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On the Lower Semicontinuous Quasiconvex Envelope for Unbounded Integrands (II): Representation by Generalized Controls

Marcus Wagner
Brandenburg University of Technology, Department of Mathematics,
P. O. Box 10 13 44, 03013 Cottbus, Germany
wagner@math.tu-cottbus.de; www.thecitytocome.de

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Motivated by the study of multidimensional control problems of Dieudonné-Rashevsky type, e.g. nonconvex correspondence problems from image processing, we raise the question how to understand to notion of quasiconvexity for a continuous function f with a convex body $\mathrm{K} \subset \mathbb{R}^{nm}$ instead of the whole space $\mathbb{R}^{nm}$ as the range of definition. Extending f by $(+\infty)$ to the complement $\mathbb{R}^{nm} \setminus \mathrm{K}$, the appropriate quasiconvex envelope turns out to be

$$f^{(qc)}(w) = \sup \Big\{ g(w) \mid g: \ \mathbb{R}^{nm} \to \mathbb{R} \cup \{ (+\infty) \} \text{ quasiconvex and lower semicontinuous, } g(v) \leqslant f(v) \ \forall v \in \mathbb{R}^{nm} \Big\}.$$

In the present paper, we prove that $f^{(qc)}$ admits a representation as

$$f^{(qc)}(w) = \mathrm{Min} \Big\{ \int_{\mathrm{K}} f(v)\, d\nu(v) \mid \nu \in \mathrm{S}^{(qc)}(w) \Big\} \quad \forall w \in \mathrm{K}$$

where the sets $\mathrm{S}^{(qc)}(w)$ are nonempty, convex, weak*-sequentially compact subsets of probability measures. This theorem, forming a natural counterpart to the author's previous results about the representation of $f^{(qc)}$ in terms of Jacobi matrices, has been proven indispensable for the derivation of Jensens' integral inequality as well as of differentiability theorems for the envelope $f^{(qc)}$. The paper is mainly concerned with a detailed analysis of the set-valued map $\mathrm{S}^{(qc)}$, which will be explicitely described in terms of averages of generalized controls.

Keywords: Unbounded function, quasiconvex envelope, probability measure, generalized control, mean value theorem, set-valued map, representation theorem

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1. Introduction

a) Nonconvex relaxation of multidimensional control problems.

With the present paper, we continue a series of publications concerned with existence and relaxation theorems for multidimensional control problems of *Dieudonné-Rashevsky*

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type:

$$(\mathrm{P}): \qquad F(x) = \int_\Omega f_0(\,t, x(t), Jx(t)\,)\,dt \longrightarrow \inf!\,; \quad x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)\,; \tag{1}$$

$$Jx(t) = \begin{pmatrix} \dfrac{\partial x_1}{\partial t_1}(t) & \dots & \dfrac{\partial x_1}{\partial t_m}(t) \\ \vdots & & \vdots \\ \dfrac{\partial x_n}{\partial t_1}(t) & \dots & \dfrac{\partial x_n}{\partial t_m}(t) \end{pmatrix} \in \mathrm{K} \subset \mathbb{R}^{n\times m} \;\; (\forall)\, t \in \Omega\,. \tag{2}$$

Here the dimensions are $n \geqslant 1$, $m \geqslant 2$, while $\Omega \subset \mathbb{R}^m$ is the closure of a bounded Lipschitz domain, $\mathrm{K} \subset \mathbb{R}^{nm}$ is a convex body with $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$, and $f_0(t,\xi,v):\ \Omega \times \mathbb{R}^n \times \mathrm{K} \to \mathbb{R}$ is a continuous, in general nonconvex integrand.

Problems of this kind arise in different contexts. We mention the study of underdetermined boundary value problems for nonlinear first-order PDE's,[1] optimization problems for convex bodies under geometrical restrictions,[2] and applications in elasticity theory (torsion problems)[3] as well as in population dynamics (age-structured models).[4] Recently, a further application area has been opened in mathematical image processing. The incorporation of a gradient constraint of type (2) into the variational formulation of the problems of image restoration, optical flow, image matching etc. allows in a very natural way the detection of edge structures within the image data.[5] In this context, there is a particular interest in the treatment of vectorial problems with nonconvex integrands.[6] Consequently, these problems in their variational formulation as well as in their recently studied formulation as multidimensional control problems (P) require a quasiconvex relaxation instead of a convex one. Motivated by these applications, we continue in the present paper the study of the relaxation of nonconvex Dieudonné-Rashevsky type problems in the vectorial case, particularly from the point of view of generalized controls ("Young measures").

b) The lower semicontinuous quasiconvex envelope.

In order to extend the known relaxation results in multidimensional control to the vectorial case ($n \geqslant 2$), the author introduced an appropriate quasiconvex envelope for unbounded integrands $f:\ \mathbb{R}^{nm} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\,(+\infty)\,\}$. For such functions, the notion of quasiconvexity must be precised in the following way:

Definition 1.1 (Quasiconvex function with values in $\overline{\mathbb{R}}$).[7] A function $f:\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ with the following properties is said to be quasiconvex:

1) $\mathrm{dom}\,(f) \subseteq \mathbb{R}^{nm}$ is a (nonempty) Borel set;
2) $f\,\big|\,\mathrm{dom}\,(f)$ is Borel measurable and bounded from below on every bounded subset of $\mathrm{dom}\,(f)$;

[1] [11], [12], [13].
[2] [1] and [2], p. 149 f.
[3] [19], pp. 531 ff., [24], pp. 240 ff., [29], p. 531 f., [30] and [31], pp. 76 ff.
[4] [7], [17].
[5] [8], [18], [35].
[6] E.g. polyconvex integrands carried over from hyperelasticity models (cf. [14]) or nonconvex regularization terms of Perona-Malik type (cf. [4], pp. 90 ff., and [21]).
[7] [34], p. 73, Definition 2.9, as a specification of [5], p. 228, Definition 2.1, in the case $p = (+\infty)$.

3) for all $v \in \mathbb{R}^{nm}$, f satisfies Morrey's integral inequality:

$$f(v) \leqslant \frac{1}{|\,\Omega\,|} \int_\Omega f(\,v + Jx(t)\,)\,dt \quad \forall\, x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)\,; \tag{3}$$

or equivalently

$$f(v) \,=\, \inf \left\{ \frac{1}{|\,\Omega\,|} \int_\Omega f\,(\,v + Jx(t)\,)\,\,dt \;\Big|\; x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n) \right\}. \tag{4}$$

Here $\Omega \subset \mathbb{R}^m$ is the closure of a bounded strongly Lipschitz domain.

The lower semicontinuous quasiconvex envelope of an unbounded function is then defined as follows:

Definition 1.2 (Lower semicontinuous quasiconvex envelope $f^{(qc)}$ for functions with values in $\overline{\mathbb{R}}$).[8] To any function $f:\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ bounded from below, we define

$$\begin{aligned} f^{(qc)}(v) = \sup \Big\{\, g(v) \;\Big|\; g:\ \mathbb{R}^{nm} \to \overline{\mathbb{R}} \text{ quasiconvex and lower semicontinuous,} \\ g(v) \leqslant f(v)\ \ \forall\, v \in \mathbb{R}^{nm} \,\Big\}. \end{aligned} \tag{5}$$

Obviously, Definition 1.2 generalizes the usual formation of a quasiconvex envelope since the quasiconvex functions g below a finite function f must be continuous from the outset.[9] Assume now that $\mathrm{K} \subset \mathbb{R}^{nm}$ is a convex body with $\mathfrak{o} \in \operatorname{int}(\mathrm{K})$ and $f:\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ is a function with $f\,\big|\,\mathrm{K} \in C^0(\mathrm{K}, \mathbb{R})$ and $f\,\big|\,(\,\mathbb{R}^{nm} \setminus \mathrm{K}\,) \equiv (+\infty)$. In this situation, the author proved that $f^{(qc)}$ may be represented in terms of Jacobi matrices in the following way:

Theorem 1.3 (Representation of $f^{(qc)}$ in terms of Jacobi matrices).[10] *Under the assumptions mentioned above, the lower semicontinuous quasiconvex envelope $f^{(qc)}:\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ admits the representation*

$$f^{(qc)}(v_0) = \begin{cases} f^*(v_0) \;\big|\; v_0 \in \operatorname{int}(\mathrm{K})\,; \\ \displaystyle\lim_{v \to v_0\,,\, v \in \mathrm{R} \cap \operatorname{int}(\mathrm{K})} f^*(v) \;\big|\; v_0 \in \partial \mathrm{K}\,; \\ (+\infty) \;\big|\; v_0 \in \mathbb{R}^{nm} \setminus \mathrm{K}\,, \end{cases} \tag{6}$$

where $\mathrm{R} = \overrightarrow{\mathfrak{o}\ v_0}$ denotes the ray through v_0 starting from the origin, and $f^*(v_0)$ is defined by

$$\begin{aligned} f^*(v_0) = \inf \Big\{ \frac{1}{|\,\Omega\,|} \int_\Omega f(\,v_0 + Jx(t)\,)\,dt \;\Big|\; x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)\,, \\ v_0 + Jx(t) \in \mathrm{K}\ \ (\forall)\, t \in \Omega \Big\} \in \overline{\mathbb{R}}\,. \end{aligned} \tag{7}$$

[8] [34], p. 76, Definition 2.14, (2).
[9] [10], p. 47, Theorem 2.31.
[10] [34], p. 95, Theorem 4.1.

c) Representation of $f^{(qc)}$ by probability measures.

Although Theorem 1.3 allows to prove an appropriate generalization of Ekeland/Témam's relaxation theorem,[11] the knowledge of the representation of $f^{(qc)}$ in terms of Jacobi matrices turns out to be insufficient for further analysis of nonconvex control problems (P). Namely, the proofs of a Jensen type integral inequality[12] and differentiability theorems for $f^{(qc)}$,[13] require an alternative representation of $f^{(qc)}$ in terms of probability measures. In the situation of Theorem 1.3, it is well-known that for all $w \in \mathrm{K}$, the convex envelope of f admits the description[14]

$$f^c(w) \;=\; \mathrm{Min}\left\{ \int_{\mathrm{K}} f(v)\,d\nu(v) \;\Big|\; \nu \in \mathrm{S}^c(w) \right\} \tag{8}$$

where $\mathrm{S}^c(w)$ is defined by

$$\mathrm{S}^c(w) = \Bigg\{ \nu \in rca\,(\mathbb{R}^{nm}) \;\Big|\; \nu \text{ is a probability measure,} \\ \mathrm{supp}\,(\nu) \subseteq \mathrm{K}\,,\; w \;=\; \begin{pmatrix} \int_{\mathrm{K}} v_{11}\,d\nu(v) & \dots & \int_{\mathrm{K}} v_{1m}\,d\nu(v) \\ \vdots & & \vdots \\ \int_{\mathrm{K}} v_{n1}\,d\nu(v) & \dots & \int_{\mathrm{K}} v_{nm}\,d\nu(v) \end{pmatrix} \Bigg\}\,. \tag{9}$$

In the present paper, we search for an analogous description of $f^{(qc)}$, depending on subsets $\mathrm{S}^{(qc)}(w) \subseteq \mathrm{S}^c(w)$.[15] As the main result, we obtain the following representation theorem for $f^{(qc)}$ as a natural counterpart to Theorem 1.3.

Theorem 1.4 (Representation of $f^{(qc)}$ in terms of probability measures). *Assume that $\mathrm{K} \subset \mathbb{R}^{nm}$ is a convex body with $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$, and $f:\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ is a function with $f\,\big|\,\mathrm{K} \in C^0(\mathrm{K},\mathbb{R})$ and $f\,\big|\,(\mathbb{R}^{nm} \setminus \mathrm{K}) \equiv (+\infty)$. Then for all $w \in \mathrm{K}$, $f^{(qc)}(w)$ may be represented as*

$$f^{(qc)}(w) \;=\; \mathrm{Min}\left\{ \int_{\mathrm{K}} f(v)\,d\nu(v) \;\Big|\; \nu \in \mathrm{S}^{(qc)}(w) \right\}. \tag{10}$$

Here $\mathrm{S}^{(qc)}:\ \mathrm{K} \to \mathfrak{P}(rca\,(\mathbb{R}^{nm}))$ is an upper semicontinuous set-valued map with nonempty, convex, weak-sequentially compact images $\mathrm{S}^{(qc)}(w) \subseteq \mathrm{S}^c(w)$. $\mathrm{S}^{(qc)}$ coincides with a map $\mathrm{S}^{\#}$, which is explicitely described in Definition 3.8 below.*

The real importance of Theorem 1.4 lies not in the mere fact that $f^{(qc)}$ admits a representation of the claimed type but in the *explicit description* of the underlying set-valued map $\mathrm{S}^{(qc)}$, which turns out to be indispensable for the subsequent analysis of $f^{(qc)}$ in [37] and [38]. It will be established by a meticulous examination of convexity and continuity properties of $\mathrm{S}^{(qc)}$, particularly when approaching the points of $\partial\mathrm{K}$ where f is discontinuous. Consequently, the desired set-valued map $\mathrm{S}^{(qc)}$ will be constructed in two steps: at

[11] See [36], p. 309, Theorem 1.3, and its generalization to integrands $f(t,\xi,v)$ in [39], p. 4, Theorem 1.4.
[12] [37], p. 608 f., Theorem 1.6.
[13] [38], p. 3 f., Theorem 1.6.
[14] Cf. [33], p. 131, Theorem 10.19.
[15] See [26], pp. 8 ff., Section 1.3.

first we define a map S^* on int (K) and then its extension $\mathrm{S}^\#$ to ∂K by an explicit limit passage.[16] We prove then that the resulting set-valued map $\mathrm{S}^\#$ is upper semicontinuous. Adopting the averaging technique for Young measures introduced by Kinderlehrer and Pedregal,[17] we are finally able to confirm that $\mathrm{S}^\# = \mathrm{S}^{(qc)}$ fulfills the claims of Theorem 1.4.

d) Outline of the paper.

In order to provide the main tools needed in our analysis, we go in *Section 2* through a condensed review of the theory of generalized controls, considering first the metric space rca^{pr} (K) of the probability measures supported on K, then the set $\boldsymbol{\mathcal{Y}}$(K) of generalized controls ("Young measures") and the subset $\boldsymbol{\mathcal{G}}$(K) of those generalized controls, which can be generated by Jacobi matrices (generalized gradient controls, "gradient Young measures"). Moreover, the section contains an appropriate generalization of Kinderlehrer/Pedregal's mean value theorem. Since most of the facts are well-known and can be easily adapted to the situation of problem (P), the proofs have been omitted completely in this section. The main part of our analysis will be performed in *Section 3*. As mentioned before, we define the map $\mathrm{S}^{(qc)}$ in two steps: at first as a continuous set-valued map S^* on int (K), and then as the upper semicontinuous extension $\mathrm{S}^\# = \mathrm{S}^{(qc)}$ of S^* to ∂K. It turns out that the ε-δ relations for the functions f^* and $f^\#$ from [34][18] may be rephrased within the context of set-valued maps where the upper semicontinuity of the set-valued map $\mathrm{S}^\#$ takes the place of the lower semicontinuity of the function $f^\# = f^{(qc)}$. *Section 4* is devoted to the proof of Theorem 1.4. Finally, in the *Appendix* the concepts from the theory of set-valued maps (Painlevé-Kuratowski limits, semicontinuity and continuity) have been summarized.

e) Notations and abbreviations.

Let $k \in \{0, 1, \dots, \infty\}$ and $1 \leqslant p \leqslant \infty$. Then $C^k(\Omega, \mathbb{R}^r)$, $L^p(\Omega, \mathbb{R}^r)$ and $W^{k,p}(\Omega, \mathbb{R}^r)$ denote the spaces of r-dimensional vector functions whose components are k-times continuously differentiable, belong to $L^p(\Omega)$ or to the Sobolev space of $L^p(\Omega)$-functions with weak derivatives up to kth order in $L^p(\Omega)$, respectively. In addition, functions within the subspaces $C_0^k(\Omega, \mathbb{R}^r) \subset C^k(\Omega, \mathbb{R}^r)$ are compactly supported while functions within the subspace $W_0^{1,\infty}(\Omega, \mathbb{R}^r) \subset W^{1,\infty}(\Omega, \mathbb{R}^r)$ admit a (Lipschitz-) continuous representative[19] with zero boundary values. The symbol $\partial x / \partial t_j$ may denote the classical as well as the weak partial derivative of x by t_j. The Jacobi matrix of x is abbreviated as Jx.

The space of Radon measures (signed regular measures) acting on the σ-algebra of the Borel subsets of a compact set $\mathrm{K} \subset \mathbb{R}^{nm}$ is denoted by rca (K). Endowed with the total variation norm $\vee\mu(\mathrm{K})$, it forms a Banach space.[20] Due to the compactness of K, the dual space $(C^0(\mathrm{K}, \mathbb{R}))^*$ and rca (K) are isomorphic,[21] consequently, every linear, continuous functional on $C^0(\mathrm{K}, \mathbb{R})$ may be represented as an integral with respect to some Radon measure $\nu \in rca$ (K). The subset of the probability measures, equipped with a suitable

[16] The limit has to be understood in the sense of Painlevé/Kuratowski, see the Appendix.
[17] [22], [26].
[18] See [34], p. 82, Theorem 3.5, p. 88, Theorem 3.12, and p. 89, Theorem 3.15.
[19] [16], p. 131, Theorem 5.
[20] [15], p. 161 f.
[21] Ibid., p. 265, Theorem 3.

metric, will be denoted by $rca^{pr}(\mathrm{K})$ (see Definition 2.1 below). The Dirac measure concentrated in $v \in \mathbb{R}^{nm}$ is denoted by δ_v.

