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Convexity on Abelian Groups

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Let A be a subset of an Abelian group G. We say that $f : A \to \mathbb{R}$ is convex if

$$2f(x) \leq f(x+h) + f(x-h)$$

holds for every $x, h \in G$ such that $x, x+h, x-h \in A$. We show that several classical theorems on convex functions defined on $\mathbb{R}^n$ can be proved in this general setting. We study extendibility of convex functions defined on subgroups of G. We show that a convex function need not have a convex extension, not even if it is defined on a subgroup of a linear space over $\mathbb{Q}$. We give a sufficient condition of extendibility which is also necessary in groups divisible by 2. We also investigate the continuity and measurability of convex functions defined on topological Abelian groups.

Introduction

Let G be an Abelian group and let A be a subset of G. We say that $f : A \to \mathbb{R}$ is *convex* if

$$2f(x) \leq f(x+h) + f(x-h) \tag{1}$$

holds whenever $x, h \in G$ and x, $x+h$, $x-h \in A$. In this paper we show that several classical theorems on convex functions defined on $\mathbb{R}^n$ or on (topological) linear spaces can be generalized to this general setting.

In Section 1 we prove that convexity implies

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{2}$$

for every rational $\lambda \in [0,1]$, whenever $\lambda x + (1-\lambda)y$ makes sense in G.

Section 2 is devoted to the problem of extendibility of convex functions defined on subgroups of G. As we shall see in Theorem 2.2, a convex function need not have a convex extension, not even if it is defined on a subgroup of a linear space over $\mathbb{Q}$. In Theorem 2.3

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we give a sufficient condition of extendibility which is also necessary in groups divisible by 2 (see Theorem 1.5).

In Section 3 we consider convex functions defined on topological groups. By a classical theorem of Bernstein and Doetsch, if a convex function f is defined on a normed linear space and is bounded from above on a nonempty open set, then f is continuous, moreover, locally Lipschitz (see [9, Chap. VII, Sec. 71, Theorem C and Chap. IV, Sec. 41, Theorem C], also [1]). As we shall see in Theorem 3.6, the same is true for convex functions defined on metric groups. On topological groups we obtain local uniform continuity. As a corollary we find that if f is convex on a dense subgroup H of a topological Abelian group G and if f is bounded from above in a nonempty relatively open subset of H, then can be extended to G as a continuous convex function (see Corollary 3.8).

In Section 4 we consider convex functions defined on open subsets of locally compact Abelian groups. We generalize well-known theorems of Blumberg [2], Sierpiński [11] and Ostrowski [7]. We also prove the category versions of these theorems.

1. The inequality of convexity with rational coefficients

Let D be a convex subset of a linear space over the rationals. It is well-known that if $f : D \to \mathbb{R}$ is convex, then it satisfies (2) for every $x, y \in D$ and $\lambda \in \mathbb{Q} \cap [0, 1]$; see [9, Chap. VII. Sec. 71, Theorem A] and [5, Chap. V, Sec. 3, Theorem 5]. In this section we prove that (2) holds in every Abelian group for every $\lambda \in \mathbb{Q}\cap[0, 1]$ whenever $\lambda x+(1-\lambda)y$ can be interpreted in G.

Let G be an Abelian group. Given $x, y \in G$ and $\lambda \in \mathbb{Q}$ we shall write $u \sim \lambda x + (1-\lambda)y$, if there are coprime integers k and $n \neq 0$ such that $\lambda = k/n$ and $nu = kx + (n-k)y$. (Note that $u \sim \lambda x + (1-\lambda)y$ does not define a relation on G; it is just an abbreviation of the statement above.) In general neither the existence nor the uniqueness of such an element $u \in G$ can be claimed. Nevertheless, in the next theorem we show that whenever an element $u \in G$ satisfies $u \sim \lambda x + (1-\lambda)y$, then (2) holds if we replace $\lambda x + (1-\lambda)y$ by u.

Theorem 1.1. *Let G be an Abelian group, and let $f : G \to \mathbb{R}$ be convex. Then*

$$f(u) \leq \lambda f(x) + (1-\lambda)f(y) \tag{3}$$

holds for every $\lambda \in \mathbb{Q} \cap [0, 1]$ and $x, y, u \in G$ such that $u \sim \lambda x + (1-\lambda)y$.

Lemma 1.2. *Let G be an Abelian group, let $x, h \in G$, and suppose that f is convex on the set $\{x, x + h, \ldots, x + nh\}$. Then*

$$nf(x + kh) \leq (n-k)f(x) + kf(x + nh) \tag{4}$$

holds for every $k = 0, \ldots, n$.

