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Cauchy Transforms of Arens Bounded Measures for a Vitushkin Amendment. I

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We amend the theorem of Vitushkin [20], who solves the problem of the rational approximation by constructive methods and a quasi-geometric notation - the so called analytic continuous capacity -, by a new sufficient condition. The proof is functional-analytic abstract and at the same time we can almost see the effects of the condition. We start with measures - a structure bearing level beneath the AC-capacity - and go by Cauchy transforms directly to the level of functions. Moreover we confirm a conjecture of Garnett [8] for functions, which are continuous on $\mathbb{C}$ including infinity and analytic off a Cantor-set.

Keywords: Cauchy-Transform, a new norm for measures, Vitushkin, rational approximation, Garnett's conjecture

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1. Introduction and Conventions

1.1. Introduction

We consider a compact set K in the complex plane and there on one hand the supnorm closure of the rational functions with poles off K, on the other hand the set of all continuous functions on K which are analytic on the topological inner of K. Both sets of functions are algebras, commonly the first is denoted by R and the second by A. The problem of the rational approximation is the question, when is $A = R$.

Vitushkin [20] considers functions, which are continuous on the hole of $\mathbb{C}$ including infinity, where they vanish, and which are analytic off compact subsets of the intersection of open sets with the topological inner of K resp. of K itself, that means pairs of sets are considered. The functions are scaled and evaluated by $|f'(\infty)|$. The open sets are evaluated by the suprema of the function evaluations. The theorem of Vitushkin says, that equal evaluations of all considered pairs of sets is equivalent to $A = R$ The theorem of Vitushkin is found in [24].

The constructive method is ahead of the abstract methods since the work of Mergelyan [12] and Vitushkin [20]. Mergelyan answers the question for finitely many "holes", Vitushkin independently of the number of holes, while the abstract methods essentially only can solve the case of finitely many holes [3, p. 233, Theorem 8.4], [9, Theorem 3.13],

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[11, Théorème 7.5]. They mostly take the way via the real parts of both of the algebras and use a result of [21], see [22]. The special case of countably many holes, which [9, Lemma 3.15] considers, has a hidden finite dimensional structure.

Many papers on this theme work with the Cauchy transform of finite measures (see [24], who gives a surway for the rational approximation), these transforms are functions, which can be unbounded. [1, 2.4 Lemma] gives a special condition for measures, in order to obtain continuous Cauchy transforms. These are analytic off the support of the measure, from which they stem by Cauchy transformation, and therefore they are candidates for functions in A. We consider measures, which satisfy a modified Arens-condition, we call them "wide".

Related fields of work are the investigation of properties of the AC-capacity, whose definition is found in [24], especially its subadditivity and regularity - we refer to [4], as well as function spaces of different integrabilities and differentiabilities - we refer to the papers of H. Triebel and the work group "function spaces" of the university of Jena, e.g. [17] and [18] - and real Hardy-spaces on $\mathbb{R}^n$ - we refer to [13].

In this paper we consider the question of the rational approximation by the more flexible measures and their Cauchy transforms instead of the less flexible AC-capacity, that means we step from a structure bearing level beneath the AC-capacity whith help of the Cauchy transforms directly to the level of functions. This is as far as we know a method which is not yet used in the literature.

How useful our methods are in a general sense, that we find confirmed in many places. Measures are tailored, [13] defines measures with linear growth ("creixement lineal"), [18] defines "d-sets". [17] considers "fractal characteristics of measures".

In Chapter 2 we prove technical lemmata, which we have not found ready to cite. In Chapter 3 we investigate the wide measures. In Chapter 4 we investigate the Cauchy transforms of these measures and their supnorm limits. In Chapter 5 we give our amendment of the theorem of Vitushkin, which considers the wide measures on the inner boundary, the definition is found below. The main theorem of Chapter 5 says, that $A = R$ holds, when the inner boundary does not support any wide measure. And in two corollaries a known and a weakened sufficient geometric condition for $A = R$ are given. Further we confirm a conjecture of [8] for functions, which are continuous on $\mathbb{C}$ including infinity and analytic off a Cantor-set.

Up to here we have represented motivation and results of this paper. In the light of the paper of R. W. Hilger und J. F. Michaliček [10], which comprehensively clarifies how to represent analytic functions by generalized Cauchy transforms, we can aspect, that also other classes than only functions analytic on the topological inner of a bounded subset of the plane, and continuous on the closure can be investigated by our method. The investigation of other classes of functions, the question of adjunction and the Corona problem are left to further work.