We denote by $\mathrm{int}\,(\mathrm{A})$, $\partial\mathrm{A}$, $\mathrm{cl}\,(\mathrm{A})$, $\mathrm{co}\,(\mathrm{A})$ and $|\,\mathrm{A}\,|$ the interior, boundary, closure, the convex hull and the r-dimensional Lebesgue measure of the set $\mathrm{A} \subseteq \mathbb{R}^r$, respectively. $\mathbb{1}_{\mathrm{A}} : \mathbb{R}^r \to \mathbb{R}$ with $\mathbb{1}_{\mathrm{A}}(t) = 1 \iff t \in \mathrm{A}$ and $\mathbb{1}_{\mathrm{A}}(t) = 0 \iff t \notin \mathrm{A}$ is the characteristic function of the set $\mathrm{A} \subseteq \mathbb{R}^r$. We set $\overline{\mathbb{R}} = \mathbb{R} \cup \{\,(+\infty)\,\}$ and equip $\overline{\mathbb{R}}$ with the natural topological and order structures where $(+\infty)$ is the greatest element. Throughout the whole paper, we consider only *proper functions* $f : \mathbb{R}^{nm} \to \overline{\mathbb{R}}$, assuming that $\mathrm{dom}\,(f) = \{\, v \in \mathbb{R}^{nm} \mid f(v) < (+\infty)\,\}$ is always nonempty. The restriction of the function f to the subset A of its range of definition is denoted by $f \,\big|\, \mathrm{A}$.

Definition 1.5 (Function class $\mathcal{F}_{\mathrm{K}}$). Let $\mathrm{K} \subset \mathbb{R}^{nm}$ be a given convex body with $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$. We say that a function $f : \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ belongs to the class $\mathcal{F}_{\mathrm{K}}$ iff $f \,\big|\, \mathrm{K} \in C^0(\mathrm{K}, \mathbb{R})$ and $f \,\big|\, (\,\mathbb{R}^{nm} \setminus \mathrm{K}\,) \equiv (+\infty)$.

Consequently, any function $f \in \mathcal{F}_{\mathrm{K}}$ is bounded and uniformly continuous on K, and the class $\mathcal{F}_{\mathrm{K}}$ and the Banach space $C^0(\mathrm{K}, \mathbb{R})$ are isomorphic and isometric.

If X is an arbitrary set then $\mathfrak{P}(\mathrm{X})$ denotes the set of all subsets of X. For the definition of the Painlevé-Kuratowski limits $\liminf^K{}_{N\to\infty} \mathrm{E}_N$, $\limsup^K{}_{N\to\infty} \mathrm{E}_N$ and $\lim^K{}_{N\to\infty} \mathrm{E}_N$ of a set sequence $\{\, \mathrm{E}_N \,\}$, we refer to the Appendix.

We close this subsection with *three nonstandard notations.* "$\{\, x^N \,\}$, A" denotes a sequence $\{\, x^N \,\}$ with members $x^N \in \mathrm{A}$. If $\mathrm{A} \subseteq \mathbb{R}^r$ then the abbreviation "$(\forall)\, t \in \mathrm{A}$" has to be read as "for almost all $t \in \mathrm{A}$" resp. "for all $t \in \mathrm{A}$ except a r-dimensional Lebesgue null set". The symbol $\mathfrak{o}$ denotes, depending on the context, the zero element resp. the zero function of the underlying space.

2. Generalized controls.

a) The metric space $rca^{pr}(\mathrm{K})$ of the probability measures supported on K.

Throughout the whole section, we assume that $\mathrm{K} \subset \mathbb{R}^{nm}$ is a fixed convex body with $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$. X denotes the (norm-)closed unit ball of the Banach space $rca\,(\mathrm{K})$. In the following, X will be equipped with the following metric:

Definition 2.1 (Metric space $rca^{pr}(\mathrm{K})$). [22] The subset of the probability measures $\nu \in rca\,(\mathrm{K})$, endowed with the metric $\sigma : \mathrm{X} \times \mathrm{X} \to \mathbb{R}$, defined by

$$\sigma(\nu', \nu'') = \sum_{s=1}^{\infty} \frac{1}{2^{1+s}\,(1 + L_s)} \,\Big|\, \int_{\mathrm{K}} g_s(v)\,\big(\, d\nu'(v) - d\nu''(v)\,\big) \,\Big|\,, \tag{11}$$

forms the metric space $rca^{pr}(\mathrm{K})$. Here $g_s \in C^0(\mathrm{K}, \mathbb{R}) \cap W^{1,\infty}(\mathrm{K}, \mathbb{R})$ are countably many functions with $\|\, g_s \,\|_{C^0(\mathrm{K},\mathbb{R})} = 1$ and Lipschitz constants $L_s > 0$, which form a dense subset $\{\, g_s \,\}$ of the unit ball of $C^0(\mathrm{K}, \mathbb{R})$ with respect to its norm topology.

[22] [33], pp. 48 ff., Definition 4.1, Lemma 4.2 and Definition 4.3.

By Alaoglu's theorem,[23] together with the metric σ from Definition 2.1, X forms a compact metric space. The probability measures ν with $\operatorname{supp}(\nu) \subseteq \mathrm{K}$ form a convex, weak*-closed subset of X [24] and, consequently, a compact metric subspace of $[\,\mathrm{X}\,,\,\sigma\,]$. The properties of $rca^{pr}(\mathrm{K})$ are summarized in the following theorem:

Theorem 2.2 (Properties of the metric space $rca^{pr}(\mathrm{K})$).

1) $rca^{pr}(\mathrm{K})$ *forms a compact metric space.*

2)[25] *For every sequence* $\{\,\nu^N\,\}\,,\, rca^{pr}(\mathrm{K})$, *it holds that* $\lim_{N\to\infty} \sigma(\nu^N, \nu) = 0 \iff \nu^N \xrightarrow{\ *\ } {}_{rca\,(\mathrm{K})}\, \nu$. *Therefore, in* $rca^{pr}(\mathrm{K})$, *convergence with respect to the metric* σ *is equivalent to weak*-convergence.*

3)[26] *The metric space* $rca^{pr}(\mathrm{K})$ *is separable with the countable, dense subset*

$$\Big\{ \sum_{k=1}^{K} \lambda_k\, \delta_{v_k} \;\Big|\; \sum_{k=1}^{K} \lambda_k = 1\,,\ \lambda_k \in [\,0\,,\,1] \cap \mathbb{Q}\,, \qquad v_k \in \mathrm{K} \cap \mathbb{Q}^{nm}\,,\ 1 \leqslant k \leqslant K, K \in \mathbb{N} \Big\}. \tag{12}$$

b) Generalized controls ("Young measures").

We start with the definition of the set $\mathcal{Y}(\mathrm{K})$ of generalized controls for (P). Throughout this subsection, we assume that $\Omega \subset \mathbb{R}^m$ is the closure of a strongly Lipschitz domain.

Definition 2.3 (Generalized controls, "Young measures"). A measure-valued map $\boldsymbol{\mu}:\ \Omega \to rca^{pr}(\mathrm{K})$ with $t \longmapsto \mu_t$ is called a generalized control if, for any continuous function $g \in C^0(\mathrm{K}, \mathbb{R})$, the function $h_g(t) = \int_{\mathrm{K}} g(v)\, d\mu_t(v)$ is Borel measurable on Ω. Two generalized controls $\boldsymbol{\mu}' = \{\,\mu'_t\,\}$ and $\boldsymbol{\mu}'' = \{\,\mu''_t\,\}$ will be identified if $\mu'_t \equiv \mu''_t$ holds for almost all $t \in \Omega$. The set of all equivalence classes of generalized controls will be denoted by $\mathcal{Y}(\mathrm{K})$. The convergence of a sequence $\{\,\boldsymbol{\mu}^N\,\}\,,\,\mathcal{Y}(\mathrm{K})$ towards the limit $\boldsymbol{\mu} \in \mathcal{Y}(\mathrm{K})$ is defined through

$$\boldsymbol{\mu}^N \to \boldsymbol{\mu} \iff \int_\Omega \int_{\mathrm{K}} f(t)\, g(v)\, \big(\, d\mu_t^N(v) - d\mu_t(v)\,\big)\, dt \to 0 \qquad \text{for all } f \in L^1(\Omega, \mathbb{R}),\ g \in C^0(\mathrm{K}, \mathbb{R}). \tag{13}$$

Remarks. 1) In the literature, the elements of $\mathcal{Y}(\mathrm{K})$ are commonly called "Young measures" or "parametrized measures".[27] We prefer, however, the use of the term "generalized control" introduced by Gamkrelidze,[28] since we must carefully distinguish between (equivalence classes of) measure-valued maps and single measures, resulting from these maps by an averaging process.

2) Equipped with the topology introduced above, $\mathcal{Y}(\mathrm{K})$ becomes a sequentially compact topological space.[29] Since both spaces $L^1(\Omega, \mathbb{R})$ and $C^0(\mathrm{K}, \mathbb{R})$ are separable, the topology

[23] [15], p. 424, Theorem 2.
[24] [28], p. 47 f. , Proposition 1.5.1 (iii).
[25] [33], p. 48 f., Lemma 2.2.
[26] [33], p. 49, Theorem 4.4, 2).
[27] [22], p. 331 ff., [25], p. 115 ff., [26], p. 20 ff., etc.
[28] [20], p. 23.
[29] [6], p. 144, Proposition 1 (i); independently proved again in [23], p. 391, Theorem 4.

from Definition 2.3 is metrizable.[30]

3) The relation between generalized and "ordinary" controls for (P) will be established in Theorem 2.6 below.

Definition 2.4 (Distance function ϱ on $\boldsymbol{\mathcal{Y}}(\mathrm{K})$). [31] Assume that countably many functions $f_1 \equiv 1/|\,\Omega\,|$, $f_r \in C^0(\Omega, \mathbb{R}) \cap L^1(\Omega, \mathbb{R})$ with $\|\, f_r \,\|_{C^0(\Omega,\mathbb{R})} \cdot |\,\Omega\,| = 1$ for $r \geqslant 2$ as well as $g_s \in C^0(\mathrm{K}, \mathbb{R}) \cap W^{1,\infty}(\mathrm{K}, \mathbb{R})$ with $\|\, g_s \,\|_{C^0(\mathrm{K},\mathbb{R})} = 1$ and Lipschitz constants $L_s > 0$ for $s \geqslant 1$ are given such that $\{\, f_r \,\}$ resp. $\{\, g_s \,\}$ form dense subsets of the unit balls of $L^1(\Omega, \mathbb{R})$ resp. $C^0(\mathrm{K}, \mathbb{R})$ with respect to their norm topologies. Then the function $\varrho :\ \boldsymbol{\mathcal{Y}}(\mathrm{K}) \times \boldsymbol{\mathcal{Y}}(\mathrm{K}) \to \mathbb{R}$ defined by

$$\begin{aligned} \varrho(\boldsymbol{\mu}', \boldsymbol{\mu}'') = \sum_{s=1}^{\infty} \frac{1}{2^{1+s}\,(1 + L_s)} \cdot \frac{1}{|\,\Omega\,|} \Big| \int_\Omega \int_\mathrm{K} g_s(v) \,\big(\, d\mu_t'(v) - d\mu_t''(v) \,\big)\, dt \,\Big| \\ + \sum_{r=2}^{\infty} \sum_{s=1}^{\infty} \frac{1}{2^{r+s}\,(1 + L_s)} \,\Big| \int_\Omega \int_\mathrm{K} f_r(t)\, g_s(v) \,\big(\, d\mu_t'(v) - d\mu_t''(v) \,\big)\, dt \,\Big| \end{aligned} \tag{14}$$

is a distance function on $\boldsymbol{\mathcal{Y}}(\mathrm{K})$ with $\{\, \boldsymbol{\mu}^N \,\}\,,\, \boldsymbol{\mathcal{Y}}(\mathrm{K}) \to \boldsymbol{\mu} \iff \varrho(\boldsymbol{\mu}^N, \boldsymbol{\mu}) \to 0$.

By means of the metric ϱ, we introduce the notion of a generating (function) sequence for a generalized control $\boldsymbol{\mu}$.

Definition 2.5 (Generating sequences for generalized controls). [32] We say that the sequence $\{\, u^N \,\}\,,\, L^\infty(\Omega, \mathbb{R}^{nm})$ generates the generalized control $\boldsymbol{\mu} \in \boldsymbol{\mathcal{Y}}(\mathrm{K})$ if $u^N(t) \in \mathrm{K}\ (\forall)\, t \in \Omega\ \forall\, N \in \mathbb{N}$ and

$$\lim_{N\to\infty} \varrho\big(\,\{\, \delta_{u^N(t)} \,\}\,,\, \boldsymbol{\mu}\,\big) \;=\; 0\,, \quad \text{i.e.} \tag{15}$$

$$\begin{aligned} &\lim_{N\to\infty} \int_\Omega f(t)\, g(\, u^N(t)\,)\, dt \\ = \;&\lim_{N\to\infty} \int_\Omega \int_\mathrm{K} f(t)\, g(v)\, d\delta_{u^N(t)}(v)\, dt \;=\; \int_\Omega \int_\mathrm{K} f(t)\, g(v)\, d\mu_t(v)\, dt\,, \end{aligned} \tag{16}$$

for all $f \in L^1(\Omega, \mathbb{R})$, $g \in C^0(\mathrm{K}, \mathbb{R})$.

Theorem 2.6 (Properties of the space $\boldsymbol{\mathcal{Y}}(\mathrm{K})$).

1)[33] *Every sequence* $\{\, u^N \,\}\,,\, L^\infty(\Omega, \mathbb{R}^{nm})$ *with* $u^N(t) \in \mathrm{K}\ (\forall)\, t \in \Omega\ \forall\, N \in \mathbb{N}$ *admits a weak*-convergent subsequence, which generates a generalized control* $\boldsymbol{\mu} \in \boldsymbol{\mathcal{Y}}(\mathrm{K})$.

2)[34] *Conversely, the generalized controls of the shape* $\boldsymbol{\mu} = \{\, \delta_{u(t)} \,\}$ *with* $u \in L^\infty(\Omega, \mathbb{R}^{nm})$, $u(t) \in \mathrm{K}\ (\forall)\, t \in \Omega$, *are dense in* $\boldsymbol{\mathcal{Y}}(\mathrm{K})$ *with respect to the topology introduced above.*

3)[35] *Consider a sequence* $\{\, \boldsymbol{\mu}^M \,\}\,,\, \boldsymbol{\mathcal{Y}}(\mathrm{K}) \to \boldsymbol{\mu} \in \boldsymbol{\mathcal{Y}}(\mathrm{K})$ *together with generating sequences* $\{\, u^{M,N} \,\}\,,\, L^\infty\big(\Omega, \mathbb{R}^{nm}\big)$ *for every* $\boldsymbol{\mu}^M$. *Then* $\boldsymbol{\mu}$ *is generated by a diagonal sequence* $\{\, u^{M,N(M)} \,\}\,,\, L^\infty(\Omega, \mathbb{R}^{nm})$.

[30] Cf. [22], p. 337.
[31] [33], p. 51 f., Definition 4.8 and Lemma 4.9.
[32] Cf. [26], pp. 96 ff.
[33] [25], p. 115 f., Theorem 3.1.
[34] [6], p. 148, Proposition 4.
[35] [33], p. 52, Lemma 2.12.

Parts 1) and 2) of this assertion establish the relation between "ordinary" and generalized controls: If the control domain $\mathrm{U} = \{\, u \in L^\infty(\Omega, \mathbb{R}^{nm}) \mid u(t) \in \mathrm{K} \ (\forall)\, t \in \Omega \,\}$ of (P) is embedded into $\boldsymbol{\mathcal{Y}}(\mathrm{K})$ via $u \longmapsto \{\, \delta_{u(t)} \,\}$ then the image of U forms a dense subset of the sequentially compact space $\boldsymbol{\mathcal{Y}}(\mathrm{K})$.

c) Generalized gradient controls ("gradient Young measures").

Let us now closer investigate those generalized controls, which are generated by sequences of gradients ($n = 1$) resp. Jacobi matrices ($n > 1$). They form a sequentially compact subset $\boldsymbol{\mathcal{G}}(\mathrm{K})$ of $\boldsymbol{\mathcal{Y}}(\mathrm{K})$.

Definition 2.7 (Generalized gradient controls, "gradient Young measures").[36] A measure-valued map $\boldsymbol{\mu} \in \boldsymbol{\mathcal{Y}}(\mathrm{K})$ is called a generalized gradient control if it is generated by a sequence $\{\, Jx^N \,\}$, $L^\infty(\Omega, \mathbb{R}^{nm})$ with $x \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ and $Jx^N(t) \in \mathrm{K}\ (\forall)\, t \in \Omega$ $\forall\, N \in \mathbb{N}$. The set of (equivalence classes of) generalized gradient controls will be denoted by $\boldsymbol{\mathcal{G}}(\mathrm{K}) \subseteq \boldsymbol{\mathcal{Y}}(\mathrm{K})$.

