h^*(t^*) - 1\big\} \text{ is a bounded subset of } E.$$

So if E is reflexive, it follows from a standard weak compactness argument that $t^* - h$ attains its maximum on E, that is to say there exists $z \in E$ such that $\langle z, t^* \rangle - h(z) = h^*(t^*)$, from which $t^* \in \partial h(z)$, and so (3) gives $y^* \in Jz$ such that $(z + t, t^* - y^*) \in G(\partial f)$. Then (1) and (4) imply that $\langle z, -y^* \rangle \geq 0$ and $\langle z, y^* \rangle = \|z\|^2$, from which $z = 0$. Thus $t^* \in \partial h(0)$, and the rest of the proof of Theorem 2.1 proceeds as before. Thus Lemma 1.3 is not needed if E is reflexive.

Acknowledgements. It was pointed out to the author by L. Thibault that there is also a proof of the result of Theorem 2.1 by him using the calculus rule established in [8], and an unpublished proof by D. Zagrodny using the mean value theorem established in [9].

References

[1] A. Brøndsted, R. T. Rockafellar: On the subdifferentiability of convex functions, Proc. Amer. Math. Soc. 16 (1965) 605–611.

[2] M. Marques Alves, B. F. Svaiter: A new proof for maximal monotonicity of subdifferential operators, J. Convex Analysis 15 (2008) 345–348.

[3] J.-J. Moreau: Fonctionelles convexes, Séminaire sur les équations aux derivées partielles, Lecture Notes, Collège de France, Paris (1966–67).

[4] R. T. Rockafellar: Extension of Fenchel's duality theorem for convex functions, Duke Math. J. 33 (1966) 81–89.

[5] R. T. Rockafellar: On the maximal monotonicity of subdifferential mappings, Pac. J. Math. 33 (1970) 209–216.

[6] S. Simons: Minimax and monotonicity, Lecture Notes in Mathematics 1693, Springer, Berlin (1998).

[7] S. Simons: Five kinds of maximal monotonicity, Set-Valued Anal. 9 (2001) 391–409.

[8] L. Thibault: Limiting convex subdifferential calculus with applications to integration and maximal monotonicity of subdifferential, in: Constructive, Experimental, and Nonlinear

Analysis (Limoges, 1999), M. Théra (ed.), CMS Conf. Proc. 27, AMS, Providence (2000) 279–289.

[9] D. Zagrodny: Approximate mean value theorem for upper subderivatives, Nonlinear Anal., Theory Methods Appl. 12 (1988) 1413–1428.