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Weak and Entropy Solutions to Nonlinear Elliptic Problems with Variable Exponent

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We study the boundary value problem $-div(a(x,\nabla u)) = f(x,u)$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $div(a(x,\nabla u))$ is a $p(x)$-Laplace type operator. We obtain the existence and uniqueness of an entropy solution for L^1-data f independent of u, the existence of weak energy solution for general data f dependent of u where the variable exponent $p(.)$ is not necessarily continuous.

Keywords: Generalized Lebesgue-Sobolev spaces; weak energy solution; entropy solution; $p(x)$-Laplace operator; electrorheological fluids

1. Introduction

Consider the nonlinear Dirichlet boundary value problem

$$\begin{cases} -div(a(x,\nabla u)) = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N > 1)$ with smooth boundary.
One of the common approaches to the weak solvability of the problem (1) when f is independent of u is based on the Browder Theorem and assumes the following Leray-Lions type conditions are satisfied (see [4, 19]):

(H_1): The function $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (continuous in ξ for a.e. $x \in \Omega$ and measurable in x for every $\xi \in \mathbb{R}^N$) and there exist $p \in (1, N)$, $\lambda > 0$ such that

$$a(x,\xi).\xi \geq \lambda\,|\xi|^p$$

holds for every $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

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(H_2): For every ξ and $\eta \in \mathbb{R}^N$, $\xi \neq \eta$, and a.e. $x \in \Omega$ there holds

$$(a(x,\xi) - a(x,\eta)).(\xi - \eta) > 0.$$

(H_3): There exists $\Lambda > 0$ such that

$$|a(x,\xi)| \leq \Lambda(j(x) + |\xi|^{p-1})$$

holds for every $\xi \in \mathbb{R}^N$ with $j \in L^{p'}(\Omega)$, $p' = p/(p-1)$. It is then natural to look for a weak solution in the Sobolev space $W_0^{1,p}(\Omega)$.

Consider a more general situation, when $\Omega = \Omega_1 \cup \Omega_2$, $1 < p_1 < p_2 < N$, and conditions $(H_1) - (H_3)$ are satisfied with p_i on Ω_i. If we simply use the above scheme to find the weak solution of (1) in $W^{k,p}(\Omega)$, we see that the validity of conditions (H_1)–(H_3) requires $p = max\{p_1, p_2\}$ and $p = min\{p_1, p_2\}$, respectively. Even more difficult situations occur when p is a function of $x \in \Omega$.

If p is a function of $x \in \Omega$, appropriate analogues of the Lebesgue spaces L^p and of the Sobolev spaces $W^{k,p}$ may be suggested. It is clear that we cannot simply replace p by $p(x)$ in the usual definition of the norm in L^p. Kovacik and Rakosnik in [16] studied the spaces $L^{p(.)}(\Omega)$ and $W^{k,p(.)}(\Omega)$ where they proved that $L^{p(.)}(\Omega)$ and $L^p(\Omega)$ have many common properties except a very important one: the p-mean continuity. In general, $L^{p(.)}(\Omega)$ is not invariant with respect to translation (cf. [16, Ex. 2.9]). They also showed that, in general, the Sobolev space $W^{1,p(.)}(\Omega)$ is not embedded in $L^{p^*(.)}(\Omega)$, where $p^*(.) = \frac{Np(.)}{N-p(.)}$ is the variable Sobolev exponent (see [16, Ex. 3.2]), but when p is continuous, they showed that the embedding $W^{1,p(.)}(\Omega)$ in $L^{q(.)}(\Omega)$ for $1 \leq q(.) \leq p^*(.) - \epsilon$ holds true for some $\epsilon > 0$. Ruzicka (cf. [26]) proved another interesting result by considering the level sets of p and using the power series expansion of the exponential function. Afterwards, Edmunds and Rakosnik showed (cf. [9]) that if $p(.)$ is a Lipschitz function such that $\sup_\Omega p(.) < N$, then the embedding $W^{1,p(.)}(\Omega)$ in $L^{p^*(.)}(\Omega)$ holds true. To compare the three results mentioned above in the study of Lebesgue and Sobolev spaces with variable exponent, we first note that each concerns a different class of functions $p(.)$. The function $p(.)$ in [16] is assumed only continuous but the target space is rather far from the desired optimal case. The function $p(.)$ in [26] can be even discontinuous but there is the Logarithmic defect on the estimation. On the other hand, Lipschitz (and even C^∞) functions $p(.)$ do not, in general satisfy the assumptions in [26]. In [10], Edmunds and Rakosnik improved their result in [9] by showing that if $p(.) : \overline{\Omega} \to [1, N)$ is a function in $W^{1,\sigma}(\Omega)$ for a $\sigma \in (N, \infty)$ and such that $\sup_\Omega p(.) < N$, the embedding $W^{1,p(.)}(\Omega)$ in $L^{p^*(.)}(\Omega)$ holds true. Recently, Diening (cf. [7]) improved the result by Edmunds and Rakosnik in [10] by showing that for a merely log-Hölder continuous $p(.)$, the embedding $W^{1,p(.)}(\Omega)$ in $L^{p^*(.)}(\Omega)$ holds true and that whenever $q(.) \leq p^*(.) - \epsilon$ for some $\epsilon > 0$, the compact embedding $W^{1,p(.)}(\Omega)$ in $L^{q(.)}(\Omega)$ holds true.

The interest of the study of Lebesgue and Sobolev spaces with variable exponent lies on the fact that most materials can be modelled with sufficient accuracy using classical Lebesgue and Sobolev spaces L^p and $W^{1,p}$ where p is a fixed constant, but for some materials with inhomogeneities, for instance electrorheological fluids (sometimes referred to as “smart fluids”), this is not adequate, but rather the exponent p should be able to vary (cf. [26]). These fluids are smart materials which are concentrated suspensions of polarizable particles in a non-conducting dielectric liquid. By applying an electric field, the viscosity can be changed by a factor up to 10^5, and the fluid can be transformed

from liquid state into semi-solid state within milliseconds. The process is reversible. An example of electrorheological fluids are alumina Al_2O_3 particles.
Using the results by Edmunds and Rakosnik (cf. [9]), Sanchon and Urbano (cf. [27]) studied problem (1) for $f \in L^1(\Omega)$ independent of u and showed that this problem has a unique entropy solution with some regularities results of the entropy solution and then, extended the results by Bénilan et al (cf. [4]) for variable exponent case. Their work was done under the following condition on $p(.)$:

(H_4): $p(.)$ is a measurable function such that

$$p(.) \in W^{1,\infty}(\Omega) \quad \text{and} \quad 1 < ess \inf_{x\in\Omega} p(x) \leq ess \sup_{x\in\Omega} p(x) < N.$$

Assumption (H_4) allowed them, in particular to exploit the embeddings Theorems of Lebesgue and Sobolev spaces with variable exponent (as in the constant exponent case) arising in the study of problem (1). Note also that in (H_4), the Lipschitz condition allowed Sanchon and Urbano in particular to perform some estimates needed for existence, uniqueness and regularities results in [27] since they can differentiate $p(.)$. In this paper, we study problem (1) with less regularity on the variable exponent $p(.)$, more precisely, we assume that

$$\begin{cases} p(.) : \overline{\Omega} \to \mathbb{R} \text{ is a measurable function such that} \\ 1 < ess \inf\limits_{x\in\Omega} p(x) \leq ess \sup\limits_{x\in\Omega} p(x) < +\infty. \end{cases} \tag{2}$$

For the vector fields $a(.,.)$, we assume (cf. [21]) that $a(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is the continuous derivative with respect to ξ of the mapping $A : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $A = A(x,\xi)$, i.e. $a(x,\xi) = \nabla_\xi A(x,\xi)$ such that:

- The following equality holds
$$A(x,0) = 0, \tag{3}$$
for almost every $x \in \Omega$.
- There exists a positive constant C_1 such that
$$|a(x,\xi)| \leq C_1 \left(j(x) + |\xi|^{p(x)-1} \right) \tag{4}$$
for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, where j is a nonnegative function in $L^{p'(.)}(\Omega)$, with $1/p(x) + 1/p'(x) = 1$.
- The following inequalities hold
$$\left(a(x,\xi) - a(x,\eta)\right).(\xi - \eta) > 0, \tag{5}$$
for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$, with $\xi \neq \eta$, and
$$|\xi|^{p(x)} \leq a(x,\xi).\xi \leq p(x)A(x,\xi) \tag{6}$$
for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$.

As examples of models with respect to above assumptions, we can give the following:

(i) Set $A(x,\xi) = (1/p(x))\,|\xi|^{p(x)}$, $a(x,\xi) = |\xi|^{p(x)-2}\,\xi$ where $p(x) \geq 2$. Then we get the $p(x)$-Laplace operator

$$div(|\nabla u|^{p(x)-2}\,\nabla u).$$

(ii) Set $A(x,\xi) = (1/p(x))\left[\left(1+|\xi|^2\right)^{p(x)/2} - 1\right]$, $a(x,\xi) = \left(1+|\xi|^2\right)^{(p(x)-2)/2}\xi$, where $p(x) \geq 2$. Then we obtain the generalized mean curvature operator

$$div\left(\left(1+|\nabla u|^2\right)^{(p(x)-2)/2}\nabla u\right).$$

Remark 1.1. (a) Since for almost every $x \in \Omega, a(x,.)$ is a gradient and is monotone then the primitive $A(x,.)$ of $a(x,.)$ is necessarily convex.

(b) Stationary PDE's with operator A satisfying (3)–(6) (as well as examples (i) and (ii)) have been studied for the first time by Mihailescu and Radulescu (cf. [21]) in the framework of existence and multiplicity of weak solutions.
The novelty of this work is on three orders. Firstly, note that in [27], for the proof of the existence of entropy solution, the authors used a classical approximation method and assumed that the approximated problem is well posed by the work of Fan and Zhang (cf. [11, Theorem 4.2]). But in [11], the authors studied the model case of example (i) above which is a particular case of the problem considered in [27]. Consequently, to avoid this error, we have made in this work some additionnal assumptions on the vector field $a(.,.)$ (cf. assumptions (3) and (6)) in order to study first, existence result of (1) when the right-hand side $f = f(x) \in L^\infty(\Omega)$ which will permit us following [27] to use approximation method in a right way. Secondly, the exponent $p(.)$ is assumed less regular than in [27] since it can be discontinuous. Thirdly it is the study of weak energy solution when the right-hand side in (1) depends on u.
The paper contains five sections. In Section 2, we recall the definitions of Lebesgue and Sobolev spaces with variable exponent and some of their properties. In Section 3, we prove the existence and uniqueness of weak energy solution of (1) when the right-hand side f is independent of u and has enough integrability i.e., $f \in L^\infty(\Omega)$. Using the results of Section 3, we study in Section 4, the question of the existence and uniqueness of entropy solution of (1) for $f \in L^1(\Omega)$ and independent of u. Finally, in Section 5, we prove some existence results of weak energy solution of problem (1) for an f assuming to be dependent of u.

2. Lebesgue and Sobolev spaces with variable exponent

In this section, we define Lebesgue and Sobolev spaces with variable exponent and give some of their properties. Since $p(.)$ is not continuous or merely log-Hölder continuous, we don't expect embedding Theorems (cf. [16, Ex. 3.2]).
Given a measurable function $p(.) : \Omega \to [1,\infty)$, we will use the following notation throughout the paper:

$$p_- := ess\inf_{x\in\Omega} p(x) \quad \text{and} \quad p_+ := ess\sup_{x\in\Omega} p(x).$$

We define the Lebesgue space with variable exponent $L^{p(.)}(\Omega)$ as the set of all measurable function $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)}\,dx$$

is finite. If the exponent is bounded, i.e., if $p_+ < \infty$, then the expression

$$|u|_{p(.)} := \inf\left\{\lambda > 0 : \rho_{p(.)}(u/\lambda) \leq 1\right\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(\cdot)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Moreover, if $p_- > 1$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\left|\int_\Omega uvdx\right| \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) |u|_{p(.)} |v|_{p'(.)}, \tag{7}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.
Now, let

$$W^{1,p(.)}(\Omega) := \left\{u \in L^{p(.)}(\Omega) : |\nabla u| \in L^{p(.)}(\Omega)\right\},$$

which is a Banach space equipped with the norm

$$\|u\|_{1,p(.)} := |u|_{p(.)} + |\nabla u|_{p(.)}.$$

Next, we define $W_0^{1,p(.)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$ under the norm

$$\|u\| := |\nabla u|_{p(.)}.$$

The space $\left(W_0^{1,p(.)}(\Omega), \|u\|\right)$ is a separable and reflexive Banach space.
An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(.)}(\Omega)$.
We have the following result (cf. [12]):

Lemma 2.1. *If $u_n, u \in L^{p(.)}(\Omega)$ and $p_+ < +\infty$ then the following relations hold:*

(*i*) $|u|_{p(.)} > 1 \Rightarrow |u|_{p(.)}^{p_-} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p_+}$;

(*ii*) $|u|_{p(.)} < 1 \Rightarrow |u|_{p(.)}^{p_+} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p_-}$;

(*iii*) $|u|_{L^{p(\cdot)}(\Omega)} < 1$ (*respectively* $= 1; > 1$) $\Leftrightarrow \rho_{p(.)}(u) < 1$ (*respectively* $= 1; > 1$);

(*iv*) $|u_n|_{L^{p(\cdot)}(\Omega)} \to 0$ (*respectively* $\to +\infty$) $\Leftrightarrow \rho_{p(.)}(u_n) \to 0$ (*respectively* $\to +\infty$);

(*v*) $\rho_{p(.)}\left(u/|u|_{L^{p(\cdot)}(\Omega)}\right) = 1$.