1.2. Conventions

We consider a compact set K in the complex plane with countably many holes. *Hole* always stands for a connected component Y_n of the complement of K, one of them is the unbounded hole Y_0. Let K be a subset of the open disc centered in 0 with radius r_0 with

$0 < r_0 \leq 1$, we denote this disc by V_0. We suppose $r_0 = 1$, this means no restriction. We name *rational support set* ρ the subset in $\overline{V_0}$ of the union of the closures of the Y_n. (The inner boundary is not part of it.) We remind that inner boundary denotes the boundary of K without the union of the boundaries of the holes, we denote it by ι. We name *analytic support set* $\alpha = \rho \cup \iota$.

We have chosen a notation for the topological inner and outer of a set $G \subseteq \mathbb{C}$, which is at the same time visualizing and symmetric: $\overset{\frown}{G}$ and $\overset{\smile}{G}$. From now "inner" always means the topological inner, other meanings as e.g. inner boundary are stated explicitly. We write q_z for the function with $q_z(\zeta) = (\zeta - z)^{-1}$. We mark by $\square$ the end of a proof, an example or a definition.

2. Technical Lemmata

Lemma 2.1. *Let be $0 \leq g_1 \leq g_2$ isotone bounded functions on a compact interval with $0 = \inf g_1 = \inf g_2$ and let be $0 \leq f$ an antitone continuous function on the same interval, then there holds for the Stieltjes-integrals*

$$\int f(s)\, d\, g_1(s) \leq \int f(s)\, d\, g_2(s)\,.$$

Proof. With $g = g_2 - g_1 \geq 0$ and by partial integration

$$\begin{aligned}\int_a^b f(s)\, d\, g(s) &\geq \int_a^b f(s)\, d\, g(s) + \int_a^b g(s)\, d\, f(s) \\ &= f(b)\ g(b) - f(a)\ g(a) = f(b)\ g(b) \geq 0.\end{aligned}$$

$\square$

Definition 2.2. We denote by $A_0(\mathbb{C}, G)$ the set of the continuous complex valued functions on $\mathbb{C}$ vanishing at infinity, which are analytic on an open subset $G \subseteq \mathbb{C}$. $\square$

Lemma 2.3. *Let be Q the set of all q_z with $z \in \overset{\smile}{K} \cap V_0$, then Q generates a supnorm dense vector subspace of R. (We recall our notation: V_0 is the open disc centred in 0 with radius 1 containing K.)*

Proof. Let be U the supnorm closure of the vector space, which is generated by the functions q_z with $z \in Y_0 \cap V_0$ (We recall our notation: Y_0 is the unbounded component of the complement of K.). Let be $T = \{z \in Y_0; q_z \in U\}$.
Then we get the following by calculations:

(1) T is relatively closed in Y_0,
(2) U is an algebra,
(3) T is open.

Since by our setting T contains the non empty $Y_0 \cap V_0$ and Y_0 is connected, we get from (1) and (3) by a common connectedness argument $T = Y_0$.
That means the closed vector spaces, which are generated by the q_z with $z \in Y_0$ resp. $z \in Y_0 \cap V_0$, are the same. That further means the closed vector spaces, which are

generated by the q_z with $z \in \overset{\smile}{K}$ resp. q_z with $z \in \overset{\smile}{K} \cap V_0$, are the same too, so they are both equal to R. $\square$

Lemma 2.4. *Each element of the dual space of* $A_0\left(\mathbb{C}, \overset{\frown}{K}\right)$ *can be represented as a finite Radon-measure on* $\overline{\overset{\smile}{K}}$*, which annihilates the* q_z $\forall z \in \overset{\frown}{K}$*. (We recall our notation* $\overset{\frown}{K}$ *for the inner of* K *and* $\overset{\smile}{K}$ *for the outer of* K*).*

Proof. For $\overset{\frown}{K} = \emptyset$ this is known [2, Korollar 29.13]. Otherwise the proof consists of the repeated application of the Hahn-Banach theorem, decomposition of a measure by the characteristic functions of $\overset{\smile}{K}$ and K, replacement of a measure on K by one on the boundary of K with the use of the maximum principle for analytic functions. Here we note that $A_0\left(\mathbb{C}, \overset{\frown}{K}\right) \cap V = 0$, where V is the supnorm closure of the vector space, which is generated by the q_z with $z \in \overset{\frown}{K}$, and all functions are restricted to $\overline{\overset{\smile}{K}}$. For each function $q \in V$ has a representation $q = \sum_{n=1}^{\infty} a_n\, q_{zn}$ with $|z_n - z_1| > \delta$ for suitable $z_1 \in \overset{\frown}{K}$ and for $n > 1$, the proof repeats parts of the proof of our Lemma 2.3. $A_0\left(\mathbb{C}, \overset{\frown}{K}\right)$ is supnorm closed. The direct sum of $A_0\left(\mathbb{C}, \overset{\frown}{K}\right)$ and V is complete in the direct sum topology, see [11, Satz 24, p. 102] and therefore closed in the supnorm topology too. The projection of $A_0\left(\mathbb{C}, \overset{\frown}{K}\right) \oplus V$ onto $A_0\left(\mathbb{C}, \overset{\frown}{K}\right)$ is continuous by the closed-graph theorem. $\square$