Proof. We put

$$F(k) = nf(x + kh) - [(n-k)f(x) + kf(x + nh)]$$

for every $k = 0, \ldots, n$, and $M = \max_{0\leq k\leq n} F(k)$. We have to prove $M \leq 0$. Suppose $M > 0$, and let $i \in \{0, \ldots, n\}$ be the smallest integer with $M = F(i)$. Since $F(0) = F(n) = 0$

we have $0 < i < n$. By the convexity of f we have $2M = 2F(i) \le F(i-1) + F(i+1) \le F(i-1) + M$, and thus $F(i-1) \ge M$. Then $F(i-1) = M$, which contradicts the minimality of i. $\square$

Lemma 1.3. *Let A be a subset of an Abelian group G, and let $f : A \to \mathbb{R}$ be a convex function. If $a \in G$ is of finite order, then $f(x+a) = f(x)$ for every $x \in G$ with $x + \mathbb{Z}a \subset A$.*

Proof. Clearly, it is enough to show $f(x+a) \le f(x)$. Take any $n \in \mathbb{N}$ such that $na = 0$. Applying Lemma 1.2 we obtain

$$nf(x+a) \le (n-1)f(x) + f(x+na) = nf(x),$$

which completes the proof. $\square$

Proof of Theorem 1.1. Let k, n be coprime integers with $\lambda = k/n$ and $nu = kx + (n-k)y$. Then we have $n(y-u) = k(y-x)$. We prove that there is an element $h \in G$ such that $y - x = nh$. Indeed, as k, n are coprime, there are integers a and b such that $ak + bn = 1$. Putting $h = a(y-u) + b(y-x)$ we obtain

$$nh = an(y-u) + bn(y-x) = ak(y-x) + bn(y-x) = (ak+bn)(y-x) = y - x$$

as we stated.

Then we have $n(x + (n-k)h) = nx + (n-k)(y-x) = nu$, and thus, by Lemma 1.3, we have $f(x + (n-k)h) = f(u)$. Therefore, by Lemma 1.2 we obtain

$$\begin{aligned} nf(u) &= nf(x + (n-k)h) \\ &\le kf(x) + (n-k)f(x+nh) \\ &= kf(x) + (n-k)f(y) \\ &= n\lambda f(x) + n(1-\lambda)f(y), \end{aligned}$$

which completes the proof. $\square$

If G is a torsion free Abelian group, then for every $x, y \in G$ and $\lambda \in \mathbb{Q}$ there is at most one $u \in G$ such that $u \sim \lambda x + (1-\lambda)y$. If there is such a u then we say that $\lambda x + (1-\lambda)y$ *exists in* G and write $\lambda x + (1-\lambda)y = u$. The next theorem is an immediate corollary of Theorem 1.1.

Theorem 1.4. *Suppose that G is a torsion free Abelian group, and let $f : G \to \mathbb{R}$ be convex. Then*

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) \tag{5}$$

holds for every $\lambda \in \mathbb{Q} \cap [0,1]$ and for every $x, y \in G$ such that $\lambda x + (1-\lambda)y$ exists in G. $\square$

Let A be a subset of an Abelian group. We say that $f : A \to \mathbb{R}$ *has property C_n* if

$$nf(x) \le f(x_1) + \ldots + f(x_n) \tag{6}$$

holds whenever $x, x_1, \ldots, x_n \in A$ and $nx = x_1 + \ldots + x_n$. Thus f is convex if and only if it satisfies C_2. It is easy to see that C_{n+1} implies C_n for every $n \in \mathbb{N}$. Indeed, if

$x, x_1, \ldots, x_n \in A$ and $nx = x_1 + \ldots + x_n$, then $(n+1)x = x_1 + \ldots + x_n + x$. Now C_{n+1} implies

$$(n+1)f(x) \leq f(x_1) + \ldots + f(x_n) + f(x),$$

which proves (6). In particular, if f satisfies C_n for some $n \geq 2$, then it is convex. In the next theorem we shall prove the converse under suitable conditions on the set A.

A subset A of an Abelian group G is said to be *convex* if $x + h, x - h \in A$ implies $x \in A$ for every $x, h \in G$. Note that if G is divisible by 2 and $A \subset G$ is convex, then $A + A = 2A$. If G is uniquely divisible by 2, then the convexity of A is equivalent to $A + A = 2A$. In general, the condition of convexity of the set A is independent of the condition $A + A = 2A$ (cf. Examples 1.7 and 1.8).