Let us introduce the following notation: given two bounded measurable functions $p(.), q(.) : \Omega \to \mathbb{R}$, we write

$$q(.) \ll p(.) \text{ if } ess \inf_{x\in\Omega}(p(x) - q(x)) > 0.$$

3. Existence and uniqueness of weak energy solution for $f \in L^\infty(\Omega)$

In this section, we study the weak energy solution of (1).

Definition 3.1. A weak solution of (1) is a function $u \in W_0^{1,1}(\Omega)$ such that $a(.,\nabla u) \in (L^1_{loc}(\Omega))^N$ and

$$\int_\Omega a(x,\nabla u).\nabla\varphi dx = \int_\Omega f(x)\varphi dx, \quad \text{for all } \varphi \in C_0^\infty(\Omega). \tag{8}$$

A weak energy solution is a weak solution such that $u \in W_0^{1,p(.)}(\Omega)$.

The main result of this section is the following

Theorem 3.2. *Assume* (2)–(6) *and* $f \in L^\infty(\Omega)$. *Then, there exists a unique weak energy solution of* (1).

Proof. $*$ **Existence**. Let E denote the generalized Sobolev space $W_0^{1,p(.)}(\Omega)$. Define the energy functional $I : E \to \mathbb{R}$ by

$$I(u) = \int_\Omega A(x,\nabla u)dx - \int_\Omega f u dx.$$

We first establish some basic properties of I.

Proposition 3.3. *The functional I is well-defined on E and $I \in C^1(E,\mathbb{R})$ with the derivative given by*

$$\langle I'(u),\varphi\rangle = \int_\Omega a(x,\nabla u).\nabla\varphi dx - \int_\Omega f\varphi dx,$$

for all $u,\varphi \in E$.

To prove Proposition 3.3, we define the functonal $\Lambda : E \to \mathbb{R}$ by

$$\Lambda(u) = \int_\Omega A(x,\nabla u)dx, \quad \text{for all } u \in E.$$

Lemma 3.4.

(*i*) *The functional Λ is well-defined on E.*
(*ii*) *The functional Λ is of class $C^1(E,\mathbb{R})$ and*

$$\langle \Lambda'(u),\varphi\rangle = \int_\Omega a(x,\nabla u).\nabla\varphi dx,$$

for all $u,\varphi \in E$.

Proof of Lemma 3.4. (*i*) For any $x \in \Omega$ and $\xi \in \mathbb{R}^N$, we have

$$A(x,\xi) = \int_0^1 \frac{d}{dt}A(x,t\xi)dt = \int_0^1 a(x,t\xi).\xi dt.$$

Then by (4),

$$A(x,\xi) \le C_1 \int_0^1 \left(j(x) + |\xi|^{p(x)-1}\, t^{p(x)-1}\right)|\xi|\, dt \le C_1 j(x)\,|\xi| + \frac{C_1}{p(x)}\,|\xi|^{p(x)}.$$

The above inequality and (6) imply

$$0 \leq \int_\Omega A(x,\nabla u)dx \leq C_1 \int_\Omega j(x)\,|\nabla u|\,dx + \frac{C_1}{p_-}\int_\Omega |\nabla u|^{p(x)}\,dx, \quad \text{for all } u \in E.$$

Using (7) and Lemma 2.1, we deduce that Λ is well-defined on E.

(ii) Existence of the Gâteaux derivative. Let $u, \varphi \in E$. Fix $x \in \Omega$ and $0 < |r| < 1$. Then by the mean value Theorem, there exists $\nu \in [0,1]$ such that

$$\begin{aligned} &|a(x,\nabla u(x) + \nu r \nabla\varphi(x))|\,|\nabla\varphi(x)| = |A(x,\nabla u(x) + r\nabla\varphi(x)) - A(x,\nabla u(x))|\,/\,|r| \\ \leq\ & \left[C_1 j(x) + C_1 2^{p_+}(|\nabla u(x)|^{p(x)-1} + |\nabla\varphi(x)|^{p(x)-1})\right]|\nabla\varphi(x)|\,. \end{aligned}$$

Next, by (7), we have

$$\int_\Omega C_1 j(x)\,|\nabla\varphi(x)|\,dx \leq \beta\,|C_1 j|_{p'(x)}\,.\,|\nabla\varphi|_{p(x)}$$

and

$$\int_\Omega |\nabla u|^{p(x)-1}\,|\nabla\varphi|\,dx \leq \alpha\left||\nabla u|^{p(x)-1}\right|_{p'(x)}.\,|\nabla\varphi|_{p(x)}\,.$$

The above inequalities imply

$$C_1\left[j(x) + 2^{p_+}\left(|\nabla u(x)|^{p(x)-1} + |\nabla\varphi(x)|^{p(x)-1}\right)\right]|\nabla\varphi(x)| \in L^1(\Omega).$$

It follows from the Lebesgue Theorem that

$$\langle\Lambda'(u),\varphi\rangle = \int_\Omega a(x,\nabla u).\nabla\varphi dx.$$

Assuming now $u_n \to u$ in E. Let us define $\psi(x,u) = a(x,\nabla u)$. Using assumption (4), Theorems 4.1 and 4.2 in [17], we deduce that $\psi(x,u_n) \to \psi(x,u)$ in $(L^{p'(x)}(\Omega))^N$. By (7), we obtain

$$|\langle\Lambda'(u_n) - \Lambda'(u),\varphi\rangle| \leq C\,|\psi(x,u_n) - \psi(x,u)|_{p'(x)}\,|\nabla\varphi|_{p(x)}$$

and so

$$\|\Lambda'(u_n) - \Lambda'(u)\| \leq C\,|\psi(x,u_n) - \psi(x,u)|_{p'(x)} \to 0, \quad \text{as } n \to \infty.$$

The proof of Lemma 3.4 is complete.