3. The Arens bounded Measures

Definition 3.1. For finite Radon measures μ on $\mathbb{C}$ we define their *crowding function* $\mu^{\approx} : \mu^{\approx}(r) = \sup\left\{|\mu|\left(V\left(z, r\right)\right); z \in \mathbb{C}\right\} \forall\ 0 \leq r \leq \infty$. It is isotone. For these measures μ we consider the Stieltjes integral $\int_0^{\infty} \frac{r+1}{r} d\, \mu^{\approx}(r)$.
If the integral is finite, we call μ a *wide measure* and define its *wide norm* by $\langle\mu\rangle = 2\int_0^{\infty} \frac{r+1}{r} d\, \mu^{\approx}(r)$. The name is justified by our Proposition 3.3. $\square$

Compare [1, (2.2.2)] to this definition. The plane measure on a bounded set is wide, since $\lambda^{\approx}(r) \leq \pi\, r^2$ for small r and $\lambda^{\approx}(r) = \mathit{const}$ for big r. Following our Lemma 2.1 and the definition a measure with bounded density to the plane measure is then wide too.

Estimations 3.2.

3.2.1. $\mu^{\approx}(r) \leq \mu^{\approx}(\infty) = \|\mu\| \leq \frac{1}{2}\langle\mu\rangle\ \ \forall\, r \geq 0$.

3.2.2. $\left|\int \frac{d\,\mu(\zeta)}{\zeta - z}\right| \leq \int \frac{d\,|\mu|(\zeta)}{|\zeta - z|} \leq \frac{1}{2}\langle\mu\rangle\ \ \forall z \in \mathbb{C}$.

3.2.3. $|\mu| \leq |\nu| \Rightarrow \mu^{\approx}(r) \leq \nu^{\approx}(r)\ \ \forall\, r \geq 0$ and $\langle\mu\rangle \leq \langle\nu\rangle$.

3.2.4. The crowding function is homogeneous for the modulus.

3.2.5. For finite sums and series converging in the variation norm we have $\left(\sum_n \mu_n\right)^{\approx}(r)$ $\leq \sum_n \mu_n^{\approx}(r)$.

Proof. All can be calculated, for 3.2.3 we use our Lemma 2.1. $\square$

Proposition 3.3. *$\langle \mu \rangle = 2\int_0^\infty \frac{r+1}{r} d\,\mu^{\approx}(r)$ is a norm on the finite Radon measures with finite Stieltjes integral $\int_0^\infty \frac{r+1}{r} d\mu^{\approx}(r)$. Let be $S \subseteq \mathbb{C}$. In the wide norm the wide measures on S form a Banach space.*

Proof. $\langle \mu \rangle = 0$ implies $\mu = 0$ following 3.2.2 and [22, page 18, Lemma 6]. The remaining properties of a norm are calculated with our Estimations 3.2.
We show the completeness. Let be (μ_n) a Cauchy sequence of wide measures. We choose if necessary an equally named subsequence of the (μ_n) with $\sum_{k=0}^\infty \frac{1}{2}\langle \mu_{k+1} - \mu_k \rangle < \infty$ and note, that following 3.2.1 the partial sums of the series $\sum_{k=1}^\infty (\mu_{k+1} - \mu_k)$ in the variation norm and the total variations of the $\sum_{k=1}^n (\mu_{k+1} - \mu_k)^{\approx}$ have the common bound $\sum_{k=0}^\infty \frac{1}{2}\langle \mu_{k+1} - \mu_k \rangle < \infty$ and $\frac{r+1}{r}$ is continuous on finite intervals off 0 and is positive and antitone, so that the conditions of [23, Theorem I 16.4] are fullfilled and we can estimate for $\mu = \mu_1 + \sum_{k=1}^\infty (\mu_{k+1} - \mu_k)$ and $\varepsilon > 0$ as follows.

$$2\int_\varepsilon^\infty \frac{r+1}{r} d(\mu - \mu_n)^{\approx}(r) = 2\int_\varepsilon^\infty \frac{r+1}{r} d\left(\sum_{k=n}^\infty (\mu_{k+1} - \mu_k)\right)^{\approx}(r)$$
$$\leq 2\int_\varepsilon^\infty \frac{r+1}{r} d\sum_{k=n}^\infty (\mu_{k+1} - \mu_k)^{\approx}(r) = \sum_{k=n}^\infty 2\int_\varepsilon^\infty \frac{r+1}{r} d(\mu_{k+1} - \mu_k)^{\approx}(r)$$
$$\leq \sum_{k=n}^\infty 2\int_0^\infty \frac{r+1}{r} d(\mu_{k+1} - \mu_k)^{\approx}(r) = \sum_{k=n}^\infty \langle \mu_{k+1} - \mu_k \rangle.$$