Remarks. 1) In the literature, the elements of $\boldsymbol{\mathcal{G}}(\mathrm{K})$ are referred to as "gradient Young measures" or "gradient parametrized measures". This notion has been avoided fore the same reasons as mentioned after Definition 2.3 above.[37]

2) Assume that a generating sequence satisfies $x^N \rightarrow_{C^0(\Omega, \mathbb{R}^n)} x \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ and $Jx^N \xrightarrow{\ *\ }_{L^\infty(\Omega, \mathbb{R}^{nm})} Jx \in L^\infty(\Omega, \mathbb{R}^{nm})$. If we replace in (P) the Jacobi matrix Jx by a formal control variable u then we obtain a (weakly formulated) state equation $Jx = u$. Then from Definition 2.5, it follows that

$$\begin{aligned} & \frac{\partial x_i^N}{\partial t_j}(t) = u_{ij}^N(t) \quad (\forall)\, t \in \Omega \\ \iff \quad & \int_\Omega \psi_i(t) \left(\frac{\partial x_i^N}{\partial t_j}(t) - u_{ij}^N(t) \right) dt = 0 \quad \forall\, \psi_i \in C_0^\infty(\Omega, \mathbb{R}) \implies \end{aligned} \tag{17}$$

$$\begin{aligned} & \lim_{N \to \infty} \int_\Omega \psi_i(t) \left(\frac{\partial x_i^N}{\partial t_j}(t) - \int_\mathrm{K} v_{ij}\, d\delta_{u^N(t)}(v) \right) dt \\ = \quad & \int_\Omega \psi_i(t) \left(\frac{\partial x_i}{\partial t_j}(t) - \int_\mathrm{K} v_{ij}\, d\mu_t(v) \right) dt = 0 \quad \forall\, \psi_i \in C_0^\infty(\Omega, \mathbb{R})\,. \end{aligned} \tag{18}$$

Consequently, the elements of $\boldsymbol{\mathcal{G}}(\mathrm{K})$ are those generalized controls, which may appear on the right-hand side of the relaxed state equation of (P). Moreover, under the assumptions mentioned above, they satisfy the integrability conditions

$$\begin{aligned} & \int_\Omega \left(\frac{\partial x_i}{\partial t_j}(t)\, \frac{\partial \psi_i}{\partial t_k}(t) - \frac{\partial x_i}{\partial t_k}(t)\, \frac{\partial \psi_i}{\partial t_j}(t) \right) dt \\ = & \int_\Omega \int_\mathrm{K} \left(\frac{\partial \psi_i}{\partial t_k}(t)\, v_{ij} - \frac{\partial \psi_i}{\partial t_j}(t)\, v_{ik} \right) d\mu_t(v)\, dt \;=\; 0 \quad \forall\, \psi_i \in C_0^\infty(\Omega, \mathbb{R}) \end{aligned} \tag{19}$$

in the distributional sense.[38]

[36] [22], p. 333, [25], p. 126, Definition 4.1.
[37] In [33], p. 54, Definition 4.13, the German notion "verallgemeinerte Jacobi-Steuerungen" has been proposed.
[38] Cf. [32], p. 169 f., Theorem 1.4.

Theorem 2.8 (Properties of the space $\mathcal{G}(\mathrm{K})$). [39]

1) *Every sequence* $\{x^N\}$, $W^{1,\infty}(\Omega,\mathbb{R}^n)$ *with* $\|x^N\|_{L^\infty(\Omega,\mathbb{R}^n)} \leqslant C$, $Jx^N(t)\in \mathrm{K}$ $(\forall)\, t\in\Omega$ $\forall\, N\in\mathbb{N}$ *admits a subsequence* $\{x^{N'}\}$ *with* $x^{N'} \to_{C^0(\Omega,\mathbb{R}^n)} x\in W^{1,\infty}(\Omega,\mathbb{R}^n)$ *and* $Jx^{N'} \xrightarrow{*}_{L^\infty(\Omega,\mathbb{R}^{nm})} Jx\in L^\infty(\Omega,\mathbb{R}^{nm})$. *Consequently,* $\{Jx^{N'}\}$ *generates a generalized gradient control* $\boldsymbol{\mu}\in\mathcal{G}(\mathrm{K})$.
2) *The subset* $\mathcal{G}(\mathrm{K})\subseteq\mathcal{Y}(\mathrm{K})$ *of the generalized gradient controls is sequentially compact as well.*

d) An adapted version of the mean value theorem.

Shortly spoken, the mean value theorem assigns to any generalized control $\boldsymbol{\mu}\in\mathcal{Y}(\mathrm{K})$ resp. $\boldsymbol{\mu}\in\mathcal{G}(\mathrm{K})$ a probability measure $\nu\in rca^{pr}(\mathrm{K})$ (the "average of $\boldsymbol{\mu}$"), which satisfies the variational equality

$$\int_\Omega\int_\mathrm{K} g(v)\,d\mu_t(v)\,dt \;=\; \int_\Omega\int_\mathrm{K} g(v)\,d\nu(v)\,dt \quad \forall\, g\in C^0(\mathrm{K},\mathbb{R})\,. \tag{20}$$

For the original statement of the theorem, we refer to the literature.[40] Here we provide a generalized version, which is particularly adapted to generalized gradient controls $\boldsymbol{\mu}\in\mathcal{G}(\mathrm{K})$.

Theorem 2.9 (Mean value theorem for generalized gradient controls). [41]
Assume that $\Omega\subset\mathbb{R}^m$ *is the closure of a strongly Lipschitz domain with* $\mathfrak{o}\in\mathrm{int}\,(\Omega)$. *We consider sequences* $\{w^N\}$, K *and* $\{x^N\}$, $W_0^{1,\infty}(\Omega,\mathbb{R}^n)$, *which satisfy*

a) $w^N\to w\in\mathrm{K}$ *(*w^N, $w\in\mathbb{R}^{nm}$ *have to be understood as* (n,m)*-matrices),*
b) $w^N+Jx^N(t)\in\mathrm{K}$ $(\forall)\,t\in\Omega$ $\forall\,N\in\mathbb{N}$,
c) $\{w^N+Jx^N\}$ *generates a generalized gradient control* $\boldsymbol{\mu}\in\mathcal{G}(\mathrm{K})$.

Then there exists a sequence of Lipschitz functions $\{\widetilde{x}^N\}$, $W_0^{1,\infty}(\Omega,\mathbb{R}^n)$ *with the following properties:*

1) $\lim_{N\to\infty}\|\widetilde{x}^N\|_{C^0(\Omega,\mathbb{R}^n)} = 0$.
2) $w^N+J\widetilde{x}^N(t)\in\mathrm{K}$ $(\forall)\,t\in\Omega$ $\forall\,N\in\mathbb{N}$.
3) $\{w^N+J\widetilde{x}^N\}$ *generates a constant generalized gradient control* $\boldsymbol{\nu}=\{\nu\}\in\mathcal{G}(\mathrm{K})$, *which may be understood as the average of* $\boldsymbol{\mu}$ *with respect to* t:

$$\begin{aligned}&\lim_{N\to\infty}\int_\Omega g(\,w^N+Jx^N(t)\,)\,dt \;=\; \int_\Omega\int_\mathrm{K} g(v)\,d\mu_t(v)\,dt\\ =\;&\lim_{N\to\infty}\int_\Omega g(\,w^N+J\widetilde{x}^N(t)\,)\,dt \;=\; \int_\Omega\int_\mathrm{K} g(v)\,d\nu(v)\,dt \quad \forall\, g\in C^0(\mathrm{K},\mathbb{R})\,.\end{aligned} \tag{21}$$

4) *It holds that*

$$w=\begin{pmatrix}\int_\mathrm{K} v_{11}\,d\nu(v) & \ldots & \int_\mathrm{K} v_{1m}\,d\nu(v)\\ \vdots & & \vdots\\ \int_\mathrm{K} v_{n1}\,d\nu(v) & \ldots & \int_\mathrm{K} v_{nm}\,d\nu(v)\end{pmatrix}. \tag{22}$$

[39] [33], p. 54, Theorem 4.14.
[40] [22], p. 334, Theorem 2.1.
[41] [33], p. 55 f., Theorem 4.16.

Remark. If the sequence $\{ w^N + Jx^N \}$ in Theorem 2.9 itself generates a constant generalized gradient control $\boldsymbol{\nu} = \{ \nu \} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$ then the limit point $w = \lim_{N\to\infty} w^N$ obeys equation (22) immediately.

The original mean value theorem justifies the definition of an average operator, which assigns to any generalized control a probability measure as its t-average. We will see that this operator is continuous.

Definition 2.10 (Average operator for generalized controls).[42] Assume that $\Omega \subset \mathbb{R}^m$ is the closure of a strongly Lipschitz domain with $\mathfrak{o} \in \mathrm{int}(\Omega)$. We define an operator $A: \boldsymbol{\mathcal{Y}}(\mathrm{K}) \to rca^{pr}(\mathrm{K})$, which assigns to every generalized control $\boldsymbol{\mu} \in \boldsymbol{\mathcal{Y}}(\mathrm{K})$ its average $A(\boldsymbol{\mu}) = \nu$ according to the mean value theorem.

Theorem 2.11 (Continuity of the average operator A). *Assume again that $\Omega \subset \mathbb{R}^m$ is the closure of a strongly Lipschitz domain with $\mathfrak{o} \in \mathrm{int}(\Omega)$. We endow $rca^{pr}(\mathrm{K})$ with the distance function σ from Definition* 2.1 *and $\boldsymbol{\mathcal{Y}}(\mathrm{K})$ with the distance function ϱ from Definition* 2.4.

1)[43] *For all $\boldsymbol{\mu}', \boldsymbol{\mu}'' \in \boldsymbol{\mathcal{Y}}(\mathrm{K})$ it holds that*

$$\sigma(A(\boldsymbol{\mu}') , A(\boldsymbol{\mu}'')) \leqslant \varrho(\boldsymbol{\mu}', \boldsymbol{\mu}'') . \tag{23}$$

In particular, we have the implication

$$\varrho(\boldsymbol{\mu}^N, \boldsymbol{\mu}) \to 0 \implies \sigma(A(\boldsymbol{\mu}^N) , A(\boldsymbol{\mu})) \to 0 , \tag{24}$$

and the average operator A is continuous with respect to the introduced topologies.

2)[44] *The measures of the shape $\nu = A(\{ \delta_{u(t)} \})$ with $u \in L^\infty(\Omega, \mathbb{R}^{nm})$, $u(t) \in \mathrm{K}$ $(\forall)\, t \in \Omega$, are dense in the set $\{ A(\boldsymbol{\mu}) \in rca^{pr}(\mathrm{K}) \mid \boldsymbol{\mu} \in \boldsymbol{\mathcal{Y}}(\mathrm{K}) \}$ with respect to its weak* topology (which is generated by the distance function σ).*

3. The set-valued maps S^* and $\mathrm{S}^\#$.

Throughout the whole section, we assume that $\Omega \subset \mathbb{R}^m$ is the closure of a strongly Lipschitz domain with $\mathfrak{o} \in \mathrm{int}(\Omega)$, what guarantees the applicability of Theorem 2.9. Further, we fix a convex body $\mathrm{K} \subset \mathbb{R}^{nm}$ with $\mathfrak{o} \in \mathrm{int}(\mathrm{K})$ and the quantities $c_\mathrm{K} = \mathrm{Dist}(\mathfrak{o}, \partial\mathrm{K})$ and $C_\mathrm{K} = \mathrm{Max}\big(1, \mathrm{Max}_{v\in\mathrm{K}} |v|\big)$, thus $0 < c_\mathrm{K} \leqslant C_\mathrm{K}$ and $\mathrm{Diam}(\mathrm{K}) \leqslant 2\, C_\mathrm{K}$.

a) The set-valued map S^*.

We start with the definition of a set-valued map $w \longmapsto \mathrm{S}^*(w) \subseteq rca^{pr}(\mathrm{K})$ on the points $w \in \mathrm{int}(\mathrm{K})$. S^* possesses nonempty, convex, weak*-sequentially compact images, whose elements result as averages of generalized gradient controls.

[42] [22], p. 336 f.
[43] [22], p. 337, Proposition 2.2.
[44] [33], p. 56, Theorem 4.18, 2).

Definition 3.1 (Definition of $\mathrm{S}^*(w)$ for $w \in \operatorname{int}(\mathrm{K})$). For $w \in \operatorname{int}(\mathrm{K})$, we define the following set of probability measures:

$$\mathrm{S}^*(w) = \Big\{ \nu \in rca^{pr}(\mathrm{K}) \;\Big|\; \text{there exist sequences } \{ w^N \} \subset \mathrm{K} \text{ and } \{ x^N \} \subset W_0^{1,\infty}(\Omega, \mathbb{R}^n) \text{ with } \begin{array}{l} a)\ \lim_{N\to\infty} w^N = w, \\ b)\ \lim_{N\to\infty} \| x^N \|_{C^0(\Omega,\mathbb{R}^n)} = 0, \\ c)\ w^N + Jx^N(t) \in \mathrm{K}\ (\forall)\, t \in \Omega\ \forall\, N \in \mathbb{N}, \\ d)\ \{ w^N + Jx^N \} \text{ generates the constant generalized} \\ \qquad \text{gradient control } \boldsymbol{\nu} = \{ \nu \}\,. \end{array} \Big\}. \tag{25}$$

Lemma 3.2 (Special generating sequences in Definition 3.1). *Let $w \in \operatorname{int}(\mathrm{K})$. For every $\nu \in \mathrm{S}^*(w)$, there exist sequences $\{ w^N \}$ and $\{ x^N \}$ with the properties a)–d) from Definition 3.1, which satisfy additionally $w^N \in \operatorname{int}(\mathrm{K})$ as well as $w^N + Jx^N(t) \in \operatorname{int}(\mathrm{K})$ $(\forall)\, t \in \Omega\ \forall\, N \in \mathbb{N}$.*

Lemma 3.3 (Dense subset of $\mathrm{S}^*(w)$). *If $w \in \operatorname{int}(\mathrm{K})$ then $\delta_w \in \mathrm{S}^*(w)$, and the measures of the shape $\nu = A(\{ \delta_{(w+Jx(t))} \})$, obtained from functions $x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ with $w + Jx(t) \in \mathrm{K}$ $(\forall)\, t \in \Omega$, are dense in $\mathrm{S}^*(w)$ with respect to the weak*-topology resp. the metric σ from Definition 2.1.*

Theorem 3.4 (Properties of the sets $\mathrm{S}^*(w)$). *For every $w \in \operatorname{int}(\mathrm{K})$, the set $\mathrm{S}^*(w) \subseteq rca^{pr}(\mathrm{K})$ is nonempty, convex and weak*-sequentially compact.*

The further investigation of the set-valued map S^* runs parallel to the investigation of the envelope f^* in [34], Sections 3.2 and 3.3. In the present subsection, we start with the proof of the continuity of S^* on $\operatorname{int}(\mathrm{K})$, which is based on a ε-δ relation depending on the distance of the given points v, $w \in \operatorname{int}(\mathrm{K})$ as well as on their distances to the boundary $\partial\mathrm{K}$.

Theorem 3.5 (ε-δ relation for S^*).[45] *For every $0 < \varepsilon < 1$ there exists $\delta_1(\varepsilon) = \frac{1}{4}\,\varepsilon / C_{\mathrm{K}} > 0$ such that for all v, $w \in \operatorname{int}(\mathrm{K})$, the following ε-δ relation holds:*

$$\begin{aligned} &\big| v - w \big| \leqslant \delta_1(\varepsilon) \cdot \operatorname{Min}\big(1\,,\ \operatorname{Dist}(v\,,\ \partial\mathrm{K})\,,\ \operatorname{Dist}(w\,,\ \partial\mathrm{K}) \big) \\ \Longrightarrow\ &\mathcal{H}\,(\, \mathrm{S}^*(v)\,,\ \mathrm{S}^*(w)\,) \leqslant \varepsilon \end{aligned} \tag{26}$$

where C_{K} is the quantity defined in the beginning of the section, and $\mathcal{H}(\,\cdot\,,\,\cdot\,)$ denotes the Hausdorff distance (cf. Definition 5.2 in the Appendix).

Theorem 3.6 (Continuity of the set-valued map S^*).[46] *The set-valued map S^* is continuous on $\operatorname{int}(\mathrm{K})$.*

Proof of Lemma 3.2. For arbitrary $\nu \in \mathrm{S}^*(w)$ there exist sequences $\{ w^N \} \subset \mathrm{K}$ and $\{ x^N \} \subset W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ with the properties a)–d) from Definition 3.1. Choose now a sequence of numbers $\{ c^N \} \subset \mathbb{R}$ with $0 < c^N < 1$ for all $N \in \mathbb{N}$ and $\lim_{N\to\infty} c^N = 1$.

[45] Compare with [34], p. 82, Theorem 3.5.
[46] Compare with [34], p. 82, Theorem 3.6, (1).