By Lemma 3.4, it is clear that Proposition 3.3 holds true and then, the proof of Proposition 3.3 is also complete.

Lemma 3.5. *The functional Λ is weakly lower semi-continuous.*

Proof of Lemma 3.5. By Corollary III.8 in [5], it is enough to show that Λ is lower semi-continuous. For this, fix $u \in E$ and $\epsilon > 0$. Since Λ is convex (by Remark 1.1 (a)), we deduce that for any $v \in E$, the following inequality holds

$$\int_\Omega A(x,\nabla v)dx \geq \int_\Omega A(x,\nabla u)dx + \int_\Omega a(x,\nabla u).(\nabla v - \nabla u)dx.$$

Using (4) and (7), we have

$$\begin{aligned}
&\int_\Omega A(x,\nabla v)dx \geq \int_\Omega A(x,\nabla u)dx - \int_\Omega |a(x,\nabla u)|\,|\nabla v - \nabla u|\,dx \\
&\geq \int_\Omega A(x,\nabla u)dx - C_1\int_\Omega j(x)\,|\nabla(v-u)|\,dx - C_1\int_\Omega |\nabla u|^{p(x)-1}\,|\nabla(v-u)|\,dx \\
&\geq \int_\Omega A(x,\nabla u)dx - C_2\,|j|_{p'(x)}\,|\nabla(v-u)|_{p(x)} - C_3\left||\nabla u|^{p(x)-1}\right|_{p'(x)}|\nabla(v-u)|_{p(x)} \\
&\geq \int_\Omega A(x,\nabla u)dx - C_4\,\|v-u\| \geq \int_\Omega A(x,\nabla u)dx - \epsilon,
\end{aligned}$$

for all $v \in E$ with $\|v-u\| < \delta = \epsilon/C_4$, where C_2, C_3 and C_4 are positive constants. We conclude that Λ is weakly lower semi-continuous. The proof of Lemma 3.5 is complete.

Proposition 3.6. *The functional I is bounded from below, coercive and weakly lower semi-continuous.*

Proof of Proposition 3.6. Using (6), we have

$$\begin{aligned}
I(u) &= \int_\Omega A(x,\nabla u)dx - \int_\Omega fudx \geq \int_\Omega \frac{1}{p(x)}\,|\nabla u|^{p(x)}\,dx - \int_\Omega fudx \\
&\geq \int_\Omega \frac{1}{p(x)}\,|\nabla u|^{p(x)}\,dx - \|f\|_{(p_-)'}\,\|u\|_{p_-} \geq \frac{1}{p_+}\int_\Omega |\nabla u|^{p(x)}\,dx - \|f\|_{(p_-)'}\,\|u\|_{p_-},
\end{aligned}$$

where $\|u\|_{p_-} = \left(\int_\Omega |u|^{p_-}\,dx\right)^{\frac{1}{p_-}}$.
For the coerciveness of I, we want to show that $\lim_{\|u\|\to+\infty} I(u) = +\infty$. For this, we can assume that $\|u\| = |\nabla u|_{p(x)} > 1$ and then, by Lemma 2.1 we obtain from above inequalities that

$$I(u) \geq \frac{1}{p_+}\,|\nabla u|_{p(x)}^{p_-} - C\,\|u\|_{p_-} \geq \frac{1}{p_+}\,\|u\|^{p_-} - C'\,\|u\|,$$

since E is continuously embedded in $L^{p_-}(\Omega)$.
As $p_- > 1$, then I is coercive. It is obvious that I is bounded from below. By Lemma 3.5, Λ is weakly lower semi-continuous. We show that I is weakly lower semi-continuous. Let $(u_n) \subset E$ be a sequence which converges weakly to u in E. Since Λ is weakly lower semi-continuous, we have

$$\Lambda(u) \leq \liminf_{n\to+\infty} \Lambda(u_n). \tag{9}$$

On the other hand, E is embedded in $L^{p_-}(\Omega)$. This fact together with relation (9) imply

$$I(u) \leq \liminf_{n\to+\infty} I(u_n).$$

Therefore, I is weakly lower semi-continuous. The proof of Proposition 3.6 is complete. Since I is proper, lower semi-continuous and coercive, then I has a minimizer which is a weak energy solution of (1). The proof of existence is then complete.

$*$ **Uniqueness**. Let u_1, u_2 be two weak energy solutions of (1). Then

$$\int_\Omega \left(a(x,\nabla u_1)-a(x,\nabla u_2)\right).(\nabla u_1-\nabla u_2)dx=0. \tag{10}$$

Using (5) in (10), we obtain

$$\int_\Omega |\nabla u_1-\nabla u_2|^{p(x)}\,dx=0. \tag{11}$$

We deduce from (11) by using Poincaré Inequality (since $W_0^{1,p(.)}(\Omega)\subset W_0^{1,p-}(\Omega)$) that

$$\int_\Omega |u_1-u_2|^{p-}\,dx=0. \tag{12}$$

From (12) it follows

$$u_1=u_2.$$

4. Entropy solutions

In this section, we study the problem (1) for a right-hand side $f\in L^1(\Omega)$. In the L^1 setting, the suitable notion of solution for the study of (1) is the notion of entropy solution. We refer to [4] for more details.
We first define the troncation function T_t by $T_t(s):=max\{-t,min\{t,s\}\}$.

Definition 4.1. A measurable function u is an entropy solution to problem (1) if, for every $t>0$, $T_t(u)\in W_0^{1,p(.)}(\Omega)$ and

$$\int_\Omega a(x,\nabla u).\nabla T_t(u-\varphi)dx\le\int_\Omega f(x)T_t(u-\varphi)dx \tag{13}$$

for all $\varphi\in W_0^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$.

Remark 4.2. A function u such that $T_t(u)\in W_0^{1,p(.)}(\Omega)$ for all $t>0$ does not necessarily belong to $W_0^{1,1}(\Omega)$. However, it is possible to define its weak gradient, still denoted by ∇u.

Our main result in this section is the following:

Theorem 4.3. *Assume* (2)–(6) *and* $f\in L^1(\Omega)$. *There exists a unique entropy solution* u *to problem* (1).

Proof. $*$ **A priori estimates**. We start with the existence of the weak gradient for every measurable function u such that $T_t(u)\in W_0^{1,p(.)}(\Omega)$ for all $t>0$.