We take $\varepsilon \to 0$ and obtain $\langle \mu - \mu_n \rangle = 2\int_0^\infty \frac{r+1}{r} d(\mu - \mu_n)^{\approx}(r) \leq \sum_{k=n}^\infty \langle \mu_{k+1} - \mu_k \rangle$. So we have $\langle \mu \rangle = \langle \mu - \mu_n + \mu_n \rangle \leq \langle \mu - \mu_n \rangle + \langle \mu_n \rangle < \infty$ and $\langle \mu - \mu_n \rangle \to 0$ for the chosen subsequence and for the original sequence as well. □

Definition 3.4. We write W for the set of all finite Radon measures on $\mathbb{C}$ with $\langle \mu \rangle < \infty$. We write W_c for the set of all finite Radon measures on $\mathbb{C}$ with compact support and with $\langle \mu \rangle < \infty$.
We define $W(S) = \{\mu \in W; \textit{support } \mu \subseteq S\}$.
We define $W_c(S) = \{\mu \in W_c; \textit{support } \mu \subseteq S\}$. □

Proposition 3.5. *Let be $\mu \in W$ and E the support of μ, let be (E_n) a sequence of bounded disjoint Baire sets with $E = \bigcup_{n=0}^\infty E_n$ and let be $\mu_n = \chi_{E_n}\mu$, then $\mu = \sum_{n=0}^\infty \mu_n$ in the wide norm.*

Proof. Let be $\nu_n = \sum_{k=0}^n \mu_k$, then we have $\|\nu_n - \mu\| \to 0$ and $|\nu_n| \leq |\mu|\ \forall\ n \in \mathbb{N}$. We have $(\mu - \nu_n)^{\approx}(r) \leq \|\mu - \nu_n\|$ and $(\mu - \nu_n)^{\approx}(r) \leq (\mu^{\approx} + \nu_n^{\approx})(r) \leq 2\mu^{\approx}(r)$. We choose for $\varepsilon > 0$ a $\delta > 0$ with $\int_0^\delta \frac{r+1}{r} d\,\mu^{\approx}(r) \leq \varepsilon$ and further we choose a ν_n with $\frac{\delta+1}{\delta}\|\mu - \nu_n\| \leq \varepsilon$, so that we can estimate as follows.

$$\int_0^\infty \frac{r+1}{r} d\ (\mu - \nu_n)^{\approx}(r) = \int_0^\delta \frac{r+1}{r} d\ (\mu - \nu_n)^{\approx}(r) + \int_\delta^\infty \frac{r+1}{r} d\ (\mu - \nu_n)^{\approx}(r)$$
$$\leq \int_0^\delta \frac{r+1}{r} d\,2\,\mu^{\approx}(r) + \frac{\delta+1}{\delta}\|\mu - \nu_n\| \leq 3\varepsilon.$$

We send $\varepsilon \to 0$, for a subsequence of the ν_n we then obtain $\langle \mu - \nu_n \rangle = 2\int_0^\infty \frac{r+1}{r} d\ (\mu - \nu_n)^{\approx}(r) \to 0$. We obtain the same result by our Lemma 2.1 for the original sequence as well, since $\mu - \nu_n \geq \mu - \nu_k \geq 0$ and so $(\mu - \nu_n)^{\approx} \geq (\mu - \nu_k)^{\approx} \geq 0$ for $k \geq n$. $\square$

As a special case we get the following corollary.

Corollary 3.6. *Let be $\mu \in W$, then there is a sequence $(\mu_n) \subseteq W_c$ with $\mu = \sum_{n=0}^{\infty} \mu_n$ in the wide norm.*

Proof. Let be E the support of μ. We choose $E_n = E \cap \{z \in \mathbb{C}; n \leq |z| < n+1\}$ and apply 3.5. $\square$

Example 3.7. The unit intervall of the real numbers does not support any wide measure.