Obviously, the sequences $\{ c^N w^N \}$ and $\{ c^N x^N \}$ possess the properties a)–d) from Definition 3.1 as well, while $c^N w^N \in \operatorname{int}(\mathrm{K})$ as well as $c^N \big(w^N + Jx^N(t) \big) \in \operatorname{int}(\mathrm{K})$ $(\forall)\, t \in \Omega$ for all $N \in \mathbb{N}$. It holds further that

$$\begin{aligned} &\varrho(\{ \delta_{c^N (w^N + Jx^N(t))} \}, \{ \nu \}) \\ &\leqslant \varrho(\{ \delta_{w^N + Jx^N(t)} \}, \{ \nu \}) + \varrho(\{ \delta_{c^N (w^N + Jx^N(t))} \}, \{ \delta_{w^N + Jx^N(t)} \}), \end{aligned} \tag{27}$$

where, by definition of ν, the first member converges to zero, while the second member can be estimated by

$$\begin{aligned} &\varrho(\{ \delta_{c^N (w^N + Jx^N(t))} \} , \{ \delta_{w^N + Jx^N(t)} \}) \\ &= \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{2^{r+s}(1+L_s)} \Big| \int_\Omega f_r(t) \big(g_s(c^N w^N + c^N Jx^N(t)) \\ &\qquad\qquad - g_s(w^N + Jx^N(t)) \big)\, dt \Big| \end{aligned} \tag{28}$$

$$\leqslant \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{2^{r+s}} \cdot \frac{L_s}{(1+L_s)} \cdot \| f_r \|_{L^1(\Omega,\mathbb{R})} \cdot \big| 1 - c^N \big| \cdot \operatorname*{ess\,sup}_{t\in\Omega} \big| w^N + Jx^N(t) \big| . \tag{29}$$

Since $\| f_r \|_{L^1(\Omega,\mathbb{R})} \leqslant 1$ for all $r \in \mathbb{N}$ and $w^N + Jx^N(t) \in \mathrm{K}$ $(\forall)\, t \in \Omega$, the second member converges to zero as well. Consequently, $\nu \in \mathrm{S}^*(w)$ can be generated in the claimed way. □

Proof of Lemma 3.3. Let $w \in \operatorname{int}(\mathrm{K})$ be given. We observe first that the constant sequences $\{ w^N \}$ and $\{ x^N \}$ with $w^N = w$ and $x^N \equiv \mathfrak{o}$ satisfy a)–c) from Definition 3.1, and $\{ w^N + Jx^N \}$ generates the constant generalized gradient control $\boldsymbol{\mu} = \{ \delta_w \}$. Thus $\delta_w \in \mathrm{S}^*(w)$. For arbitrary $\nu \in \mathrm{S}^*(w)$, we choose sequences $\{ w^N \}$ and $\{ x^N \}$ with the properties a)–d) from Definition 3.1. We have $\operatorname{Dist}(w, \partial\mathrm{K}) = C > 0$ since $w \in \operatorname{int}(\mathrm{K})$. Define now the functions

$$y^N = \frac{C}{C + | w^N - w |}\, x^N \in W_0^{1,\infty}(\Omega, \mathbb{R}^n) . \tag{30}$$

For all $N \in \mathbb{N}$ we obtain:

$$\begin{aligned} & w^N + Jx^N(t) \in \mathrm{K} \quad (\forall)\, t \in \Omega \\ \Longrightarrow\quad & w + Jx^N(t) \in \mathrm{K} + \mathrm{K}(\mathfrak{o}, | w^N - w |) \quad (\forall)\, t \in \Omega \end{aligned} \tag{31}$$

$$\Longrightarrow\quad w + \frac{C}{C + | w^N - w |} Jx^N(t) = w + Jy^N(t) \in \mathrm{K} \quad (\forall)\, t \in \Omega . \tag{32}$$

By definition of ν, the first member within the inequality

$$\begin{aligned} &\varrho(\{ \delta_{w + Jy^N(t)} \} , \{ \nu \}) \\ &\leqslant \varrho(\{ \delta_{w^N + Jx^N(t)} \} , \{ \nu \}) + \varrho(\{ \delta_{w^N + Jx^N(t)} \} , \{ \delta_{w + Jy^N(t)} \}) \end{aligned} \tag{33}$$

converges to zero, and the second member obeys the estimate

$$\begin{aligned}
&\varrho(\{\delta_{w^N+Jx^N(t)}\}, \{\delta_{w+Jy^N(t)}\}) \\
= &\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{1}{2^{r+s}(1+L_s)}\Big|\int_\Omega f_r(t)\Big(g_s\big(w^N+Jx^N(t)\big) \\
&\quad - g_s\Big(w+\frac{C}{C+|w^N-w|}Jx^N(t)\Big)\Big)\,dt\Big| \qquad (34)\\
\leqslant &\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{1}{2^{r+s}(1+L_s)}\int_\Omega |f_r(t)|\cdot L_s \\
&\quad\cdot\Big|w-w^N+\Big(1-\frac{C}{C+|w^N-w|}\Big)Jx^N(t)\Big|\,dt \qquad (35)\\
\leqslant &\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{1}{2^{r+s}}\cdot\frac{L_s}{1+L_s}\cdot\|f_r\|_{L^1(\Omega,\mathbb{R})} \\
&\quad\cdot\Big(|w^N-w|+\frac{|w^N-w|}{C+|w^N-w|}\cdot\operatorname*{ess\,sup}_{t\in\Omega}|Jx^N(t)|\Big). \qquad (36)
\end{aligned}$$

By $\|f_r\|_{L^1(\Omega,\mathbb{R})}\leqslant 1\ \forall r\in\mathbb{N}$ (Definition 2.4) and the uniform essential boundedness of $Jx^N(t)$, this member converges to zero as well. By Theorem 2.11, 1), from $\varrho(\{\delta_{w+Jy^N(t)}\}, \{\nu\})\to 0$ it follows that $A(\{\delta_{w+Jy^N(t)}\}) \xrightarrow{*} A(\{\nu\})=\nu$. On the other hand, by the mean value theorem (Theorem 2.9), the probability measures of the shape $A(\{\delta_{w+Jy^N(t)}\})$ belong to $S^*(w)$ as well. Thus ν can be approximated in the claimed way. □

Proof of Theorem 3.4. • *Step 1. The set* $S^*(w)$ *is nonempty.* By Lemma 3.3, we have $\delta_w\in S^*(w)$ for all $w\in \operatorname{int}(K)$.

• *Step 2.*[47] *The set* $S^*(w)$ *is convex.* Let $w\in\operatorname{int}(K)$, ν', $\nu''\in S^*(w)$ and $0<\lambda<1$ be given; we will prove that the convex combination $\nu=\lambda\,\nu'+(1-\lambda)\,\nu''$ belongs to $S^*(w)$ as well. We choose first a subset $E\subset\operatorname{int}(\Omega)$, which forms the closure of a strongly Lipschitz domain as well, and satisfies $|E|=\lambda|\Omega|$. (E may be obtained e.g. as the image of Ω under a homothety with center $\mathfrak{o}\in\Omega$.) Define further subsets

$$E^K=\{t\in E\mid \operatorname{Dist}(t,\partial E)\geqslant 1/K\} \qquad (37)$$

and functions $\eta^K\in W^{1,\infty}(\Omega,\mathbb{R})$ with

$$\eta^K(t)\begin{cases}=0 & \forall t\in\Omega\setminus E,\\ \in[0,1] & \forall t\in E\setminus E^K,\\ =1 & \forall t\in E^K\end{cases}\quad\text{and}\quad |\nabla\eta^K(t)|\leqslant C_1\cdot K\quad(\forall)\,t\in\Omega. \qquad (38)$$

We will investigate now the measure-valued map

$$\boldsymbol{\mu}=\{\mu_t\}=\{\mathbb{1}_E(t)\cdot\nu'+\mathbb{1}_{(\Omega\setminus E)}(t)\cdot\nu''\}. \qquad (39)$$

Obviously, $\boldsymbol{\mu}$ is a generalized control; we will show that $\boldsymbol{\mu}$ belongs to $\boldsymbol{\mathcal{G}}(K)$. For this purpose, we choose sequences $\{w'_N\}$, K and $\{x'_N\}$, $W_0^{1,\infty}(\Omega,\mathbb{R}^n)$ resp. $\{w''_N\}$, K and

[47] Cf. [22], p. 339 f.

$\{ x''_N \}$, $W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ with the properties a)–c) from Definition 3.1 such that $\{ w'_N + Jx'_N \}$ and $\{ w''_N + Jx''_N \}$ generate the constant generalized gradient controls $\boldsymbol{\mu}' = \{ \nu' \}$ and $\boldsymbol{\mu}'' = \{ \nu'' \}$. Define now the functions

$$\begin{aligned} x^{N,K}(t) = &\big(\eta^K(t)\, w'_N\, t + (\, 1 - \eta^K(t)\,)\, w''_N\, t \big) \\ &+ \eta^K(t)\, x'_N(t) + (\, 1 - \eta^K(t)\,)\, x''_N(t) \quad \text{with} \end{aligned} \tag{40}$$

$$\begin{aligned} Jx^{N,K}(t) = &\eta^K(t) \big(w'_N + Jx'_N(t) \big) + (\, 1 - \eta^K(t)\,) \big(w''_N + Jx''_N(t) \big) \\ &+ \big((x'_N(t) + w'_N\, t - w\, t) + (w\, t - w''_N\, t - x''_N(t)) \big) \cdot \nabla \eta^K(t) \end{aligned} \tag{41}$$

(the points w, w'_N, w''_N have to be understood as (n, m)-matrices). The first both members form a convex combination, which belongs to K for all $t \in \Omega$. By assumptions a) and b) from Definition 3.1, we find to every K an index $N(K)$ with

$$\begin{aligned} \big\| \, x'_N + w'_N\, t - w\, t \, \big\|_{C^0(\Omega, \mathbb{R}^n)} &\leqslant \frac{1}{2\, C_1\, K^2} \quad \text{and} \\ \big\| \, x''_N + w''_N\, t - w\, t \, \big\|_{C^0(\Omega, \mathbb{R}^n)} &\leqslant \frac{1}{2\, C_1\, K^2} \quad \forall\, N \geqslant N(K)\,. \end{aligned} \tag{42}$$

With a constant $C_2 > 0$, depending on the matrix norm in $\mathbb{R}^{n \times m}$, it holds that

$$\begin{aligned} &\Big| \big((x'_{N(K)}(t) + w'_N\, t - w\, t) + (w\, t - w''_N\, t - x''_{N(K)}(t)) \big) \cdot \nabla \eta^K(t) \Big| \\ &\leqslant C_2 \cdot \big(\big\| \, x'_N + w'_N\, t - w\, t \, \big\|_{C^0(\Omega, \mathbb{R}^n)} + \big\| \, x''_N + w''_N\, t - w\, t \, \big\|_{C^0(\Omega, \mathbb{R}^n)} \big) \cdot \big| \, \nabla \eta^K(t) \, \big| \\ &\leqslant C_2 / K\,; \end{aligned} \tag{43}$$

consequently, $Jx^{N(K),K}(t) \in \mathrm{K} + \mathrm{K}(\mathfrak{o}, C_2/K)$ for all $t \in \Omega$. With the number $c_{\mathrm{K}} = \operatorname{Dist}(\mathfrak{o}, \partial \mathrm{K}) > 0$, we obtain

$$\frac{c_{\mathrm{K}}\, K}{c_{\mathrm{K}}\, K + C_2} \big(\mathrm{K} + \mathrm{K}(\mathfrak{o}, C_2/K) \big) \subseteq \mathrm{K}\,. \tag{44}$$

We claim that the sequences

$$w^K = \frac{c_{\mathrm{K}}\, K}{c_{\mathrm{K}}\, K + C_2} \cdot w''_{N(K)} \quad \text{and} \quad y^K(t) = \frac{c_{\mathrm{K}}\, K}{c_{\mathrm{K}}\, K + C_2} \cdot \Big(x^{N(K),K} - w''_{N(K)}\, t \Big) \tag{45}$$

satisfy the assumptions a)–c) from Theorem 2.9, and that $\{ w^K + Jy^K \}$ generates $\boldsymbol{\mu}$ as well. It is clear that $w^K \in \mathrm{K}$ for all $K \in \mathbb{N}$ and

$$\lim_{K \to \infty} w^K = \lim_{K \to \infty} \frac{c_{\mathrm{K}}\, K}{c_{\mathrm{K}}\, K + C_2} \cdot w''_{N(K)} = w\,. \tag{46}$$

For all $t \in \partial\Omega$, we have $\eta^K(t) = 0$ and $y^K(t) = \frac{c_{\mathrm{K}}\, K}{c_{\mathrm{K}}\, K + C_2} \cdot x''_{N(K)}(t) = \mathfrak{o}$; consequently, y^K belongs to $W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ together with the functions x'_N und x''_N. From the construction above, it follows that $w^K + Jy^K(t) \in \mathrm{K}\ (\forall)\, t \in \Omega\ \forall\, K \in \mathbb{N}$ as well. After all, $\{ w^K + Jy^K \}$

generates the generalized control $\boldsymbol{\mu}$. To prove this, we fix $f \in C^0(\Omega, \mathbb{R}) \subset L^1(\Omega, \mathbb{R})$ and $g \in C^0(\mathrm{K}, \mathbb{R})$ and calculate

$$\int_\Omega f(t)\, g\left(\, w^K + Jy^K(t)\,\right)\, dt \;=\; J_{1,K} + J_{2,K} + J_{3,K} \quad \text{with} \tag{47}$$

$$J_{1,K} \;=\; \int_{\mathrm{E}^K} f(t)\, g\left(\, \frac{c_{\mathrm{K}}\, K}{c_{\mathrm{K}}\, K + C_2} \cdot \left(\, w'_{N(K)} + Jx'_{N(K),K}(t)\,\right)\right)\, dt\,; \tag{48}$$

$$J_{2,K} \;=\; \int_{\Omega\,\setminus\,\mathrm{E}} f(t)\, g\left(\, \frac{c_{\mathrm{K}}\, K}{c_{\mathrm{K}}\, K + C_2} \cdot \left(\, w''_{N(K)} + Jx''_{N(K),K}(t)\,\right)\right)\, dt\,; \tag{49}$$

$$J_{3,K} \;=\; \int_{\mathrm{E}\,\setminus\,\mathrm{E}^K} f(t)\, g\left(\, w^K + Jy^K(t)\,\right)\, dt\,. \tag{50}$$

Passing to the limit $K \to \infty$, we obtain

$$\lim_{K\to\infty} J_{1,K} = \int_{\mathrm{E}} \int_{\mathrm{K}} f(t)\, g(v)\, d\nu'(v)\, dt\,; \tag{51}$$

$$\lim_{K\to\infty} J_{2,K} = \int_{\Omega\,\setminus\,\mathrm{E}} \int_{\mathrm{K}} f(t)\, g(v)\, d\nu''(v)\, dt\,; \tag{52}$$

$$\lim_{K\to\infty} J_{3,K} = 0\,, \tag{53}$$

since the integrands $f(t) \cdot g(\, w^K + Jy^K(t)\,)$ are uniformly bounded on Ω. Consequently, it holds

$$\begin{aligned} &\lim_{K\to\infty} \int_\Omega f(t)\, g(\, w^K + Jy^K(t)\,)\, dt \\ &= \int_{\mathrm{E}} \int_{\mathrm{K}} f(t)\, g(v)\, d\nu'(v)\, dt \;+\; \int_{\Omega\,\setminus\,\mathrm{E}} \int_{\mathrm{K}} f(t)\, g(v)\, d\nu''(v)\, dt\,, \end{aligned} \tag{54}$$

and $\{\, w^K + Jy^K \,\}$ generates $\boldsymbol{\mu}$. Let us apply now the mean value theorem (Theorem 2.9) to $\{\, w^K \,\}$ and $\{\, y^K \,\}$. The constant generalized gradient control $\widetilde{\boldsymbol{\nu}} = \{\, \widetilde{\nu}\,\}$, which is generated by $\{\, w^K + J\widetilde{y}^K \,\}$, satisfies

$$\begin{aligned} &\int_\Omega \int_{\mathrm{K}} g(v)\, d\widetilde{\nu}(v)\, dt = \int_\Omega \int_{\mathrm{K}} g(v)\, d\mu_t(v)\, dt \\ &= \int_{\mathrm{E}} \int_{\mathrm{K}} g(v)\, d\nu'(v)\, dt \;+\; \int_{\Omega\,\setminus\,\mathrm{E}} \int_{\mathrm{K}} g(v)\, d\nu''(v)\, dt \end{aligned} \tag{55}$$

$$\begin{aligned} \Longrightarrow \int_{\mathrm{K}} g(v)\, d\widetilde{\nu}(v) &= \frac{|\,\mathrm{E}\,|}{|\,\Omega\,|} \cdot \int_{\mathrm{K}} g(v)\, d\nu'(v) + \frac{|\,\Omega\,\setminus\,\mathrm{E}\,|}{|\,\Omega\,|} \cdot \int_{\mathrm{K}} g(v)\, d\nu''(v) \\ &= \int_{\mathrm{K}} g(v)\, d\big(\, \lambda\, \nu'(v) + (1-\lambda)\, \nu''(v)\,\big) \end{aligned} \tag{56}$$

for all $g \in C^0(\mathrm{K}, R)$. We see that ν and $\widetilde{\nu}$ coincide, and ν belongs to $\mathrm{S}^*(w)$.