Proposition 4.4. *If u is a measurable function such that $T_t(u)\in W_0^{1,p(.)}(\Omega)$ for all $t>0$, then there exists a unique measurable function $v:\Omega\to\mathbb{R}^N$ such that*

$$v\chi_{\{|u|<t\}}=\nabla T_t(u)\ \ \textit{for a.e. } x\in\Omega,\ \ \textit{and for all } t>0,$$

where χ_B denotes the characteristic function of a measurable set B. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v coincides with the standard distributional gradient of u.

Proof. As $T_t(u) \in W_0^{1,p(.)}(\Omega) \subset W_0^{1,p_-}(\Omega)$ for all $t > 0$, then by Theorem 1.5 in [3], the result follows.

Proposition 4.5. *Assume (2)–(6) and $f \in L^1(\Omega)$. Let u be an entropy solution of (1). If there exists a positive constant M such that*

$$\int_{\{|u|>t\}} t^{q(x)} dx \leq M, \quad \text{for all } t > 0, \tag{14}$$

then

$$\int_{\left\{|\nabla u|^{\alpha(.)}>t\right\}} t^{q(x)} dx \leq \|f\|_1 + M \quad \text{for all } t > 0,$$

where $\alpha(.) = p(.)/(q(.) + 1)$.

Proof. We know that

$$\int_\Omega |\nabla T_t(u)|^{p(x)} dx \leq t \|f\|_1, \quad \text{for all } t > 0.$$

Therefore, defining $\psi := T_t(u)/t$, we have, for all $t > 0$,

$$\int_\Omega t^{p(x)-1} |\nabla \psi|^{p(x)} dx = \frac{1}{t} \int_\Omega |\nabla T_t(u)| \, dx \leq \|f\|_1 .$$

From the above inequality, the definition of $\alpha(.)$ and (14), we have

$$\int_{\left\{|\nabla u|^{\alpha(.)}>t\right\}} t^{q(x)} dx \leq \int_{\left\{|\nabla u|^{\alpha(.)}>t\right\} \cap \{|u| \leq t\}} t^{q(x)} dx + \int_{\{|u|>t\}} t^{q(x)} dx$$

$$\leq \int_{\{|u|\leq t\}} t^{q(x)} \left(\frac{|\nabla u|^{\alpha(x)}}{t} \right)^{\frac{p(x)}{\alpha(x)}} dx + M \leq \|f\|_1 + M, \quad \text{for all } t > 0.$$

Proposition 4.6. *Assume (2)–(6) and $f \in L^1(\Omega)$. Let u be an entropy solution of (1), then*

$$\frac{1}{h} \int_{\{|u|\leq h\}} |\nabla T_h(u)|^{p(x)} dx \leq M$$

for every $h > 0$, with M a positive constant. More precisely, there exists $D > 0$ such that

$$meas\,\{|u| > h\} \leq D^{p_-} \frac{1+h}{h^{p_-}}.$$

Proof. Taking $\varphi = 0$ in the entropy inequality (13) and using (6), we obtain

$$\int_{\{|u|\leq h\}} |\nabla T_h(u)|^{p(x)} dx \leq \int_{\{|u|\leq h\}} a(x, \nabla u).\nabla u dx \leq \int_\Omega f(x) T_h(u) dx \leq h \|f\|_1 \leq Mh$$

for all $h > 0$. Next,

$$\int_{\{|u|\leq h\}} |\nabla T_h(u)|^{p(x)} dx \leq Mh \Rightarrow \int_{\{|u|\leq h\}} |\nabla T_h(u)|^{p_-} dx \leq C(1 + h).$$

By the Poincaré inequality in constant exponent, we obtain

$$\|T_h(u)\|_{L^{p_-}(\Omega)} \leq D(1+h)^{\frac{1}{p_-}}.$$

The above inequality implies that

$$\int_\Omega |T_h(u)|^{p_-}\, dx \leq D^{p_-}(1+h),$$

from which we obtain

$$meas\,\{|u|>h\} \leq D^{p_-}\frac{1+h}{h^{p_-}}.$$

$*$ **Uniqueness of entropy solution**. Let $h>0$ and u, v two entropy solutions of (1). We write the entropy inequality (13) corresponding to the solution u, with $T_h v$ as test function, and to the solution v, with $T_h u$ as test function. Upon addition, we get

$$\begin{cases} \displaystyle\int_{\{|u-T_hv|\leq t\}} a(x,\nabla u).\nabla(u-T_hv)dx + \int_{\{|v-T_hu|\leq t\}} a(x,\nabla v).\nabla(v-T_hu)dx \\ \displaystyle\leq \int_\Omega f(x)(T_t(u-T_hv)+T_t(v-T_hu))dx. \end{cases} \tag{15}$$

Define

$$E_1 := \{|u-v|\leq t, |v|\leq h\}, \qquad E_2 := E_1\cap\{|u|\leq h\} \quad \text{and} \quad E_3 := E_1\cap\{|u|>h\}.$$

We start with the first integral in (15). By (6), we have

$$\begin{cases} \displaystyle\int_{\{|u-T_hv|\leq t\}} a(x,\nabla u).\nabla(u-T_hv)dx \\ \displaystyle= \int_{\{|u-T_hv|\leq t\}\cap(\{|v|\leq h\}\cup\{|v|>h\})} a(x,\nabla u).\nabla(u-T_hv)dx \\ \displaystyle= \int_{\{|u-T_hv|\leq t,|v|\leq h\}} a(x,\nabla u).\nabla(u-T_hv)dx \\ \displaystyle\qquad + \int_{\{|u-T_hv|\leq t,|v|>h\}} a(x,\nabla u).\nabla(u-T_hv)dx \\ \displaystyle= \int_{\{|u-v|\leq t,|v|\leq h\}} a(x,\nabla u).\nabla(u-v)dx + \int_{\{|u-h|\leq t,|v|>h\}} a(x,\nabla u).\nabla u dx \\ \displaystyle\geq \int_{\{|u-v|\leq t,|v|\leq h\}} a(x,\nabla u).\nabla(u-v)dx = \int_{E_1} a(x,\nabla u).\nabla(u-v)dx \\ \displaystyle= \int_{E_1\cap(\{|u|\leq h\}\cup\{|u|>h\})} a(x,\nabla u).\nabla(u-v)dx \\ \displaystyle= \int_{E_2} a(x,\nabla u).\nabla(u-v)dx + \int_{E_3} a(x,\nabla u).\nabla(u-v)dx \\ \displaystyle= \int_{E_2} a(x,\nabla u).\nabla(u-v)dx + \int_{E_3} a(x,\nabla u).\nabla u dx - \int_{E_3} a(x,\nabla u).\nabla v dx \\ \displaystyle\geq \int_{E_2} a(x,\nabla u).\nabla(u-v)dx - \int_{E_3} a(x,\nabla u).\nabla v dx. \end{cases} \tag{16}$$