Proof. The crowding function $\mu^{\approx}$ of a finite Radon measure μ on $[0,1]$ is isotone and bounded. Without restriction let be $\mu \geq 0$ and $\mu^{\approx}(1) = \|\mu\| = 1$. We observe that the supremum is realized for centres on the unit intervall and that the crowding function is isotone and give for an element of the sequence (2^{-n}) and for $0 \leq t \leq 1$ only the most important steps of the calculation: $\mu\left([t - 2^{-n+1}, t + 2^{-n+1}]\right) \leq \mu\left([t - 2^{-n+1}, t]\right) + \mu\left([t, t + 2^{-n+1}]\right) \leq 2\mu^{\approx}(2^{-n})$ and further $\mu^{\approx}(2^{-n+1}) = \sup_{0 \leq t \leq 1} \mu\left([t - 2^{-n+1}, t + 2^{-n+1}]\right) \leq 2\mu^{\approx}(2^{-n})$ for all natural n and $\mu^{\approx}(r) \geq \mu^{\approx}\left(2^{-k}\right) \geq 2^{-k+1}\mu^{\approx}(1) \geq r\mu^{\approx}(1)$ for all $0 \leq r \leq 1$ and for a k depending on r. That means following our Lemma 2.1 for an arbitrary measure on the unit intervall $\int_0^1 \frac{r+1}{r} d\,\mu^{\approx}(r) \geq \int_0^1 \frac{r+1}{r} d\,r\mu^{\approx}(1) = \mu^{\approx}(1) \int_0^1 \frac{r+1}{r} d\,r = \infty$. $\square$

4. The Cauchy Transforms of Arens bounded measures

Definition 4.1. For a measure $\mu \in W$ we define the *Cauchy transform* by $\hat{\mu} : \hat{\mu}(z) = \int q_z \, d\mu$. We denote $\hat{W}_c = \{\hat{\mu}; \mu \in W_c\}$ and $\hat{W} = \{\hat{\mu}; \mu \in W\}$. $\square$

Estimation 4.2. $\|\hat{\mu}\| \leq \frac{1}{2} \langle \mu \rangle \;\; \forall \mu \in W$.

Proof. 3.2.2. $\square$

Properties 4.3.

4.3.1. The transform $\hat{\mu}$ of $\mu \in W_c$ is defined on $\mathbb{C}$, is continuous on $\mathbb{C}$, vanishes at ∞ and is analytic off the support of μ.

4.3.2. For a measure $\mu \in W$ there is a representation $\mu = \sum_{n=0}^{\infty} \mu_n$ in the wide norm with $\mu_n \in W_c$

4.3.3. For each representation $\mu = \sum_{n=0}^{\infty} \mu_n$ in the wide norm with $\mu_n \in W$ we have $\hat{\mu} = \sum_{n=0}^{\infty} \hat{\mu}_n$ in the supnorm.

4.3.4. The transform $\hat{\mu}$ of $\mu \in W$ is defined on $\mathbb{C}$, is continuous on $\mathbb{C}$, vanishes at ∞ and is analytic off the support of μ.

Proof. 1) see [1, 2.4], 2) see our 3.5, 3) because of our 4.2, 4) because of 1)–3). $\square$

Proposition 4.4. *$\hat{W}_c$ is supnorm dense in $C_0(\mathbb{C})$.*

Proof. First we show that $\hat{W}_c$ is σ^*-dense in $C_0(\mathbb{C})$ by showing that all measures in the dual space of $C_0(\mathbb{C})$, which are annihilated by $\hat{W}_c$, vanish themselves. Let be ν a finite Radon measure on $\mathbb{C}$, then $\int\int |q_z|\, d|\mu|\, d|\nu|(z) \leq \langle\mu\rangle \|\nu\| < \infty$ for all $\mu \in W_c$. By Fubini's theorem we obtain $0 = \int \hat{\mu}(z)\, d\nu(z) = \int\int \frac{1}{\zeta - z} d\nu(z)\, d\mu(\zeta) = -\int \hat{\nu} d\mu$. Follwing the remark after 3.1 especially $\int_D \hat{\nu} d\lambda = 0$ holds for all measurable bounded subsets D of the plane, when we choose for D the sets $\{z \in \mathbb{C}; |z| \leq n, Re\,\hat{\nu} \geq 0\}$, $\{z \in \mathbb{C}; |z| \leq n, Im\,\hat{\nu} \geq 0\}$ etc., we see that with $\hat{\nu} = 0$ Lebesgue-almost everwhere also $\nu = 0$ holds following [22, page 18, Lemma 6].