• *Step 3. The set* $\mathrm{S}^*(w)$ *is weak*-sequentially compact.* Let a sequence $\{\, \nu^N \,\}$, $\mathrm{S}^*(w) \xrightarrow{\;*\;} \nu \in rca\,(\mathrm{K})$ be given. Since $rca^{\,pr}\,(\mathrm{K})$ itself is weak*-sequentially compact, the limit element ν is a probability measure as well. Consider now the sequence of constant generalized gradient controls $\boldsymbol{\mu}^N = \{\,\nu^N\,\}$. By Theorem 2.8, 2), it admits a subsequence $\{\,\boldsymbol{\mu}^{N'}\,\}$, which converges to a generalized gradient control $\boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$. With the aid of Theorem 2.6, 3), from the generating sequences $\{\,w^{N'} + Jx^{N',K}\}$ for the $\boldsymbol{\mu}^{N'}$ we may select a diagonal sequence $\{\,w^{N'} + Jx^{N',K(N')}\,\}$ as generating sequence for $\boldsymbol{\mu}$ where $\{\,w^{N'}\,\}$ and $\{\,x^{N',K(N')}\,\}$ possess the properties a)–c) from Definition 3.1. Applying the mean value theorem (Theorem 2.9) to these sequences, we find a sequence $\{\,w^{N'} + J\widetilde{x}^{N',K(N')}\,\}$, which generates a constant generalized gradient control $\boldsymbol{\nu}' = \{\,\nu'\,\}$. From $\boldsymbol{\mu}^{N'} \to \boldsymbol{\mu}$ and the continuity of the average operator (Theorem 2.11, 1)), it follows that $\nu^{N'} = A(\boldsymbol{\mu}^{N'}) \xrightarrow{\ *\ } A(\boldsymbol{\mu}) = \nu'$. Since $\{\,\nu^{N'}\,\}$ is a subsequence of the weak*-convergent sequence $\{\,\nu^N\,\} \xrightarrow{\ *\ } \nu$, we arrive at $\nu' = \nu$. Consequently, $\{\,\nu\,\}$ can be generated by sequences with the properties a)–d) from Definition 3.1, ν belongs to $\mathrm{S}^*(w)$, and the set $\mathrm{S}^*(w)$ is weak*-closed. By Theorem 2.8, 2), the images $\mathrm{S}^*(w)$ are weak*-sequentially compact. □

Proof of Theorem 3.5. Let $0 < \varepsilon < 1$ and $v,\, w \in \mathrm{int}\,(\mathrm{K})$ with

$$\big|\, v - w \,\big| \;\leqslant\; \frac{\varepsilon}{4\,C_{\mathrm{K}}} \cdot \mathrm{Min}\,\big(\,1\,,\,\mathrm{Dist}\,(v\,,\,\partial\mathrm{K})\,\big) \tag{57}$$

be given. Then we choose arbitrary $\nu' \in \mathrm{S}^*(v)$ and show that there exists $\nu'' \in \mathrm{S}^*(w)$ with $\sigma(\nu',\nu'') \leqslant \varepsilon$ (cf. Definition 5.2). For ν' there exist sequences $\{\,v^N\,\}$, K and $\{\,x^N\,\}$, $W_0^{1,\infty}(\Omega,\mathbb{R}^n)$ with the properties a)–d) from Definition 3.1. By Lemma 3.2 and 3.3, we may assume that $v^N = v$ and $v + Jx^N(t) \in \mathrm{int}\,(\mathrm{K})$ $(\forall)\, t \in \Omega$ for all $N \in \mathbb{N}$. We invoke the following geometrical lemma:

Lemma 3.7.[48] *Let a nonempty, convex, compact set* $\mathrm{K} \subset \mathbb{R}^{nm}$ *with* $v_0 \in \mathrm{int}\,(\mathrm{K})$ *and a function* $x \in W_0^{1,\infty}(\Omega,\mathbb{R}^n)$ *be given. Then it holds for all* $0 < \lambda < 1$*:*

$$\begin{aligned} & v_0 + Jx(t) \in \mathrm{int}\,(\mathrm{K}) \quad (\forall)\, t \in \Omega \\ \Longrightarrow\;& \lambda \cdot \mathrm{Dist}\,(v_0\,,\,\partial\mathrm{K}) \leqslant \mathrm{Dist}\,\big(\,v_0 + (1-\lambda)\,Jx(t)\,,\,\partial\mathrm{K}\,\big)\ (\forall)\, t \in \Omega\,. \end{aligned} \tag{58}$$

We define $\delta_1(\varepsilon) = \frac{1}{4}\,\varepsilon/C_{\mathrm{K}} = \lambda$ and find

$$\begin{aligned} & v + Jx^N(t) \in \mathrm{int}\,(\mathrm{K})\ \ (\forall)\, t \in \Omega \\ \Longrightarrow\;& v + (1-\lambda)\,Jx^N(t) \in \mathrm{K}\ \ (\forall)\, t \in \Omega\ \ \forall\, N \in \mathbb{N}\,; \end{aligned} \tag{59}$$

From Lemma 3.7, we get for all $w \in \mathrm{int}\,(\mathrm{K})$ the implication

$$\begin{aligned} & \big|\, v - w \,\big| \;\leqslant\; \lambda \cdot \mathrm{Min}\,\big(\,1\,,\,\mathrm{Dist}\,(v\,,\,\partial\mathrm{K}\big) \;\leqslant\; \mathrm{Dist}\,\big(\,v + (1-\lambda)\,Jx^N(t)\,,\,\partial\mathrm{K}\,\big) \\ \Longrightarrow\quad & w + (1-\lambda)\,Jx^N(t) \;=\; (w-v) \;+\; \big(\,v + (1-\lambda)\,Jx^N(t)\,\big) \in \mathrm{K} \\ & \qquad\qquad (\forall)\, t \in \Omega\ \ \forall\, N \in \mathbb{N}\,, \end{aligned} \tag{60}$$

from which follows that

$$\begin{aligned} & \big|\, v - w \,\big| \;\leqslant\; \lambda \cdot \mathrm{Min}\,\big(\,1\,,\,\mathrm{Dist}\,(v\,,\,\partial\mathrm{K})\,\big) \\ \Longrightarrow\quad & \big|\,\big(\,v + Jx^N(t)\,\big) - \big(\,w + (1-\lambda)\,Jx^N(t)\,\big)\,\big| \;\leqslant\; \varepsilon \quad (\forall)\, t \in \Omega\ \ \forall\, N \in \mathbb{N}\,. \end{aligned} \tag{61}$$

[48] [34], p. 83, Lemma 3.10.

We obtain for all $N \in \mathbb{N}$:

$$\begin{aligned} & \varrho(\{\,\delta_{v+Jx^N(t)}\,\}\,,\,\{\,\delta_{w+(1-\lambda)\,Jx^N(t)}\,\}\,) \\ = & \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{1}{2^{r+s}}\cdot\frac{1}{1+L_s}\cdot\left|\int_\Omega f_r(t)\,\big(g_s(v+Jx^N(t))-g_s(w+(1-\lambda)\,Jx^N(t))\big)\,dt\right| \quad (62) \\ \leqslant & \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{1}{2^{r+s}}\cdot\frac{1}{1+L_s}\cdot\int_\Omega |f_r(t)|\cdot L_s\cdot\big|\big(v+Jx^N(t)\big)-\big(w+(1-\lambda)Jx^N(t)\big)\big|\,dt \quad (63) \\ \leqslant & \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{1}{2^{r+s}}\cdot\frac{L_s}{1+L_s}\cdot\|\,f_r\,\|_{L^1(\Omega,\mathbb{R})}\cdot\varepsilon\leqslant\varepsilon\,. \quad (64) \end{aligned}$$

Passing now to a subsequence $\{\,\delta_{w+(1-\lambda)\,Jx^{N'}(t)}\,\}$, which converges to a generalized gradient control $\boldsymbol{\mu}\in\boldsymbol{\mathcal{G}}(\mathrm{K})$, we arrive at

$$\lim_{N'\to\infty}\varrho(\{\,\delta_{v+Jx^{N'}(t)}\,\}\,,\,\{\,\delta_{w+(1-\lambda)\,Jx^{N'}(t)}\,\}\,)\;=\;\varrho(\{\,\nu'\,\}\,,\,\boldsymbol{\mu}\,)\leqslant\varepsilon\,. \quad (65)$$

From Theorem 2.11, 1) it follows that

$$\sigma(\nu',A(\boldsymbol{\mu}))\;\leqslant\;\varrho(\{\,\nu'\,\}\,,\,\boldsymbol{\mu}\,)\;\leqslant\;\varepsilon \quad (66)$$

where $A(\boldsymbol{\mu})=\nu''$ belongs to $\mathrm{S}^*(w)$. Exchanging the roles of v, $w\in\mathrm{int}\,(\mathrm{K})$ and assuming that

$$\big|\,v-w\,\big|\;\leqslant\;\frac{\varepsilon}{4\,C_{\mathrm{K}}}\cdot\mathrm{Min}\,\big(\,1\,,\,\mathrm{Dist}\,(w\,,\,\partial\mathrm{K})\,\big)\,, \quad (67)$$

we find, conversely, for arbitrary $\nu''\in\mathrm{S}^*(w)$ some $\nu'\in\mathrm{S}^*(v)$ with $\sigma(\nu',\nu'')\leqslant\varepsilon$. The proof is complete. □

Proof of Theorem 3.6. The assertion follows in complete analogy to [34], p. 85, Proof of Theorem 3.6, (1), from Theorem 3.5, Definition 5.6 and Theorem 5.7. □

b) Semicontinuous extension $\mathrm{S}^\#$ of the set-valued map $\mathrm{S}^*\big|\,\mathrm{int}\,(\mathrm{K})$ to $\partial\mathrm{K}$.

In a second step, we extend the set-valued map S^* to the boundary of K. For this purpose, we will show that, along every ray R starting from the origin, the Painlevé-Kuratowski limit $\lim^K_{v\to v_0\,,\,v\,\in\,\mathrm{R}\,\cap\,\mathrm{int}\,(\mathrm{K})}\,\mathrm{S}^*(v)$ in the point $v_0\in\mathrm{R}\cap\partial\mathrm{K}$ exists. We start with

Definition 3.8 ($\mathrm{S}^\#$ **as extension of** S^* **to the boundary** $\partial\mathrm{K}$**).** We define the set-valued map $\mathrm{S}^\#:\ \mathrm{K}\to\mathfrak{P}\big(\,rca^{\,pr}\,(\mathrm{K})\,\big)$ by

$$\mathrm{S}^\#(v_0)=\begin{cases}\mathrm{S}^*(v_0)\;\big|\;v_0\in\mathrm{int}\,(\mathrm{K})\,;\\ \lim\limits^K_{v\to v_0\,,\,v\,\in\,\mathrm{R}\,\cap\,\mathrm{int}\,(\mathrm{K})}\;\mathrm{S}^*(v)\;\big|\;v_0\in\partial\mathrm{K}\,.\end{cases} \quad (68)$$

This definition will be justified by the following Theorem 3.9. $\mathrm{S}^\#$ is a set-valued map with nonempty, convex, weak*-sequentially compact images, and we obtain a representation of $\mathrm{S}^\#(v_0)$ for $v_0\in\partial\mathrm{K}$ in analogy to Definition 3.1. Finally, we prove that the set-valued map $\mathrm{S}^\#$ is upper semicontinuous (Theorem 3.12).

Theorem 3.9.[49]

1) **(ε-δ relation for f^* along rays starting from the origin)** *Assume that two points v, $w \in \mathrm{int}\,(\mathrm{K})$ admit the following properties: a) v, w are situated on the same ray R starting from $\mathfrak{o}$, and b) $0 < \mathrm{Dist}\,(w\,,\,\partial\mathrm{K}) < \mathrm{Dist}\,(v\,,\,\partial\mathrm{K}) < \frac{1}{2}\,c_{\mathrm{K}}$. Then S^* obeys the following ε-δ estimate, which holds uniformly for all rays R starting from $\mathfrak{o}$:*

$$\begin{aligned} &\mathrm{Dist}\,(w\,,\,v) \leqslant \delta_2(\varepsilon) \\ \Longrightarrow\; &\textit{for every } \nu'' \in \mathrm{S}^*(w) \textit{ there exists } \nu' \in \mathrm{S}^*(v) \textit{ with } \sigma(\nu'',\nu') \leqslant \varepsilon \end{aligned} \tag{69}$$

with $\delta_2(\varepsilon) = \frac{1}{6}\,\delta(\varepsilon)\cdot c_{\mathrm{K}}/C_{\mathrm{K}}$ where c_{K} and C_{K} are the quantities defined in the beginning of the section.

2) **(Justification of Definition 3.8)** *Along every ray R starting from the origin, the following Painlevé-Kuratowski limit in the point $v_0 \in \mathrm{R} \cap \partial\mathrm{K}$ exists:*

$$\lim^K_{v\to v_0\,,\,v\,\in\,\mathrm{R}\,\cap\,\mathrm{int}\,(\mathrm{K})} \mathrm{S}^*(v)\,. \tag{70}$$

3) **(ε-δ relation for $\mathrm{S}^\#$ along rays starting from the origin)** *Under the assumptions of Part 1), we consider two points v, $w \in \mathrm{K}$, which a) are situated on the same ray R starting from $\mathfrak{o}$ and b) satisfy $0 \leqslant \mathrm{Dist}\,(w\,,\,\partial\mathrm{K}) \leqslant \mathrm{Dist}\,(v\,,\,\partial\mathrm{K}) < \frac{1}{2}\,c_{\mathrm{K}}$. Then the ε-δ estimate from Part 1) can be extended to $\mathrm{S}^\#$:*

$$\begin{aligned} &\mathrm{Dist}\,(w\,,\,v) \leqslant \delta_2(\varepsilon) \\ \Longrightarrow\; &\textit{for every } \nu'' \in \mathrm{S}^\#(w) \textit{ there exists } \nu' \in \mathrm{S}^\#(v) \textit{ with } \sigma(\nu'',\nu') \leqslant \varepsilon\,, \end{aligned} \tag{71}$$

and again the estimate holds uniformly for all rays R starting from $\mathfrak{o}$.

Theorem 3.10 (Properties of the sets $\mathrm{S}^\#(w)$ for $w \in \partial\mathrm{K}$).

1) *For every $w \in \partial\mathrm{K}$, the set $\mathrm{S}^\#(w) \subseteq rca^{pr}\,(\mathrm{K})$ is nonempty, convex and weak*-sequentially compact.*
2) *For every $w \in \partial\mathrm{K}$, the set $\mathrm{S}^\#(w)$ may be represented as*

$$\begin{aligned} \mathrm{S}^\#(w) = \Big\{\, \nu \in rca^{pr}\,(\mathrm{K}) \;\Big|\; &\textit{there exist sequences } \{\, w^N \,\}\,,\ \mathrm{int}\,(\mathrm{K}) \\ &\textit{and } \{\, x^N \,\}\,,\ W_0^{1,\infty}(\Omega,\mathbb{R}^n) \textit{ with} \\ &a)\ \lim_{N\to\infty} w^N = w, \\ &b)\ \lim_{N\to\infty} \|\, x^N \,\|_{C^0(\Omega,\mathbb{R}^n)} = 0, \\ &c)\ w^N + Jx^N(t) \in \mathrm{K}\ (\forall)\, t \in \Omega\ \forall\, N \in \mathbb{N}, \\ &d)\ \{\, w^N + Jx^N \,\} \textit{ generates the constant generalized} \\ &\qquad \textit{gradient control } \boldsymbol{\nu} = \{\, \nu \,\} \,\Big\}\,. \end{aligned} \tag{72}$$

The proof of the upper semicontinuity of the set-valued map $\mathrm{S}^\#$ is based on the following assertion:

[49] Compare with [34], p. 88, Theorem 3.12.

Theorem 3.11 (ε-δ relation for $\mathrm{S}^\#$ in points $v \in \partial\mathrm{K}$).[50] *Let a point $v \in \partial\mathrm{K}$ be given. Then for arbitrary $\varepsilon > 0$ there exists $\delta_4(\varepsilon, v) > 0$ with*

$$\begin{aligned} &\mathrm{Dist}\,(w, v) \leqslant \delta_4(\varepsilon, v) \\ \Longrightarrow\ &\textit{for every } \nu_w \in \mathrm{S}^\#(w) \textit{ there exists } \nu_v \in \mathrm{S}^\#(v) \textit{ with } \sigma(\nu_v, \nu_w) \leqslant 3\,\varepsilon \end{aligned} \tag{73}$$

for all $w \in \mathrm{K}$.