Using (4) and (7), we estimate the last integral in (16) as follows

$$\begin{cases} \left|\int_{E_3} a(x,\nabla u).\nabla v dx\right| \leq C_1 \int_{E_3} \left(j(x) + |\nabla u|^{p(x)-1}\right) |\nabla v|\, dx \\ \leq C\left(|j|_{p'(.)} + \left||\nabla u|^{p(x)-1}\right|_{p'(.),\{h<|u|\leq h+t\}}\right) |\nabla v|_{p(.),\{h-t<|v|\leq h\}}. \end{cases} \tag{17}$$

where $\left||\nabla u|^{p(x)-1}\right|_{p'(.),\{h<|u|\leq h+t\}} = \left\||\nabla u|^{p(x)-1}\right\|_{L^{p'(.)}(\{h<|u|\leq h+t\})}$.

The quantity $C\left(|j|_{p'(.)} + \left||\nabla u|^{p(x)-1}\right|_{p'(.),\{h<|u|\leq h+t\}}\right)$ is finite, since $u \in W_0^{1,p(.)}(\Omega)$ and $j \in L^{p'(.)}$; then by Proposition 4.6, the last expression converges to zero as h tends to infinity. Therefore, from (16) and (17), we obtain

$$\int_{\{|u-T_h v|\leq t\}} a(x,\nabla u).\nabla(u - T_h v)dx \geq I_h + \int_{E_2} a(x,\nabla u).\nabla(u-v)dx, \tag{18}$$

where I_h converges to zero as h tends to infinity. We may adopt the same procedure to treat the second term in (15) to obtain

$$\int_{\{|v-T_h u|\leq t\}} a(x,\nabla v).\nabla(v - T_h u)dx \geq J_h - \int_{E_2} a(x,\nabla v).\nabla(u-v)dx, \tag{19}$$

where J_h converges to zero as h tends to infinity.

Next, consider the right-hand side of inequality (15). Noting that

$$T_t(u - T_h v) + T_t(v - T_h u) = 0 \text{ in } \{|u| \leq h, |v| \leq h\};$$

we obtain

$$\begin{aligned} &\left|\int_\Omega f(x)(T_t(u-T_h v) + T_t(v - T_h u))dx\right| \\ &= \left|\int_{\{|u|>h\}} f(x)(T_t(u-T_h v) + T_t(v - T_h u))dx \right. \\ &\qquad \left. + \int_{\{|u|\leq h\}} f(x)(T_t(u-T_h v) + T_t(v - T_h u))dx\right| \\ &= \left|\int_{\{|u|>h\}} f(x)(T_t(u-T_h v) + T_t(v - T_h u))dx \right. \\ &\qquad \left. + \int_{\{|u|\leq h, |v|>h\}} f(x)(T_t(u-T_h v) + T_t(v - T_h u))dx\right| \\ &\leq 2t\left(\int_{\{|u|>h\}} |f|\, dx + \int_{\{|v|>h\}} |f|\, dx\right). \end{aligned}$$

According to Proposition 4.6, both $meas\,\{|u| > h\}$ and $meas\,\{|v| > h\}$ tend to zero as h goes to infinity, then by the inequality above, the right-hand side of inequality (15) tends to zero as h goes to infinity. From this assertion, (15), (18) and (19), we obtain, letting $h \to +\infty$,

$$\int_{\{|u-v|\leq t\}} (a(x,\nabla u) - a(x,\nabla v)).\nabla(u-v)dx \leq 0, \quad \text{for all } t > 0.$$

By assertion (5), we conclude that $\nabla u = \nabla v$, a.e. in Ω.
Finally, from Poincaré inequality, we have

$$\int_\Omega |T_t(u-v)|^{p_-}\, dx \leq C \int_\Omega |\nabla(T_t(u-v))|^{p_-}\, dx = 0, \quad \text{for all } t > 0;$$

and hence $u = v$, a.e. in Ω.

$*$ **Existence of entropy solutions**. Let $(f_n)_n$ be a sequence of bounded functions, strongly converging to $f \in L^1(\Omega)$ and such that

$$\|f_n\|_1 \leq \|f\|_1, \quad \text{for all } n. \tag{20}$$

Note that this choice is possible by taking for example $f_n = T_n(f)$.
We consider the problem

$$\begin{cases} -div(a(x,\nabla u_n)) = f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{21}$$

It follows from Theorem 3.2 that problem (21) has a unique weak energy solution $u_n \in W_0^{1,p(.)}(\Omega)$. Our interest is to prove that these approximated solutions u_n tend, as n goes to infinity, to a measurable function u which is an entropy solution of the limit problem (1). We start the proof of existence by proving that the sequence $(u_n)_{n\in\mathbb{N}}$ of solutions of problem (21) converges in measure to a measurable function u.

Proposition 4.7. *Assume* (2)–(6)*,* $f \in L^1(\Omega)$ *and* (20)*. Let* $u_n \in W_0^{1,p(.)}(\Omega)$ *be the solution of* (21)*. The sequence* $(u_n)_{n\in\mathbb{N}}$ *is Cauchy in measure. In particular, there exists a measurable function* u *and a subsequence still denoted* u_n *such that* $u_n \to u$ *in measure.*

Proof of Proposition 4.7. Let $s > 0$ and define

$$E_1 := \{|u_n| > t\}, \qquad E_2 := \{|u_m| > t\} \quad \text{and} \quad E_3 := \{|T_t(u_n) - T_t(u_m)| > s\},$$

where $t > 0$ is to be fixed. We note that

$$\{|u_n - u_m| > s\} \subset E_1 \cup E_2 \cup E_3,$$

and hence

$$meas\,\{|u_n - u_m| > s\} \leq meas\,(E_1) + meas\,(E_2) + meas\,(E_3)\,. \tag{22}$$

Let $\epsilon > 0$. Using (20) and the uniform bound given by Proposition 4.6, we choose $t = t(\epsilon)$ such that

$$meas(E_1) \leq \epsilon/3 \quad \text{and} \quad meas(E_2) \leq \epsilon/3. \tag{23}$$