Now $C_0(\mathbb{C})$ is a Banach space in the supnorm and the proof is done by [4, page 422, Theorem 13 and Corollary]. $\square$

Lemma 4.5. *Let be μ a wide measure, let be U an open subset of the plane with $\hat{\mu}$ analytic on U, then $|\mu|(U) = 0$ holds.*

Proof. Let be $V \subseteq U$, V open disc, let be $\mu = \kappa + \nu$ with $\kappa = \chi_V\,\mu$.
Then $\hat{\mu}(z) = \int q_z d\mu = \int q_z d\kappa + \int q_z d\nu$ holds and the sum as well as the second term are analytic on V, i.e. also the first term - call it f - is analytic on V. For this term therefore holds that it is continuous on the whole plane and analytic off the boundary of V. In V we have $f = \sum a_n z^n = \sum b_n \bar{z}^n$. Following [11, II 1.3] or the classic Fourier analysis on the unit circle it is constant. Because it vanishes at ∞, we have $\kappa = 0$ following [22, page 18, Lemma 6]. Since V was chosen arbitraryly, $|\mu|(U) = 0$ holds. $\square$

We generalize our Lemma 4.4.

Proposition 4.6. $\hat{W}_c\left(\overline{\breve{K}}\right)$ *is supnorm dense in* $A_0\left(\mathbb{C}, \overset{\frown}{K}\right)$.

Proof. We refine the proof of 4.4. First we show, that $\hat{W}_c\left(\overline{\breve{K}}\right)$ is σ^*-dense in $A_0\left(\mathbb{C}, \overset{\frown}{K}\right)$ by showing that a measure ν in the dual space of $A_0\left(\mathbb{C}, \overset{\frown}{K}\right)$, which is annihilated by $\hat{W}_c\left(\overline{\breve{K}}\right)$, vanishes itself. Let be ν in the dual space as assumed, then it is following our Lemma 2.4 a finite Radon measure on $\overline{\breve{K}}$ with $0 = \int \hat{\mu}(z)\, d\nu(z)$ and $\int\int |q_z|\, d|\mu|\, d|\nu|(z) \leq \langle\mu\rangle \|\nu\| < \infty$ for all $\mu \in \hat{W}_c\left(\overline{\breve{K}}\right)$. By our assumption and by Fubini's theorem we obtain $0 = \int \hat{\mu}(z)\, d\nu(z) = \int\int \frac{1}{\zeta - z} d\nu(z)\, d\mu(\zeta) = -\int \hat{\nu} d\mu$. Especially on the one hand $\int_U \hat{\nu} d\lambda = 0$ holds for all bounded measurable sets $U \subseteq \overline{\breve{K}}$ and on the other holds following our Lemma 2.4 $\hat{\nu}(z) = \int q_z d\nu = 0\ \ \forall z \in \overset{\frown}{K}$. (If $\overset{\frown}{K} = \emptyset$, then nothing besides Proposition 4.4 is to show.) We see that with $\hat{\nu} = 0$ Lebesgue-almost everywhere also $\nu = 0$ holds following [22, page 18, Lemma 6]. Now $A_0\left(\mathbb{C}, \overset{\frown}{K}\right)$ is a Banach space in the supnorm and the proof is done by [4, page 422, Theorem 13 and Corollary]. $\square$

Theorem 4.7. *The restrictions to K of the Cauchy transforms of $W_c\left(\breve{K} \cap V_0\right)$ are a dense subset of R in the supnorm.*

Proof. Let be $z \in \breve{K}$, $0 < r_1 < r_2$, $\Delta(z, r_2) \subseteq \breve{K}$ and $\mu = \frac{\lambda}{-2\pi(r_2 - r_1)r}$ on $r_1 \le r \le r_2$, where r is the distance from z, then μ is a wide measure with support in $\breve{K}$, for which $q_z(\zeta) = \frac{1}{2\pi(r_2-r_1)} \int_{r_1}^{r_2} \int_0^{2\pi} \frac{1}{\zeta - z - re^{i\varphi}} \frac{d\varphi\, d\, r}{r} = \hat{\mu}(\zeta)$ holds. We have calculated with the Cauchy integral formula.

The q_z with $z \in \breve{K}$ by definition span a supnorm dense vector subspace of R, on the other hand following a result of Runge [24, Theorem 9.1] and our Proposition 3.5 together with the property 4.3.3 the restrictions to K of the Cauchy transforms of $W_c\left(\breve{K}\right)$ are a vector subspace of R. Following our Lemma 2.3 it is sufficient to consider the q_z with $z \in \breve{K} \cap V_0$. □

Theorem 4.8. *The restrictions to K of the Cauchy transforms of $W_c\left(\overline{\breve{K}} \cap V_0\right)$ are a dense subset of A in the supnorm.*

Proof. Following [12, Satz 3] each function in A can be continued to a function in $A_0\left(\mathbb{C}, \overset{\frown}{K}\right)$. The parts off V_0 of the considered measures can be replaced for the supnorm approximatin on K by measures supported on $Y_0 \cap V_0$ following our Lemma 2.3. Our Proposition 4.6 completes the proof. □

5. A sufficient condition in the theorem of Vitushkin

We now approach our main theorem and by considering the wide measures on the boundary of K gain a new sufficient condition for $A = R$. This sufficient condition allows us to handle whole classes of examples, which are given in the literature for $A = R$, in a simpler and visualizing manner.