Theorem 3.12 (Upper semicontinuity of the set-valued map $\mathrm{S}^\#$).[51]

1) *The set-valued map $\mathrm{S}^\#$ is upper semicontinuous on K.*
2) *For all $v_0 \in \mathrm{K}$, it holds that $\mathrm{S}^\#(v_0) = \limsup^K{}_{v \to v_0\,,\, v \in \mathrm{int}\,(\mathrm{K})}\ \mathrm{S}^*(v)$.*

Finally, we state

Theorem 3.13 ($\mathrm{S}^\#$ in extremal points of K).[52] *For every $w \in \mathrm{ext}\,(\mathrm{K})$, the set $\mathrm{S}^\#(w) = \{\,\delta_w\,\}$ is a singleton.*

Proof of Theorem 3.9. *1*) Let $\mathrm{Dist}\,(\mathfrak{o}\,,\, v) = D$ and $\mathrm{Dist}\,(\mathfrak{o}\,,\, w) = D + d$. Then it follows:

$$0 < \frac{c_\mathrm{K}}{2} \leqslant D < D + d < C_\mathrm{K} \Longrightarrow \frac{c_\mathrm{K}}{2\,C_\mathrm{K}} < \frac{c_\mathrm{K}}{2\,(D+d)} \leqslant \frac{D}{D+d} < 1\,, \tag{74}$$

and the points v and w can be written as

$$v = \frac{D}{D+d}\,w \quad \text{resp. } w = \frac{D+d}{D}\,v\,. \tag{75}$$

Choose now $\varepsilon > 0$ and arbitrary $\nu'' \in \mathrm{S}^*(w)$. Then, in relation to ν'', there exists a sequence $\{\,x^N\,\}$, $W_0^{1,\infty}(\Omega, \mathbb{R}^n)$, which possesses together with the constant sequence $\{\,w\,\}$ the properties a)–d) from Definition 3.1 (cf. Lemma 3.3). Consequently, we have for all $N \in \mathbb{N}$:

$$\begin{aligned} &\frac{D+d}{D}\,v + Jx^N(t) \in \mathrm{K}\ \ (\forall)\,t \in \Omega \quad \text{resp.} \\ &v + \frac{D}{D+d}\,Jx^N(t) \in \frac{D}{D+d}\,\mathrm{K} \subset \mathrm{K}\ \ (\forall)\,t \in \Omega \end{aligned} \tag{76}$$

and

$$\begin{aligned} &\left|\,\big(\,w + Jx^N(t)\,\big) - \left(v + \frac{D}{D+d}\,Jx^N(t)\right)\right| \\ &\leqslant d\left(\frac{|\,v\,|}{D} + \frac{|\,Jx^N(t)\,|}{D+d}\right) \leqslant \frac{d}{D}\,\big(\,|\,v\,| + |\,Jx^N(t)\,|\,\big) \leqslant d \cdot \frac{6\,C_\mathrm{K}}{c_\mathrm{K}}\,. \end{aligned} \tag{77}$$

[50] Compare with [34], p. 89, Theorem 3.15.
[51] Compare with [34], p. 89, Theorem 3.16.
[52] Compare with [34], p. 89, Theorem 3.14, (2).

With (77), we obtain

$$\begin{aligned}
&\varrho(\{\,\delta_{w+Jx^N(t)}\,\},\,\{\,\delta_{v+\frac{D}{D+d}Jx^N(t)}\,\}) \\
&= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{1}{2^{r+s}}\cdot\frac{1}{1+L_s}\cdot\left|\int_\Omega f_r(t)\Big(g_s\left(w+Jx^N(t)\right)-g_s\left(v+\frac{D}{D+d}Jx^N(t)\right)\Big)\,dt\right| && (78)\\
&\leqslant \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{1}{2^{r+s}}\cdot\frac{1}{1+L_s}\cdot\int_\Omega |f_r(t)|\cdot L_s\cdot\left|\left(w+Jx^N(t)\right)-\left(v+\frac{D}{D+d}Jx^N(t)\right)\right|\,dt && (79)\\
&\leqslant \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{1}{2^{r+s}}\cdot\frac{L_s}{1+L_s}\cdot\|\,f_r\,\|_{L_1(\Omega)}\cdot\frac{6\,C_{\mathrm{K}}}{c_{\mathrm{K}}}\cdot d; && (80)
\end{aligned}$$

consequently, the implication

$$\begin{aligned}
&\big|\,w-v\,\big| \;=\; d\;\leqslant\;\delta_2(\varepsilon)=\frac{\varepsilon}{6}\cdot\frac{c_{\mathrm{K}}}{C_{\mathrm{K}}}\\
&\Longrightarrow\;\varrho(\,\{\,\delta_{w+Jx^N(t)}\,\}\;,\;\{\,\delta_{v+\frac{D}{D+d}\,Jx^N(t)}\,\}\;)\;\leqslant\;\varepsilon\quad\forall\,N\in\mathbb{N}
\end{aligned}\qquad(81)$$

holds. Passing now to a subsequence with elements $\{\,\delta_{v+\frac{D}{D+d}\,Jx^{N'}(t)}\,\}\;\in\;\boldsymbol{\mathcal{Y}}(\mathrm{K})$, which converges to a generalized gradient control $\boldsymbol{\mu}\in\boldsymbol{\mathcal{G}}(\mathrm{K})$, we arrive at

$$\lim_{N'\to\infty}\;\varrho(\,\{\,\delta_{w+Jx^{N'}(t)}\,\}\;,\;\{\,\delta_{v+\frac{D}{D+d}\,Jx^{N'}(t)}\,\}\;)\;=\;\varrho(\,\{\,\nu''\,\}\;,\;\boldsymbol{\mu}\,)\;\leqslant\;\varepsilon\,.\qquad(82)$$

By Theorem 2.11, 1), it holds that

$$\sigma(\nu'',A(\boldsymbol{\mu}))\;\leqslant\;\varrho(\,\{\,\nu''\,\}\;,\;\boldsymbol{\mu}\,)\;\leqslant\;\varepsilon\qquad(83)$$

where $A(\boldsymbol{\mu})=\nu'$ belongs to $\mathrm{S}^*(v)$.

2) Assume on the contrary that there exists some element

$$\nu''\in\left(\limsup^K_{v\to v_0\,,\,v\,\in\,\mathrm{R}\,\cap\,\mathrm{int}\,(\mathrm{K})}\;\mathrm{S}^*(v)\right)\;\setminus\;\left(\liminf^K_{v\to v_0\,,\,v\,\in\,\mathrm{R}\,\cap\,\mathrm{int}\,(\mathrm{K})}\;\mathrm{S}^*(v)\right).\qquad(84)$$

Then, by Definition 5.3, there exist sequences of points $\{\,w^N\,\}$, $\mathrm{R}\cap\mathrm{int}\,(\mathrm{K})\to v_0$ and measures $\{\,\nu''_N\,\}$, $rca^{pr}\,(\mathrm{K})$ with $\nu''_N\in\mathrm{S}(w^N)\;\forall\,N\in\mathbb{N}$ and $\lim_{\mathrm{N}\to\infty}\sigma(\nu''_N,\nu'')=0$, but at the same time there exists another sequence of points $\{\,v^N\,\}$, $\mathrm{R}\cap\mathrm{int}\,(\mathrm{K})\to v_0$ such that for any sequence of measures $\{\,\nu'_N\,\}$, $rca^{pr}\,(\mathrm{K})$ with $\nu'_N\in\mathrm{S}(v^N)\;\forall\,N\in\mathbb{N}$ and $\lim_{N\to\infty}\sigma(\nu'_N,\nu')=0$, the limits do not coincide: $\nu''\neq\nu'$. By a passage to suitable subsequences (without change of indices), we may guarantee that for all indices N, it holds at the same time that

$$\big|\,w^N-v_0\,\big|\;\leqslant\;\frac{1}{N}\cdot\frac{c_{\mathrm{K}}}{12\,C_{\mathrm{K}}}\,,\quad\big|\,v^N-v_0\,\big|\;\leqslant\;\frac{1}{N}\cdot\frac{c_{\mathrm{K}}}{12\,C_{\mathrm{K}}}\quad\text{and}\quad\big|\,w^N-v_0\,\big|\;<\;\big|\,v^N-v_0\,\big|\qquad(85)$$

and thus

$$\big|\,w^N-v^N\,\big|\;\leqslant\;\frac{1}{N}\cdot\frac{c_{\mathrm{K}}}{6\,C_{\mathrm{K}}}\qquad(86)$$

as well. By Part 1), for every $\nu''_N \in \mathrm{S}^*(w^N)$ there exists $\nu'_N \in \mathrm{S}^*(v^N)$ with $\sigma(\nu''_N, \nu'_N) \leqslant 1/N$. The sequence $\{\, \nu'_N \,\}$ admits a weak*-convergent subsequence (index M); when passing to this subsequence, we may assume

$$\sigma(\nu''_M, \nu'') \leqslant \frac{1}{M} \quad \text{and} \quad \sigma(\nu'_M, \nu') \leqslant \frac{1}{M} \tag{87}$$

as well. We arrive now at a contradiction by the limit passage $M \to \infty$ within the inequality

$$0 < \sigma(\nu'', \nu') \leqslant \sigma(\nu'', \nu''_M) + \sigma(\nu''_M, \nu'_M) + \sigma(\nu'_M, \nu') \leqslant \frac{1}{M} + \frac{1}{M} + \frac{1}{M}\,. \tag{88}$$

3) Choose $\varepsilon > 0$. In view of Part 1), it remains to prove that the relation holds in the case where $w \neq v$ with $\mathrm{Dist}\,(w, v) \leqslant \delta_2(\varepsilon)$ belongs to $\partial\mathrm{K}$. Then there exists a sequence of points $\{\, w^N \,\}$, $\mathrm{R} \cap \mathrm{int}\,(\mathrm{K}) \to w$ with $\mathrm{Dist}\,(w^N, \partial\mathrm{K}) < \mathrm{Dist}\,(v, \partial\mathrm{K})$ and, consequently, $\mathrm{Dist}\,(w^N, v) \leqslant \delta_2(\varepsilon)$ for all $N \in \mathbb{N}$. Let $\nu'' \in \mathrm{S}^\#(w)$ be given. By Part 2), we find measures $\nu''_N \in \mathrm{S}^\#(w^N)$ with $\sigma(\nu'', \nu''_N) \leqslant \varepsilon^N$ and $\{\, \varepsilon^N \,\} \to 0$. Furthermore, by Part 1), for every $\nu''_N \in \mathrm{S}^\#(w^N) = \mathrm{S}^*(w^N)$ there exists $\nu'_N \in \mathrm{S}^\#(v) = \mathrm{S}^*(v)$ with $\sigma(\nu''_N, \nu'_N) \leqslant \varepsilon$. Since $\mathrm{S}^\#(v) \subseteq rca^{pr}\,(\mathrm{K})$ is weak*-sequentially compact, the sequence $\{\, \nu'_N \,\}$, $\mathrm{S}^\#(v)$ admits a weak*-convergent subsequence with limit $\nu' \in \mathrm{S}^\#(v)$ (we keep the index N). We may further assume that $\sigma(\nu'_N, \nu') \leqslant \varepsilon^N$. Summing up, we arrive at

$$\sigma(\nu'', \nu') \leqslant \sigma(\nu'', \nu''_N) + \sigma(\nu''_N, \nu'_N) + \sigma(\nu'_N, \nu') \leqslant \varepsilon^N + \varepsilon + \varepsilon^N\,, \tag{89}$$

what proves assertion 3) since $\{\, \varepsilon^N \,\} \to 0$. □

Proof of Theorem 3.10. *1*) By Theorem 3.4, δ_v belongs to $\mathrm{S}^*(v)$ for all $v \in \mathrm{R} \cap \mathrm{int}\,(\mathrm{K})$, and for all sequences $\{\, v^N \,\}$, $\mathrm{R} \cap \mathrm{int}\,(\mathrm{K}) \to v_0$, it holds that $\lim_{N\to\infty} \sigma(\delta_{v_0}, \delta_{v^N}) = 0$. Consequently, δ_{v_0} belongs to $\mathrm{S}^\#(v_0) = \lim^K{}_{v \to v_0\,,\, v \in \mathrm{R} \cap \mathrm{int}\,(\mathrm{K})} \mathrm{S}^*(v)$, and this set is nonempty. As a Painlevé-Kuratowski limit, it is closed with respect to the topology generated by σ as well (Lemma 5.4, 2)). The convexity follows from Theorem 5.5, 2), the compactness again from Theorem 2.8, 2).

2) By Definition 3.8, it holds that

$$\begin{aligned} \mathrm{S}^\#(w) &= \lim\nolimits^K_{v \to w\,,\, v \in \mathrm{R} \cap \mathrm{int}\,(\mathrm{K})} \mathrm{S}^\#(v) = \limsup\nolimits^K_{v \to w\,,\, v \in \mathrm{R} \cap \mathrm{int}\,(\mathrm{K})} \mathrm{S}^\#(v) \\ &= \Big\{\, \nu \in \mathrm{X} \;\Big|\; \exists\, \{\, w^N \,\},\ \mathrm{R} \cap \mathrm{int}\,(\mathrm{K}) \to w \;\; \exists\, \{\, \nu^N \,\}, \\ &\qquad \mathrm{X} \to \nu \;\; \text{with } \nu^N \in \mathrm{S}^\#(w^N) \;\; \forall\, N \in \mathbb{N} \,\Big\}\,. \end{aligned} \tag{90}$$

Consider now $\nu \in \mathrm{S}^\#(w)$ together with sequences $\{\, w^N \,\}$, $\mathrm{int}\,(\mathrm{K}) \to w$ and $\{\, \nu^N \,\}$, $rca^{pr}\,(\mathrm{K})$ with $\nu^N \in \mathrm{S}^\#(w^N)$ for all $N \in \mathbb{N}$ and $\lim_{\mathrm{N}\to\infty} \sigma(\nu^N, \nu) = 0$. By Lemma 3.2 and 3.3, for every $\nu^N \in \mathrm{S}^\#(w^N)$ there exist the constant sequence $\{\, w^N \,\}$ and a sequence $\{\, x^{N,M} \,\}$, $W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ with the properties a)–d) from Definition 3.1. We select a diagonal sequence $\{\, x^{N,M(N)} \,\}$ with

$$\varrho(\, \{\, \delta_{w^N + Jx^{N,M(N)}(t)} \,\}\,,\, \{\, \nu^N \,\}\,) \leqslant \frac{1}{N}\,. \tag{91}$$

At the same time, we may assume that the sequence of generalized controls $\{\,\delta_{w^N+Jx^{N,M(N)}(t)}\,\}$ converges itself to a generalized gradient control $\boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$. Consequently, it holds that

$$\varrho(\,\{\,\delta_{w^N+Jx^{N,M(N)}(t)}\,\}\;,\;\boldsymbol{\mu}\,)\;\leqslant\;\frac{1}{N}\,. \tag{92}$$

Applying the mean value theorem (Theorem 2.9) to $\boldsymbol{\mu}$, we find that $A(\boldsymbol{\mu})$ is generated by sequences with the properties a)–d). Finally, from Theorem 2.11, 1) it follows that

$$\begin{aligned}&\sigma(\nu, A(\boldsymbol{\mu}))\;\leqslant\;\sigma(\nu,\nu^N)\;+\;\sigma(\nu^N, A(\boldsymbol{\mu}))\\ \leqslant\;&\sigma(\nu,\nu^N)\;+\;\varrho(\,\{\,\nu^N\,\}\;,\;\{\,\delta_{w^N+Jx^{N,M(N)}(t)}\,\}\,)\;+\;\varrho(\,\{\,\delta_{w^N+Jx^{N,M(N)}(t)}\,\}\;,\;\boldsymbol{\mu}\,)\to 0\,,\end{aligned} \tag{93}$$

what proves $A(\boldsymbol{\mu}) = \nu$. Thus $\nu \in \mathrm{S}^{\#}(w)$ may be represented in the claimed way. □

Proof of Theorem 3.11. Let us fix $\varepsilon > 0$. By Theorem 3.9, 2), there exists a point $v' \in \mathrm{R}_v \cap \mathrm{int}\,(\mathrm{K})$ on the ray $\mathrm{R}_v = \overrightarrow{\mathfrak{o}\,v}$ with

$$0\;<\;\mathrm{Dist}\,(v',\,\partial\mathrm{K})\;\leqslant\;\mathrm{Min}\,\left(1\,,\;\frac{\delta_2(\varepsilon)}{2}\;,\;\frac{c_\mathrm{K}}{2}\right)\quad\text{and}\quad\mathcal{H}\,(\,\mathrm{S}^{\#}(v)\,,\,\mathrm{S}^{\#}(v')\,)\leqslant\varepsilon\,. \tag{94}$$

From Theorem 3.5, we take $\delta_1(\varepsilon) = \frac{1}{4}\,\varepsilon/C_\mathrm{K} < 1$. Defining $\delta_3(\varepsilon, v) = \mathrm{Dist}\,(v',\,\partial\mathrm{K})$, we infer from the Lipschitz continuity of the distance function $\mathrm{Dist}\,(\,\cdot\,,\,\partial\mathrm{K})$ (see [9], p. 50) that the points $w' \in \mathrm{K}(v', \frac{1}{2}\,\delta_1(\varepsilon)\,\delta_3(\varepsilon, v))$ obey

$$\big|\,\mathrm{Dist}\,(w'\,,\,\partial\mathrm{K}) - \mathrm{Dist}\,(v'\,,\,\partial\mathrm{K})\,\big|\;\leqslant\;\big|\,w'-v'\,\big|\;\leqslant\;\frac{1}{2}\,\delta_1(\varepsilon)\,\delta_3(\varepsilon,v) \tag{95}$$

$$\Longrightarrow\;-\,\frac{1}{2}\,\delta_1(\varepsilon)\,\delta_3(\varepsilon,v)\;\leqslant\;\mathrm{Dist}\,(w'\,,\,\partial\mathrm{K}) - \mathrm{Dist}\,(v'\,,\,\partial\mathrm{K}) \tag{96}$$

$$\Longrightarrow\;-\,\frac{1}{2}\,\delta_1(\varepsilon)\,\delta_3(\varepsilon,v) + \mathrm{Dist}\,(v'\,,\,\partial\mathrm{K})\;=\;\delta_3(\varepsilon,v)\,\left(1-\frac{\delta_1(\varepsilon)}{2}\right)\;\leqslant\;\mathrm{Dist}\,(w'\,,\,\partial\mathrm{K})\,. \tag{97}$$