On the other hand, taking $\varphi = 0$ in the entropy condition (13) for u_n yields

$$\int_\Omega |\nabla T_t(u_n)|^{p(x)}\,dx \le t\,\|f\|_1\,, \quad \text{for all } n \ge 0,$$

by using (6) and (20). The above inequality implies that

$$\int_\Omega |\nabla T_t(u_n)|^{p_-}\,dx \le M(1+t), \quad \text{for all } n \ge 0, \tag{24}$$

therefore, by Sobolev embedding, we can assume that $(T_t(u_n))_n$ is a Cauchy sequence in $L^{p_-}(\Omega)$. Consequently, there exists a measurable function u such that

$$T_t(u_n) \to T_t(u), \quad \text{in } L^{p_-}(\Omega) \text{ and a.e.}$$

Thus,

$$meas(E_3) \le \int_\Omega \left(\frac{|T_t(u_n) - T_t(u_m)|}{s}\right)^{p_-} dx \le \frac{\epsilon}{3},$$

for all $n, m \ge n_0(s,\epsilon)$.
Finally, from (22), (23) and the last estimate, we obtain that

$$meas\,\{|u_n - u_m| > s\} \le \epsilon, \quad \text{for all } n, m \ge n_0(s,\epsilon), \tag{25}$$

i.e., $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure.
Next, in order to prove that the sequence $(\nabla u_n)_{n\in\mathbb{N}}$ converges in measure to the weak gradient of u, we need two technical Lemmas.

Lemma 4.8 (cf. [27, Lemma 5.2]). *Let $(v_n)_n$ be a sequence of measurable functions. If v_n converges in measure to v and is uniformly bounded in $L^{p(.)}(\Omega)$ for some $1 \ll p(.) \in L^\infty(\Omega)$, then $v_n \to v$ strongly in $L^1(\Omega)$.*

The second technical Lemma is a standard fact in measure theory (cf. [14]).

Lemma 4.9. *Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) < +\infty$. Consider a measurable function $\gamma : X \to [0, +\infty]$ such that*

$$\mu\left(\{x \in X : \gamma(x) = 0\}\right) = 0.$$

Then, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mu(A) < \epsilon, \quad \textit{for all } A \in \mathcal{M} \textit{ with } \int_A \gamma\,d\mu < \delta.$$

We can now prove the convergence in measure of the weak gradient, the last ingredient in the proof of existence.

Proposition 4.10. *Assume (2)–(6), $f \in L^1(\Omega)$ and (20). Let $u_n \in W_0^{1,p(.)}(\Omega)$ be the weak energy solution of (21). The following assertions hold:*

(i) ∇u_n *converges in measure to the weak gradient of* u.

(ii) $a(x, \nabla u_n)$ *converges to* $a(x, \nabla u)$ *strongly in* $\left(L^1(\Omega)\right)^N$.

(iii) $a(x,\nabla u)\in\left(L^{p'(\cdot)}(\Omega)\right)^N$.

Proof of Proposition 4.10. (i) We claim that $(\nabla u_n)_n$ is Cauchy in measure. Indeed, let $s>0$, and consider

$$E_1:=\{|\nabla u_n|>h\}\cup\{|\nabla u_m|>h\}, \qquad E_2:=\{|u_n-u_m|>t\},$$

and

$$E_3:=\{|\nabla u_n|\le h, |\nabla u_m|\le h, |u_n-u_m|\le t, |\nabla u_n-\nabla u_m|>s\},$$

where h and t will be chosen later. We note that

$$\{|\nabla u_n-\nabla u_m|>s\}\subset E_1\cup E_2\cup E_3. \tag{26}$$

Let $\epsilon>0$. By Proposition 4.5, we may choose $h=h(\epsilon)$ large enough such that $meas(E_1)\le\epsilon/3$ for all $n,m\ge 0$. On the other hand, by Proposition 4.7 (see (25)), we have that $meas(E_2)\le\epsilon/3$ for all $n,m\ge n_0(t,\epsilon)$. Moreover, by assumption (5), there exists a real valued function $\gamma:\Omega\to[0,+\infty]$ such that $meas\left(\{x\in\Omega:\gamma(x)=0\}\right)=0$ and

$$(a(x,\xi)-a(x,\xi').(\xi-\xi'))\ge\gamma(x), \tag{27}$$

for all $\xi,\xi'\in\mathbb{R}^N$ such that $|\xi|,|\xi'|\le h$, $|\xi-\xi'|\ge s$, for a.e. $x\in\Omega$. Let $\delta=\delta(\epsilon)$ be given from Lemma 4.9, replacing ϵ and A by $\epsilon/3$ and E_3, respectively. Using (27), the equation and (20), we obtain

$$\int_{E_3}\gamma dx\le\int_{E_3}\left(a(x,\nabla u_n)-a(x,\nabla u_m)\right).\nabla(u_n-u_m)dx\le 2\left\|f\right\|_1 t<\delta,$$

by choosing $t=\delta/(4\left\|f\right\|_1)$. From Lemma 4.9, it follows that $meas(E_3)<\epsilon/3$. Thus, using (26) and the estimates obtained for E_1,E_2 and E_3, it follows that

$$meas\left(\{|\nabla u_n-\nabla u_m|\ge s\}\right)\le\epsilon,\quad\text{for all } n,m\ge n_0(s,\epsilon),$$

and then the claim is proved.

As a consequence, $(\nabla u_n)_{n\in\mathbb{N}}$ converges in measure to some measurable function v.

Finally, since$(\nabla T_t(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in $\left(L^{p(\cdot)}(\Omega)\right)^N$ for all $t>0$, it converges weakly to $\nabla T_t(u)$ in $(L^1(\Omega))^N$. Therefore, by Proposition 4.4, v coincides with the weak gradient of u.

(ii)–(iii) By part (i) and Nemytskii Theorem (cf. [17, Lemma 2.1]), we obtain that $a(x,\nabla u_n)$ converges to $a(x,\nabla u)$ in measure. Moreover, using (4), we have

$$|a(x,\nabla u_n)|\le C_1\left(j(x)+|\nabla u_n|^{p(x)-1}\right),$$

with $j\in L^{p'(\cdot)}(\Omega)$. By (20), we have that $\left(|\nabla u_n|^{p(x)-1}\right)_{n\in\mathbb{N}}$ is uniformly bounded in $L^{p'(\cdot)}(\Omega)$. hence, using Lemma 4.8, we obtain that $a(x,\nabla u_n)$ converges to $a(x,\nabla u)$ strongly in $(L^1(\Omega))^N$, and $a(x,\nabla u)\in(L^{p'(\cdot)}(\Omega))^N$.