The boundary of a single hole can support a wide measure, as the Denjoy Example 5.6 shows, when we take the inner boundary there for K. But for the supnorm approximation we can neglect such measures, as the following lemma shows.

Lemma 5.1. *Let be $\complement K$ connected, then the restrictions to K of the supnorm closures of $\hat{W}_c\left(\overline{\breve{K}}\right)$ and $\hat{W}_c\left(\breve{K}\right)$ are equal.*

Proof. The polynome result of Mergelyan [12] gives here $A = R$. So our Theorems 4.7 and 4.8 finish the proof. □

Definition 5.2. We denote $\mathcal{R} = W_c(\rho)$ and we denote $\mathcal{A} = W_c(\alpha) = W_c\left(\overline{\breve{K}} \cap V_0\right)$. □

$\mathcal{R} \subseteq \mathcal{A}$ is obvious.

Main Theorem 5.3. If $\mathcal{A} = \mathcal{R}$, then $A = R$.

The condition means, that the wide measures with compact support on α and on ρ are the same, or that the inner boundary does not support any wide measure. There are examples that the condition is not necessary. See our Remark 5.9.

Proof. Let be $\mathcal{R} = \mathcal{A}$, then by the definitions and the assumptions of the theorem the following holds $\hat{W}_c\left(\breve{K} \cap V_0\right) \subseteq \hat{W}_c(\rho) = \hat{\mathcal{R}} = \hat{\mathcal{A}} = \hat{W}_c\left(\overline{\breve{K}} \cap V_0\right)$. For the supnorm closures of the restrictions to K of the first two sets equality holds following 5.1, so $A = R$ holds following our Theorems 4.7 and 4.8. □

Our first corollary is a known result, which is included as illustration of our definitions and results and also to be compared to other methods.

Corollary 5.4. *$A = R$, when the inner boundary is a countable set of points.*

Proof. Each measure on a countable set of points is a series of pointmeasures, which converges in the variation norm. A wide measure on the inner boundary therefore had a representation as a series of wide pointmeasures converging in the wide norm following 3.2.3 and 3.5. But a wide pointmeasure is 0. □

A sufficient geometric condition of Vitushkin for $A = R$ is, that "the inner boundary is contained in a countable union of Ljapunow curves" [24, Theorem 14.3]. Our sufficient condition, which uses only measures, suggests that the piecewise differentiability of the Ljapunow curve is not really used. Indeed we can weaken this condition of Vitushkin considerably in our second corollary and at the same time we can replace the proof cited and described there as "long and complicated" by a short and clear one.

Corollary 5.5. *$A = R$, when the inner boundary is contained in a countable union of continuous curves $z(t)$, each of which fullfills the following conditions:*

(5.5.1) $|z(t) - z(\tau)| \leq c|t - \tau| \ \forall t, \tau \in [0, 1]$, *where $z(t)$ is its parameter representation and $0 < c$ is a real constant, which may depend on the single curve, and*

(5.5.2) the inverse image of $V(z(t), s)$ in the parameter representation is connected $\forall t \in [0, 1]$, *where $s > 0$ is a real constant, which may depend on the single curve.*

Proof. We argue by contradiction. Because of our Proposition 3.5 one of the curves would already support a positive wide measure μ, which by the measurable inverse image $\overset{-1}{z}$ of the parameter representation z would be carried to the unit intervall, we name it ν. For $|z(t) - z(\tau)| = r$ holds $|t - \tau| \geq rc^{-1}$ following *(5.5.1)* or together with *(5.5.2)* $\overset{-1}{z}(V(z(t), r)) \supseteq V(t, rc^{-1}) \ \forall\, 0 \leq r < s$, then $\mu(V(z(t), r)) \geq \nu(V(t, rc^{-1}))$ holds and also $\mu^{\approx}(r) \geq \nu^{\approx}(rc^{-1})$ holds $\forall\, r < s$ and finally $\int_0^s \frac{r+1}{r} d\mu^{\approx}(r) \geq \int_0^s \frac{r+1}{r} d\nu^{\approx}(r\, c^{-1}) = \infty$ because of 2.1 and 3.7. □

Example 5.6. As another application we consider continuous functions on $\overline{\mathbb{C}}$, which are analytc off Cantor sets of the plane. Such functions were examined already in the early literature, see [5] and [19]. They used to be represented as limits of convex combinations of functions q_z with poles not in the inner boundary, but tending to it. Let be $0 < t < 1$ and let be K_t the Cantor set, which we obtain, when we remove the middle interval of length t from the unit interval and repeat this process inductively on the 2^n closed intervals $I_{n,j}$, which remain after the n-th step. Let be $E_t = K_t \times K_t$. Then $E_t = \bigcap J_n$, where J_n is a union of 4^n squares with sides of length s^n with $s = \frac{1}{2}(1-t)$. Let be $r_n = \frac{1}{2}\sqrt{2}\, s^n$ the radius of the circle, which surrounds one of the squares, and let be $m_{n,k}$ the centre of the k-th square at the n-th step. The circles centred in $m_{n,k}$ with radius r_n separate the squares $I_{n,j} \times I_{n,k}$.