From $\delta_1(\varepsilon) < 1$ we conclude then

$$\begin{aligned}\frac{\delta_3(\varepsilon,v)}{2}\leqslant\delta_3(\varepsilon,v)\cdot\left(1-\frac{\delta_1(\varepsilon)}{2}\right)\;&=\;\mathrm{Min}\,\left(1\,,\,\delta_3(\varepsilon,v)\,,\,\delta_3(\varepsilon,v)\cdot\left(1-\frac{\delta_1(\varepsilon)}{2}\right)\right)\\ \leqslant\;\mathrm{Min}\,\left(1\,,\,\mathrm{Dist}\,(v'\,,\,\partial\mathrm{K})\,,\,\mathrm{Dist}\,(w'\,,\,\partial\mathrm{K})\,\right)\,.&\end{aligned} \tag{98}$$

Summing up, we arrive at the implication

$$\begin{aligned}&\big|\,w'-v'\,\big|\;\leqslant\;\frac{1}{2}\,\delta_1(\varepsilon)\,\delta_3(\varepsilon,v)\\ \Longrightarrow\;&\big|\,w'-v'\,\big|\;\leqslant\;\delta_1(\varepsilon)\cdot\mathrm{Min}\,\big(1\,,\,\mathrm{Dist}\,(v',\,\partial\mathrm{K})\,,\,\mathrm{Dist}\,(w',\,\partial\mathrm{K})\,\big)\end{aligned} \tag{99}$$

for arbitrary points $w' \in \mathrm{int}\,(\mathrm{K})$, from which follows $\mathcal{H}\,(\,\mathrm{S}^{\#}(v')\,,\,\mathrm{S}^{\#}(w')\,) \leqslant \varepsilon$ (Theorem 3.5). Consider now the points $w \in \mathrm{K}$ with $|\,v - w\,| \leqslant \frac{1}{2}\,\delta_1(\varepsilon)\,\delta_3(\varepsilon, v) = \delta_4(\varepsilon, v)$. By the intercept theorems, for any of these points w there exists a further point $w'' \in \mathrm{R}_w \cap \mathrm{int}\,(\mathrm{K})$

on the ray $R_w = \overrightarrow{\mathfrak{o}\, w}$ such that w'' belongs at the same time to $K(v', \frac{1}{2}\,\delta_1(\varepsilon)\,\delta_3(\varepsilon, v))$. For such a point w'', it holds that

$$\begin{aligned} \big|\, w - w'' \,\big| \,&\leqslant\, \big|\, w - v \,\big| + \big|\, v - v' \,\big| + \big|\, v' - w'' \,\big| \\ &\leqslant \frac{1}{2}\,\delta_1(\varepsilon)\,\delta_3(\varepsilon, v) \,+\, \delta_3(\varepsilon, v) \,+\, \frac{1}{2}\,\delta_1(\varepsilon)\,\delta_3(\varepsilon, v) \\ &= \delta_3(\varepsilon, v)\,\big(\,1 + \delta_1(\varepsilon)\,\big) \,\leqslant\, 2\,\delta_3(\varepsilon, v) \,\leqslant\, \delta_2(\varepsilon)\,. \end{aligned} \tag{100}$$

Then by Theorem 3.9, 3), for every $\nu_w \in S^{\#}(w)$ there exists $\nu'' \in S^{\#}(w'')$ with $\sigma(\nu_w, \nu'') \leqslant \varepsilon$. Since $\mathcal{H}\,(\,S^{\#}(v')\,,\,S^{\#}(w'')\,) \leqslant \varepsilon$, in relation to ν'' there exists $\nu' \in S^{\#}(v')$ with $\sigma(\nu', \nu'') \leqslant \varepsilon$, and since $\mathcal{H}\,(\,S^{\#}(v)\,,\,S^{\#}(v')\,) \leqslant \varepsilon$, in relation to ν' there exists $\nu_v \in S^{\#}(v)$ with $\sigma(\nu_v, \nu') \leqslant \varepsilon$. Combining these inequalities, we find that for every $\nu_w \in S^{\#}(w)$ there exists $\nu_v \in S^{\#}(v)$ with

$$\sigma(\nu_v, \nu_w) \,\leqslant\, \sigma(\nu_v, \nu') \,+\, \sigma(\nu', \nu'') \,+\, \sigma(\nu'', \nu_w) \,\leqslant\, 3\,\varepsilon\,. \tag{101}$$

□

Proof of Theorem 3.12. *1*) It remains only to prove that $S^{\#}$ is upper semicontinuous in points $v_0 \in \partial K$. From Theorem 3.11, for all $v \in K$ it follows:

$$\begin{aligned} &\big|\, v - v_0 \,\big| \,\leqslant\, \delta_4\left(\frac{\varepsilon}{3}, v_0\right) \\ &\Longrightarrow \text{ to every } \nu \in S^{\#}(v) \text{ there exists } \nu_0 \in S^{\#}(v_0) \text{ with } \sigma(\nu, \nu_0) \leqslant \varepsilon. \end{aligned} \tag{102}$$

By Theorem 5.7, 2), this is equivalent with $\limsup^K{}_{v\to v_0} S^{\#}(v) \subseteq S^{\#}(v_0)$. According to Definition 5.6, 2), $S^{\#}$ is upper semicontinuous then in v_0.

2) Choose an arbitrary point $v_0 \in K$. Then we conclude from Definition 5.3 and Part 1):

$$\limsup^K_{v\to v_0\,,\,v\,\in\,\mathrm{int}\,(K)} S^*(v) \,=\, \limsup^K_{v\to v_0\,,\,v\,\in\,\mathrm{int}\,(K)} S^{\#}(v) \,\subseteq\, \limsup^K_{v\to v_0\,,\,v\,\in\,K} S^{\#}(v) \,\subseteq\, S^{\#}(v_0)\,. \tag{103}$$

Conversely, if R denotes the ray $\overrightarrow{\mathfrak{o}\, w}$ then it holds that

$$S^{\#}(v_0) = \lim^K_{v\to v_0\,,\,v\,\in\,R\,\cap\,\mathrm{int}\,(K)} S^{\#}(v) \,=\, \limsup^K_{v\to v_0\,,\,v\,\in\,R\,\cap\,\mathrm{int}\,(K)} S^{\#}(v) \tag{104}$$

$$\begin{aligned} = \,\,\big\{\, \nu \in X \,\big|\, \exists\,\{\, v^N \,\}\,,\, R \cap \mathrm{int}\,(K) \to v_0 \;\; \exists\,\{\, \nu^N \,\}\,, \\ X \to \nu \;\;\text{ with } \nu^N \in S^{\#}(v^N) \;\; \forall\, N \in \mathbb{N} \,\big\} \end{aligned} \tag{105}$$

$$\begin{aligned} \subseteq \,\,\big\{\, \nu \in X \,\big|\, \exists\,\{\, v^N \,\}\,,\, \mathrm{int}\,(K) \to v_0 \;\; \exists\,\{\, \nu^N \,\}\,, \\ X \to \nu \;\;\text{ with } \nu^N \in S^{\#}(v^N) \;\; \forall\, N \in \mathbb{N} \,\big\} \end{aligned} \tag{106}$$

$$= \limsup^K_{v\to v_0\,,\,v\,\in\,\mathrm{int}\,(K)} S^{\#}(v)\,, \tag{107}$$

and the claimed equality results. □

The proof of Theorem 3.13 will be postponed to the following section.

4. Proof of the representation theorem for $f^{(qc)}$.

We arrive now at the desired representation of the lower semicontinuous quasiconvex envelope $f^{(qc)}$ of a function $f \in \mathfrak{F}_{\mathrm{K}}$ by means of the set-valued map $\mathrm{S}^{\#}$.

Proof of Theorem 1.4. We have to prove that the set-valued map $\mathrm{S}^{(qc)} = \mathrm{S}^{\#}$ possess all the claimed properties.

• *Step 1.* By Theorem 3.4 and 3.10, 1), the images $\mathrm{S}^{\#}(w)$ are nonempty, convex and weak*-sequentially compact for all $w \in \mathrm{K}$. Applying the remark after Theorem 2.9 to the sequences in Definition 3.1 and Theorem 3.10, 2), we see that $\mathrm{S}^{(qc)}(w) \subseteq \mathrm{S}^{c}(w) \subseteq rca^{pr}(\mathrm{K})$ holds for all $w \in \mathrm{K}$ as well. The upper semicontinuity of the set-valued map $\mathrm{S}^{\#} : \mathrm{K} \to \mathfrak{P}\big(rca^{pr}(\mathrm{K})\big)$ has been established in Theorem 3.12, 1). Turning now to the proof of the relation

$$f^{(qc)}(w) \;=\; \mathrm{Min}\left\{ \int_{\mathrm{K}} f(v)\, d\nu(v) \;\middle|\; \nu \in \mathrm{S}^{(qc)}(w) \right\}, \tag{108}$$

we remark first that we may replace in (108) the minimum by infimum since the sets $\mathrm{S}^{\#}(w) \subseteq rca^{pr}(\mathrm{K})$ are weak*-sequentially compact. We define, consequently,

$$h(w) \;=\; \inf\left\{ \int_{\mathrm{K}} f(v)\, d\nu(v) \;\middle|\; \nu \in \mathrm{S}^{\#}(w) \right\} \tag{109}$$

and distinguish the cases $w \in \mathrm{int}\,(\mathrm{K})$ and $w \in \partial\mathrm{K}$.

• *Step 2.* Choose $w \in \mathrm{int}\,(\mathrm{K})$. Then by Theorem 1.3, $f^{(qc)}(w)$ admits the representation

$$\begin{aligned} f^{(qc)}(w) &= f^{*}(w) \\ &= \inf\left\{ \frac{1}{|\Omega|}\int_{\Omega} f(v + Jx(t))dt \,\middle|\, x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n),\ v + Jx(t) \in \mathrm{K}\ \ (\forall)\, t \in \Omega \right\}. \end{aligned} \tag{110}$$

Consequently, for every $\varepsilon > 0$ there exists $x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ with $w + Jx(t) \in \mathrm{K}$ $(\forall)\, t \in \Omega$ and

$$f^{(qc)}(w) \;\leqslant\; \frac{1}{|\,\Omega\,|}\int_{\Omega} f(\,w + Jx(t)\,)\, dt \;\leqslant\; f^{(qc)}(w) + \varepsilon\,. \tag{111}$$

Applying the mean value theorem (Theorem 2.9) to $\boldsymbol{\mu} = \{\, \delta_{w+Jx(t)} \,\}$, we find a sequence $\{\, \widetilde{x}^{N} \,\},\ W_0^{1,\infty}(\Omega,\, \mathbb{R}^n)$ with

$$\frac{1}{|\,\Omega\,|}\int_{\Omega} f(\,w + Jx(t)\,)\, dt \;=\; \lim_{N\to\infty} \frac{1}{|\,\Omega\,|}\int_{\Omega} f(\,w + J\widetilde{x}^{N}(t)\,)\, dt \;=\; \int_{\mathrm{K}} f(v)\, d\nu(v) \tag{112}$$

where the sequences $\{\, w \,\}$, $\mathrm{int}\,(\mathrm{K})$ and $\{\, \widetilde{x}^{N} \,\},\ W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ possess the properties a)–d) from Definition 3.1. Thus ν belongs to $\mathrm{S}^{\#}(w)$. It follows that for every $\varepsilon > 0$ there exists $\nu \in \mathrm{S}^{\#}(w)$ with

$$\int_{\mathrm{K}} f(v)\, d\nu(v) \;\leqslant\; f^{(qc)}(w) \;+\; \varepsilon\,, \tag{113}$$

what proves the inequality $h(w) \leqslant f^{(qc)}(w)$. Conversely, for every $\varepsilon > 0$ there exists $\nu \in \mathrm{S}^{\#}(w)$ with

$$h(w) \leqslant \int_{\mathrm{K}} f(v)\, d\nu(v) \leqslant h(w) + \frac{\varepsilon}{2}\,. \tag{114}$$

By Lemma 3.2 and 3.3, $\{\,\nu\,\}$ may be generated by sequences $\{\,w\,\}$, $\mathrm{int}\,(\mathrm{K})$ and $\{\,x^N\,\}$, $W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ with the properties a)–d) from Definition 3.1; in particular, there exists $x^N \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ with $w + Jx^N(t) \in \mathrm{K}\ (\forall)\, t \in \Omega$ and

$$\left| \int_{\mathrm{K}} f(v)\, d\nu(v) - \frac{1}{|\,\Omega\,|} \int_{\Omega} f(\,w + Jx^N(t)\,)\, dt \right| \leqslant \frac{\varepsilon}{2}\,. \tag{115}$$

It follows that

$$\begin{aligned} &\frac{1}{|\,\Omega\,|} \int_{\Omega} f(\,w + Jx^N(t)\,)\, dt \\ &\leqslant h(w) + \frac{\varepsilon}{2} + \left(\frac{1}{|\,\Omega\,|} \int_{\Omega} f(\,w + Jx^N(t)\,)\, dt - \int_{\mathrm{K}} f(v)\, d\nu(v) \right) \\ &\leqslant h(w) + \frac{\varepsilon}{2} + \left| \frac{1}{|\,\Omega\,|} \int_{\Omega} f(\,w + Jx^N(t)\,)\, dt - \int_{\mathrm{K}} f(v)\, d\nu(v) \right| \leqslant h(w) + \varepsilon\,. \end{aligned} \tag{116}$$

We obtain

$$f^{(qc)}(w) \leqslant \frac{1}{|\,\Omega\,|} \int_{\Omega} f(\,w + Jx^N(t)\,)\, dt \leqslant h(w) + \varepsilon\,, \tag{117}$$

and the reverse inequality $f^{(qc)}(w) \leqslant h(w)$ follows.

• *Step 3.* Let now $w \in \partial\mathrm{K}$ be given. Then by Theorem 1.3, we have for arbitrary sequences $\{\,w^N\,\}$, $\mathrm{R} \cap \mathrm{int}\,(\mathrm{K}) \to w$ along the ray $\mathrm{R} = \overrightarrow{\mathfrak{o}\ w}$

$$f^{(qc)}(w) = \lim_{N\to\infty} f^{(qc)}(w^N) = \lim_{N\to\infty} f^{*}(w^N)\,. \tag{118}$$

We fix $\varepsilon > 0$ and choose for every w^N a function $x^N \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ with $w^N + Jx^N(t) \in \mathrm{K}\ (\forall)\, t \in \Omega$ and

$$f^{(qc)}(w^N) \leqslant \frac{1}{|\,\Omega\,|} \int_{\Omega} f(\,w^N + Jx^N(t)\,)\, dt \leqslant f^{(qc)}(w^N) + \varepsilon\,. \tag{119}$$

The sequence of generalized controls $\{\,\delta_{w^N + Jx^N(t)}\,\} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$ admits by Theorem 2.8, 2) a subsequence, which converges to a generalized gradient control $\boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$ (we keep the index N). It follows that

$$\begin{aligned} f^{(qc)}(w) &= \lim_{N\to\infty} f^{(qc)}(w^N) \\ &\leqslant \lim_{N\to\infty} \frac{1}{|\,\Omega\,|} \int_{\Omega} f(\,w^N + Jx^N(t)\,)\, dt = \frac{1}{|\,\Omega\,|} \int_{\Omega} \int_{\mathrm{K}} f(v)\, d\mu_t(v)\, dt \\ &\leqslant \lim_{N\to\infty} f^{(qc)}(w^N) + \varepsilon = f^{(qc)}(w) + \varepsilon\,. \end{aligned} \tag{120}$$

Applying the mean value theorem (Theorem 2.9) to $\boldsymbol{\mu}$, we find a sequence $\{\,\widetilde{x}^N\,\}$, $W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ with

$$\frac{1}{|\,\Omega\,|}\int_\Omega\int_{\mathrm{K}} f(v)\,d\mu_t(v)\,dt \;=\; \lim_{N\to\infty}\frac{1}{|\,\Omega\,|}\int_\Omega f(\,w^N + J\widetilde{x}^N(t)\,)\,dt \;=\; \int_{\mathrm{K}} f(v)\,d\nu(v)\,. \quad (121)$$

By Theorem 3.10, 2), ν belongs to $\mathrm{S}^\#(w)$. Consequently, for every $\varepsilon > 0$ there exists $\nu \in \mathrm{S}^\#(w)$ with

$$\int_{\mathrm{K}} f(v)\,d\nu(v) \;\leqslant\; f^{(qc)}(w) \,+\, \varepsilon\,, \quad (122)$$

and we get the inequality $h(w) \leqslant f^{(qc)}(w)$ as in Step *2*. Conversely, for every $\varepsilon > 0$ there exists $\nu \in \mathrm{S}^\#(w)$ with

$$h(w) \;\leqslant\; \int_{\mathrm{K}} f(v)\,d\nu(v) \;\leqslant\; h(w) \,+\, \frac{\varepsilon}{4}\,. \quad (123)$$