Now, fix $t > 0$, $\varphi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$, and choose $T_t(u_n - \varphi)$ as test function in (8), with u replaced by u_n, to obtain

$$\int_\Omega a(x, \nabla u_n).\nabla \left(T_t(u_n - \varphi)\right) dx = \int_\Omega f_n(x) T_t(u_n - \varphi) dx.$$

Note that this choice can be made using a standard density argument. We now pass to the limit in the previous identity. For the right-hand side, the convergence is obvious since f_n converges strongly in L^1 to f and $T_t(u_n - \varphi)$ converges weakly-$*$ in L^∞, and a.e., to $T_t(u - \varphi)$.
Next, we write the left hand side as

$$\int_{\{|u_n - \varphi| \le t\}} a(x, \nabla u_n).\nabla u_n dx - \int_{\{|u_n - \varphi| \le t\}} a(x, \nabla u_n).\nabla \varphi dx \tag{28}$$

and note that $\{|u_n - \varphi| \le t\}$ is a subset of $\{|u_n| \le t + \|\varphi\|_\infty\}$. Hence, taking $s = t + \|\varphi\|_\infty$, we rewrite the second integral in (28) as

$$\int_{\{|u_n - \varphi| \le t\}} a(x, \nabla T_s(u_n)).\nabla \varphi dx.$$

Since $a(x, \nabla T_s(u_n))$ is uniformly bounded in $(L^{p'(\cdot)}(\Omega))^N$ (by (24) and assumption (4)), by Proposition 4.10 (i), we have that it converges weakly to $a(x, \nabla T_s(u))$ in $(L^{p'(\cdot)}(\Omega))^N$. Therefore, the last integral converges to

$$\int_{\{|u - \varphi| \le t\}} a(x, \nabla T_s(u)).\nabla \varphi dx.$$

The first integral in (28) is nonnegative by (7), and it converges a.e. by Proposition 4.10. It follows from Fatou's Lemma that

$$\int_{\{|u - \varphi| \le t\}} a(x, \nabla u).\nabla u dx \le \liminf_{n \to +\infty} \int_{\{|u_n - \varphi| \le t\}} a(x, \nabla u_n).\nabla u_n dx.$$

Gathering results, we obtain

$$\int_\Omega a(x, \nabla u).\nabla T_t(u - \varphi) dx \le \int_\Omega f(x) T_t(u - \varphi) dx,$$

i.e., u is an entropy solution of (1).

Remark 4.11. In [27], for the proof of uniqueness of entropy solution, the authors used a so-called "Poincaré inequality" (cf. [27, Proposition 2.1]) which was presented for the first time by Y. Fu (see [13, Lemma 2.14]) and also used in [6] for the study of existence of weak solution. But, the "Poincaré inequality" claimed by Y. Fu is incorrect (see [25] for more details) in the context of assumptions used by Fu and by Sanchon and Urbano and then, the proof of uniqueness in [27] may be done in the same way as in this paper.

5. Weak energy solutions for a right-hand side dependent of u

In this section, we study problem (1) for an f more general. We assume that

$$f : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function.} \tag{29}$$

Let

$$F(x,t) = \int_0^t f(x,s)ds.$$

We assume that there exists $C_1 > 0$, $C_2 > 0$ such that

$$|f(x,t)| \leq C_1 + C_2 \, |t|^{\beta-1}, \quad \text{where } 1 \leq \beta < p_-. \tag{30}$$

We have the following result.

Theorem 5.1. *Under assumptions* (2)–(6), (29) *and* (30)*, the problem* (1) *has at least one weak energy solution.*

Proof. Let $g(u) = \int_\Omega F(x,u)dx$, then $g' : E \to E^*$ is completely continuous i.e. $u_n \rightharpoonup u \Rightarrow g'(u_n) \to g'(u)$, and thus the functional g is weakly continuous.
Consequently,

$$I(u) = \int_\Omega A(x, \nabla u)dx - \int_\Omega F(x,u)dx, \quad u \in E$$

is such that $I \in C^1(E, \mathbb{R})$ and is lower semi-continuous. We then have to prove that I is bounded from below and coercive in order to complete the proof. From (30), we have $|F(x,t)| \leq C(1 + |t|^\beta)$ and then

$$I(u) \geq \frac{1}{p_+} \int_\Omega |\nabla u|^{p(x)} \, dx - C \int_\Omega |u|^\beta \, dx - C_3.$$

We know that E is continuously embedded in $L^\beta(\Omega)$. It follows that there exists $C_4 > 0$ such that

$$\|u\| > C_4 \, |u|_\beta \, , \quad \text{for all } u \in E.$$

On the other hand, by Lemma 2.1, we have

$$\int_\Omega |\nabla u|^{p(x)} \, dx \geq \|u\|^{p_-} \, , \quad \text{for all } u \in E \text{ with } \ \|u\| > 1.$$

Then, we get

$$I(u) \geq \|u\|^{p_-} - C_5 \, \|u\|^\beta - C_3 \to +\infty \ \text{ as } \ \|u\| \to +\infty.$$

Consequently, I is bounded from below and coercive. The proof is then complete.

Assume now that $F^+(x,t) = \int_0^t f^+(x,s)ds$ is such that there exists $C_1 > 0, C_2 > 0$ such that

$$\left| f^+(x,t) \right| \leq C_1 + C_2 \, |t|^{\beta-1} \, , \quad \text{where } 1 \leq \beta < p_- \tag{31}$$

Then we have the following result

Theorem 5.2. *Under assumptions* (2)–(6), (29) *and* (31), *the problem* (1) *has at least one weak energy solution.*

Proof. As $f = f^+ - f^-$, let $F^-(x,t) = \int_0^t f^-(x,s)ds$. Then

$$\begin{cases} I(u) = \displaystyle\int_\Omega A(x,\nabla u)dx + \int_\Omega F^-(x,u)dx - \int_\Omega F^+(x,u)dx \\ \qquad \geq \displaystyle\int_\Omega A(x,\nabla u)dx - \int_\Omega F^+(x,u)dx. \end{cases}$$

Then by the same way as in the proof of Theorem 5.1, the result of Theorem 5.2 follows.

Remark 5.3. There is no uniqueness of weak energy solution of (1) under assumptions (30) or (31). Indeed, the function

$$f(x,t) = \lambda(t^{\gamma-1} - t^{\beta-1}) \tag{32}$$

where $1 < \beta < \gamma < p_-$ and $\lambda > 0$ verify (29) and (30). Then by Mihailescu and Radulescu's work (see [21, Theorem 2.1]), problem (1) with particular data (32) has at least two distinct non-negative non-trivial weak energy solutions.

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