Not for all Cantor sets with parameter $0 < t < 1$ such functions exists. In [8] is shown, that for the parameters $0 < t < \frac{1}{2}$ such functions exist and do not exist for the parameters $\frac{1}{2} < t < 1$. For the parameter $t = \frac{1}{2}$ we confirm a conjecture of J. Garnett, that such functions do not exist. This could not be demonstrated in [8]. □

The direct application of our results requests as a helping construction a circle, which surrounds the Cantor set, and holes, which have the Cantor set as inner boundary. On the compact set K contructed in this way we consider only functions, whose rational parts vanish.

Lemma 5.7. *Let be μ a probability measure on E_t, then $\mu^{\approx}(r_n) \geq \frac{1}{4^n}$ holds.*
Let be β_t the probability measure on K_t with $\beta_t(I_{n,k}) = \frac{1}{2^n}$, let be $\beta_t^2 = \beta_t \otimes \beta_t$ and let be $\frac{1}{2}\left(\sqrt{2}-1\right) \leq t < 1$, then $\beta_t^{2\approx}(r_n) = \frac{1}{4^n}$ holds.

Proof. By calculations. □

Theorem 5.8. *E_t does not support any wide measure for $\frac{1}{2} \leq t < 1$.*
E_t supports the wide measure $\beta_t^2$for $\frac{1}{2}\left(\sqrt{2}-1\right) \leq t < \frac{1}{2}$.

Proof. Let be μ a probability measure on E_t and let be $\nu^{\approx}$ a crowding function with $\nu^{\approx}(t) = \frac{1}{4^n}$ for all $r_{n+1} \leq t < r_n$, then we estimate for $\frac{1}{2} \leq t < 1$ recalling $4s = 2(1-t)$ as follows:

$$\frac{1}{2}\langle\mu\rangle \geq \int_0^\infty \frac{r+1}{r}\, d\,\nu^{\approx}(r) \geq \sum_{n=1}^{\infty} \frac{1}{r_n}\left(\nu^{\approx}(r_n) - \nu^{\approx}(r_{n+1})\right)$$
$$= \sum_{n=1}^{\infty} \sqrt{2}\, s^{-n}\left(\frac{1}{4^n} - \frac{1}{4^{n+1}}\right) = \frac{3}{4}\sqrt{2}\sum_{n=1}^{\infty}\left(\frac{1}{2(1-t)}\right)^n = \infty$$

For $\frac{1}{2}\left(\sqrt{2}-1\right) \leq t < \frac{1}{2}$ we estimate as follows:

$$\frac{1}{2}\left\langle\beta_t^2\right\rangle \leq 1 + \sum_{n=1}^{\infty} \frac{1}{r_{n+1}}\left(\beta_t^{2\approx}(r_n) - \beta_t^{2\approx}(r_{n+1})\right)$$
$$= 1 + \sum_{n=1}^{\infty} \sqrt{2}\, s^{-n-1}\left(\frac{1}{4^n} - \frac{1}{4^{n+1}}\right) = 1 + \frac{3}{4}\sqrt{2}\,\frac{1}{s}\sum_{n=1}^{\infty}\left(\frac{1}{2(1-t)}\right)^n < \infty$$

We have used our Lemmata 2.1 and 5.7. □

For the Cauchy transforms of measures on Cantor sets see also [13]. Our method allows us a more exact glance at a result, which we cite from [6, page 219, 8.5 Corollary]: If K is compact and $\complement K$ has positive lower Lebesgue density at every point of the boundary of K, then $A = R$. The definition of the lower Lebesgue density is found at the cited place.

Remark 5.9. Let be K the compact set, which is formed by removing from the current middle square a concentric square of half of its area according to the construction of the Cantor set described in our Example 5.6. Then $E_t = \iota$ and $\complement K$ has positive lower Lebesgue density at every point of the boundary and $A = R$ holds independant of the parameter t of the construction. But for the parameters $\frac{1}{2} \leq t < 1$ the inner boundary does not support any wide measure following our Theorem 5.8. That means in these cases the assumption of the positive lower Lebesgue density is sufficient for $A = R$ but weaker than for the other parameters. $\square$

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