Then by Definitions 5.3 and 3.8, there exist sequences $\{\,w^N\,\}$, $\mathrm{R} \cap \mathrm{int}\,(\mathrm{K}) \to w$ and $\{\,\nu^N\,\}$, $rca^{\,pr}\,(\mathrm{K})$ with $\nu^N \in \mathrm{S}^\#(w^N)$ and $\sigma(\nu^N, \nu) \to 0$. Now we may choose an index N, a probability measure $\nu^N \in \mathrm{S}^\#(w^N)$ and a function $x^N \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ such that at the same time

$$\big|\, f^{(qc)}(w^N) - f^{(qc)}(w)\,\big| \;\leqslant\; \frac{\varepsilon}{4}\,; \quad (124)$$

$$\left|\,\int_{\mathrm{K}} f(v)\,\big(\,d\nu^N(v) - d\nu(v)\,\big)\right| \;\leqslant\; \frac{\varepsilon}{4}\,; \quad (125)$$

$$\left|\,\int_{\mathrm{K}} f(v)\,d\nu^N(v) - \frac{1}{|\,\Omega\,|}\int_\Omega f\,\big(\,w^N + Jx^N(t)\,\big)\;dt\,\right| \;\leqslant\; \frac{\varepsilon}{4} \quad (126)$$

and $w^N + Jx^N(t) \in \mathrm{K}\ (\forall)\, t \in \Omega$ hold. It follows that

$$\begin{aligned}
h(w) \;&\geqslant\; \int_{\mathrm{K}} f(v)\,d\nu(v) \,-\, \frac{\varepsilon}{4} \;=\; \frac{1}{|\,\Omega\,|}\int_\Omega f(\,w^N + Jx^N(t)\,)\,dt \\
&\quad -\left(\frac{1}{|\,\Omega\,|}\int_\Omega f(\,w^N + Jx^N(t)\,)\,dt \,-\, \int_{\mathrm{K}} f(v)\,d\nu^N(v)\right) \quad (127)\\
&\quad -\left(\int_{\mathrm{K}} f(v)\,d\nu^N(v) \,-\, \int_{\mathrm{K}} f(v)\,d\nu(v)\right) \,-\, \frac{\varepsilon}{4} \\
&\geqslant\; f^{(qc)}(w^N) \,-\, |\,...\,| \,-\, |\,...\,| \,-\, \frac{\varepsilon}{4} \quad (128)\\
&=\; f^{(qc)}(w) \,-\, \big(\,f^{(qc)}(w) - f^{(qc)}(w^N)\,\big) \,-\, \frac{3\,\varepsilon}{4} \\
&\geqslant\; f^{(qc)}(w) \,-\, |\,...\,| \,-\, \frac{3\,\varepsilon}{4} \;\geqslant\; f^{(qc)}(w) \,-\, \varepsilon\,. \quad (129)
\end{aligned}$$

We arrive at $f^{(qc)}(w) \leqslant h(w)$, and the proof is complete. □

Proof of Theorem 3.13. Let $w \in \mathrm{ext}\,(\mathrm{K})$ be given. Then $\delta_w \in \mathrm{S}^\#(w)$ by Lemma 3.3 and Proof of Theorem 3.10, 1). Let now arbitrary $\nu \in \mathrm{S}^\#(w)$ be given. By [34], p. 89, Theorem 3.14, (2), and the Theorems 1.3 and 1.4 above, we have

$$f^{(qc)}(w) = f(w) = \int_{\mathrm{K}} f(v)\,d\nu(v) \quad \forall\, f \in \mathcal{F}_{\mathrm{K}}\,. \tag{130}$$

Since $\mathcal{F}_{\mathrm{K}}$ and $C^0(\mathrm{K}, \mathbb{R})$ are isomorphic, ν is uniquely determined by this variational equality as a linear, continuous functional on $C^0(\mathrm{K}, \mathbb{R})$, and we arrive at $\nu = \delta_w$. □

5. Appendix: Set-valued maps.

a) Painlevé-Kuratowski limits.

Definition 5.1 (Painlevé-Kuratowski limits for set sequences). [53] Within a metric space $[\,\mathrm{X}\,,\,\sigma\,]$, let a sequence of sets $\{\,\mathrm{E}_N\,\}\,,\,\mathfrak{P}(\mathrm{X})$ be given. We define

$$\liminf_{N\to\infty}{}^{K}\ \mathrm{E}_N = \Big\{\nu \in \mathrm{X}\,\Big|\,\exists\{\,\nu^N\,\}\,,\mathrm{X}\text{ with }\nu^N \in \mathrm{E}_N \text{ for almost all } N \text{ and } \lim_{\mathrm{N}\to\infty}\sigma(\nu^N,\nu) = 0\Big\}\,; \tag{131}$$

$$\limsup_{N\to\infty}{}^{K}\ \mathrm{E}_N = \Big\{\nu \in \mathrm{X}\,\Big|\,\exists\{\,\nu^N\,\}\,,\mathrm{X}\text{ with }\nu^N \in \mathrm{E}_N \text{ for infinitely many } N \text{ and } \lim_{\mathrm{N}\to\infty}\sigma(\nu^N,\nu) = 0\Big\}\,; \tag{132}$$

$$\lim_{N\to\infty}{}^{K}\ \mathrm{E}_N = \mathrm{E} \iff \liminf_{N\to\infty}{}^{K}\ \mathrm{E}_N = \limsup_{N\to\infty}{}^{K}\ \mathrm{E}_N = \mathrm{E}\,. \tag{133}$$

In this definition, "for almost all N" means "except at most finitely many".

Definition 5.2 (Hausdorff distance in the metric space X).[54] Let $[\,\mathrm{X}\,,\,\sigma\,]$ be a compact metric space. Then we define the Hausdorff distance of nonempty, closed subsets $\mathrm{S}', \mathrm{S}'' \subseteq \mathrm{X}$ by

$$\mathcal{H}\,(\mathrm{S}',\mathrm{S}'') = \mathrm{Max}\Big(\mathop{\mathrm{Max}}_{\nu'\in\mathrm{S}'}\mathrm{Dist}\,(\nu',\mathrm{S}'')\,,\ \mathop{\mathrm{Max}}_{\nu''\in\mathrm{S}''}\mathrm{Dist}\,(\nu'',\mathrm{S}')\Big) \quad \text{resp.} \tag{134}$$

$\mathcal{H}\,(\mathrm{S}',\mathrm{S}'') \leqslant \varepsilon \iff$ for every $\nu' \in \mathrm{S}'$ there exists $\nu'' \in \mathrm{S}''$ with $\sigma(\nu',\nu'') \leqslant \varepsilon$, and for every $\nu'' \in \mathrm{S}''$ there exists $\nu' \in \mathrm{S}'$ with $\sigma(\nu'',\nu') \leqslant \varepsilon$.

Definition 5.3 (Painlevé-Kuratowski limits for set-valued maps).[55] Let a nonempty, compact subset $\mathrm{K} \subset \mathbb{R}^{nm}$ with $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$ and a compact metric space $[\,\mathrm{X}\,,\,\sigma\,]$ be given. We consider a set-valued map $\mathrm{S}:\ \mathrm{K} \to \mathfrak{P}(\mathrm{X})$ with nonempty, closed images,

[53] [3], p. 17, Definition 1.1.1; see also [27], p. 109, Definition 4.1. In [27], all definitions and theorems have been formulated within the framework of the euclidean space $\mathbb{R}^n$ only. However, numerous assertions presented there remain valid within arbitrary metric spaces.
[54] [27], p. 117, Example 4.13.
[55] [3], p. 41, Definition 1.4.6; see also [27], p. 152.

and define for $v_0 \in \mathrm{K}$

$$\begin{aligned} \liminf_{v\to v_0}{}^{K}\ \mathrm{S}(v) &= \bigcap_{v^N\to v_0} \liminf_{N\to\infty}{}^{K}\ \mathrm{S}(v^N) \\ &= \{\, \nu\in \mathrm{X} \mid \forall\{\, v^N \,\},\ \mathrm{K}\to v_0\ \exists\{\, \nu^N \,\},\ \mathrm{X}\to\nu \ \text{ with } \nu^N\in \mathrm{S}(v^N)\ \ \forall N\in \mathbb{N} \,\}; \end{aligned} \tag{135}$$

$$\begin{aligned} \limsup_{v\to v_0}{}^{K}\ \mathrm{S}(v) &= \bigcup_{v^N\to v_0} \limsup_{N\to\infty}{}^{K}\ \mathrm{S}(v^N) \\ &= \{\, \nu\in \mathrm{X} \mid \exists\{\, v^N \,\},\ \mathrm{K}\to v_0\ \exists\{\, \nu^N \,\},\ \mathrm{X}\to\nu \ \text{ with } \nu^N\in \mathrm{S}(v^N)\ \ \forall N\in \mathbb{N} \,\}; \end{aligned} \tag{136}$$

$$\lim_{v\to v_0}{}^{K} \mathrm{S}(v) = \mathrm{E} \iff \liminf_{v\to v_0}{}^{K}\ \mathrm{S}(v) = \limsup_{v\to v_0}{}^{K}\ \mathrm{S}(v) = \mathrm{E}\,. \tag{137}$$

Lemma 5.4 (Closedness of the Painlevé-Kuratowski limits). *Assume that* $\mathrm{K}\subset \mathbb{R}^{nm}$ *is nonempty and compact with* $\mathfrak{o}\in \mathrm{int}\,(\mathrm{K})$ *and* $[\,\mathrm{X}\,,\,\sigma\,]$ *is a compact metric space.*

1)[56] *For every set sequence* $\{\,\mathrm{E}_N\,\}\,,\,\mathfrak{P}(\mathrm{X})$*, the sets* $\liminf^K{}_{N\to\infty}\,\mathrm{E}_N$*,* $\limsup^K{}_{N\to\infty}\,\mathrm{E}_N$ *and (in the case of its existence)* $\lim^K{}_{N\to\infty}\,\mathrm{E}_N$ *are closed with respect to the topology generated by* σ*.*

2) *Assume that* $\mathrm{S}:\ \mathrm{K}\to\mathfrak{P}(\mathrm{X})$ *is a set-valued map with nonempty, closed images. Then for all* $v_0\in\mathrm{K}$*, the sets* $\liminf^K{}_{v\to v_0}\,\mathrm{S}(v)$*,* $\limsup^K{}_{v\to v_0}\,\mathrm{S}(v)$ *and (in case of its existence)* $\lim^K{}_{v\to v_0}\,\mathrm{S}(v)$ *are closed with respect to the topology generated by* σ*.*

Theorem 5.5 (Convexity of the Painlevé-Kuratowski limes inferior). *Consider a nonempty, compact set* $\mathrm{K}\subset\mathbb{R}^{nm}$ *with* $\mathfrak{o}\in\mathrm{int}\,(\mathrm{K})$ *and a linear topological space, which contains* X *as convex and sequentially compact subset. Assume further that the restriction of the topology to* X *is metrizable, and thus* $[\,\mathrm{X}\,,\,\sigma\,]$ *forms a compact metric space.*

1)[57] *If* $\{\,\mathrm{E}_N\,\}\,,\,\mathfrak{P}(\mathrm{X})$ *is a sequence of convex sets then the sets* $\liminf^K{}_{N\to\infty}\,\mathrm{E}_N$ *and (in case of its existence)* $\lim^K{}_{N\to\infty}\,\mathrm{E}_N$ *are convex as well.*

2) *If* $\mathrm{S}:\ \mathrm{K}\to\mathfrak{P}(\mathrm{X})$ *is a set-valued map with nonempty, closed, convex images then for all* $v_0\in\mathrm{K}$*, the sets* $\liminf^K{}_{v\to v_0}\,\mathrm{S}(v)$ *and (in case of its existence)* $\lim^K{}_{v\to v_0}\,\mathrm{S}(v)$ *are convex as well.*

The assumptions of Theorem 5.5 are particularly satisfied for $\mathrm{X} = rca^{pr}\,(\mathrm{K})$, endowed with the metric σ from Definition 2.1. In fact, by Theorem 2.2, 2), the restriction of the weak* topology of the space *rca* (K) to its (norm-)closed unit ball can be metrized by σ; consequently, the operations of addition and scalar multiplication are continuous with respect to this metric, and $rca^{pr}\,(\mathrm{K})$ forms a convex, weak*-sequentially compact subset of the unit ball.

Proof of Lemma 5.4. *2)* For $\liminf^K{}_{v\to v_0}\mathrm{S}(v)$ and (in case of its existence) $\lim^K{}_{v\to v_0}\mathrm{S}(v)$, the assertion follows from Part 1) together with the representations of the limits according to Definition 5.3. Consider now a sequence $\{\,\nu^K\,\}\,,\,\limsup^K{}_{v\to v_0}\mathrm{S}(v)\ \to$

[56] [27], p. 111, Proposition 4.4.
[57] See [27], p. 119, Proposition 4.15.

$\nu \in \mathrm{X}$. Then for every index K there exist sequences $\{ v^{N,K} \}$, $\mathrm{K} \to v_0$ and $\{ \nu^{N,K} \}$, $\mathrm{X} \to \nu^K$ with $\nu^{N,K} \in \mathrm{S}(v^{N,K})\ \forall N \in \mathbb{N}$, and for every $\varepsilon > 0$ there exists an index $K(\varepsilon)$ with $\sigma(\nu^{K(\varepsilon)}, \nu) \leqslant \varepsilon$ as well as an index $N(\varepsilon)$ with $\sigma(\nu^{N(\varepsilon),K(\varepsilon)}, \nu^{K(\varepsilon)}) \leqslant \varepsilon$ and $|\, v^{N(\varepsilon),K(\varepsilon)} - v_0 \,| \leqslant \varepsilon$. Consequently, there exist sequences $\{ v^M \}$, $\mathrm{K} \to v_0$ and $\{ \nu^M \}$, $\mathrm{X} \to \nu$ with $\nu^M \in \mathrm{S}(v^M)\ \forall M \in \mathbb{N}$, and ν belongs to $\limsup^K{}_{v \to v_0} \mathrm{S}(v)$ as well. □

Proof of Theorem 5.5. *1*) The proof of [27], p. 119, Proposition 4.15, can be immediately carried over to the present analytical situation.

2) The assertion follows from Part 1) together with the representation of $\liminf^K{}_{v \to v_0} \mathrm{S}(v)$ as an intersection (Definition 5.3). □

b) Semicontinuous and continuous set-valued maps.

Definition 5.6 (Semicontinuity and continuity of set-valued maps).[58] Let a nonempty, compact set $\mathrm{K} \subset \mathbb{R}^{nm}$ with $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$ and a compact metric space $[\,\mathrm{X}\,, \sigma\,]$ be given. We consider a set-valued map $\mathrm{S}:\ \mathrm{K} \to \mathfrak{P}(\mathrm{X})$ with nonempty, closed images.

1) The set-valued map S is called lower semicontinuous in $v_0 \in \mathrm{K}$ if $\mathrm{S}(v_0) \subseteq \liminf^K{}_{v \to v_0} \mathrm{S}(v)$ holds.
2) The set-valued map S is called upper semicontinuous in $v_0 \in \mathrm{K}$ if $\limsup^K{}_{v \to v_0} \mathrm{S}(v) \subseteq \mathrm{S}(v_0)$ holds.
3) The set-valued map S is called continuous in $v_0 \in \mathrm{K}$ if $\mathrm{S}(v_0) = \lim^K{}_{v \to v_0} \mathrm{S}(v)$ holds.

Theorem 5.7 (Conditions for semicontinuity and continuity of set-valued maps).[59] *Let a nonempty, compact set* $\mathrm{K} \subset \mathbb{R}^{nm}$ *with* $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$ *and a compact metric space* $[\,\mathrm{X}\,, \sigma\,]$ *be given. Assume further that* $\mathrm{S}:\ \mathrm{K} \to \mathfrak{P}(\mathrm{X})$ *is a set-valued map with nonempty, closed images, and* $\mathrm{E} \subseteq \mathrm{X}$ *is a nonempty, closed subset of* X.

1) $\mathrm{E} \subseteq \liminf^K{}_{v \to v_0} \mathrm{S}(v) \iff \forall\, \varepsilon > 0\ \exists\, \delta(\varepsilon) > 0:$
If $|\, v - v_0 \,| \leqslant \delta(\varepsilon)$ *then there exists for every* $\nu \in \mathrm{E}$ *an element* $\nu_v \in \mathrm{S}(v)$ *with* $\sigma(\nu, \nu_v) \leqslant \varepsilon$.
2) $\limsup^K{}_{v \to v_0} \mathrm{S}(v) \subseteq \mathrm{E} \iff \forall\, \varepsilon > 0\ \exists\, \delta(\varepsilon) > 0:$
If $|\, v - v_0 \,| \leqslant \delta(\varepsilon)$ *then there exists for every* $\nu_v \in \mathrm{S}(v)$ *an element* $\nu \in \mathrm{E}$ *with* $\sigma(\nu_v, \nu) \leqslant \varepsilon$.
3) $\mathrm{E} = \lim^K{}_{v \to v_0} \mathrm{S}(v) \iff \lim_{v \to v_0} \mathcal{H}\,(\mathrm{S}(v), \mathrm{E}) = 0$.

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