Theorem 1.5. *Let A be a convex subset of an Abelian group G such that $A + A = 2A$. Then every convex function defined on A satisfies C_n for every $n \in \mathbb{N}$.*

Proof. First we prove that, for every $x, h \in G$, if $x \in A$ and $x + 2^k h \in A$, then $x + ih \in A$ for every $i = 0, \ldots, 2^k$. Let $I = \{i : 0 \leq i \leq 2^k \text{ and } x + ih \in A\}$. Then $0, 2^k \in I$ by assumption. We show that if $a, b \in I$ and $c = (a+b)/2$ is an integer, then $c \in I$. Indeed, since $(x + ch) + (a - c)h = x + ah \in A$ and $(x + ch) - (a - c)h = x + bh \in A$, the convexity of A implies that $x + ch \in A$ and $c \in I$. Now it is easy to check that, whenever $I \subset \{0, \ldots, 2^k\}$ is such that $0,\ 2^k \in I$ and the average of any two elements of I of the same parity is also an element of I, then $I = \{0, \ldots, 2^k\}$. Thus $x + ih \in A$ for every $i = 0, \ldots, 2^k$.

Now let $f : A \to \mathbb{R}$ be a convex function. We prove that if $x,\ y,\ z \in A$ and $2^{k+1}x = 2^k(y + z)$ for some $k \in \mathbb{N}$, then $2f(x) \leq f(y) + f(z)$. Indeed, let $a = 2x - (y + z)$. Then $2^k a = 0$. Since $z = z + 2^k a \in A$, it follows that $z + ia \in A$ for every i. Therefore, $f(z + a) = f(z)$ by Lemma 1.3. Since $2x = y + (z + a)$, we have

$$2f(x) \leq f(y) + f(z + a) = f(y) + f(z)$$

by the convexity of f. Now we prove by induction on k that C_{2^k} holds for every $k \in \mathbb{N}$. Assume C_{2^k} for a $k \in \mathbb{N}$ and let $x,\ x_1, \ldots, x_{2^{k+1}} \in A$ be such that $2^{k+1}x = x_1 + \ldots + x_{2^{k+1}}$. Since $A + A = 2A$, there are $y,\ z \in A$ such that $2^k y = x_1 + \ldots + x_{2^k}$ and $2^k z = x_{2^k+1} + \ldots + x_{2^{k+1}}$. Then $2^{k+1}x = 2^k(y + z)$ and thus, as we proved above, $2f(x) \leq f(y) + f(z)$. Therefore, applying the induction hypothesis, we obtain

$$\begin{aligned} 2^{k+1} f(x) &\leq 2^k f(y) + 2^k f(z) \\ &\leq f(x_1) + \ldots + f(x_{2^k}) + f(x_{2^k+1}) + \ldots + f(x_{2^{k+1}}). \end{aligned}$$

This means that f satisfies $C_{2^{k+1}}$. We have proved that f satisfies C_{2^k} for every k. Since C_{n+1} implies C_n for every n, it follows that f satisfies C_n for every n. □

Corollary 1.6. *Let A be a convex subset of an Abelian group divisible by 2. Then every convex function defined on A satisfies C_n for every $n \in \mathbb{N}$.* □

The examples below show that the assumptions imposed on A in Theorem 1.5 are essential.

Example 1.7. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denote the circle group. We identify $\mathbb{T}$ with $[0, 1)$, where addition is meant modulo 1. Let $A = [0, 1/3]$. Then A is a subset of the group $\mathbb{T}$ such that $A + A = 2A$. Let $f(x) = x$ for every $x \in A$. Then $f : A \to \mathbb{R}$ is convex, but does not have property C_3. Indeed, for $a = 1/3$ we have $3a = 0 = 0 + 0 + 0$, but $3f(a) = 1 > f(0) + f(0) + f(0) = 0$. (Note that A is not convex, as $(2/3) \pm (1/3) \in A$ but $2/3 \notin A$.)

Example 1.8. Let $G = \mathbb{Z}^2$ and $A = G$. Then A is a convex subset of G. (However, it violates $A + A = 2A$.)

We construct a convex function on $\mathbb{Z}^2$ which does not satisfy C_3. Let $f : \{(0,0), (1,1)\ (-1,0), (0,-1)\} \to \mathbb{R}$ be arbitrary. We prove that f can be extended to $\mathbb{Z}^2$ as a convex function. We may assume that f takes its values in $[0,1]$. Let $x_1, x_2, \dots$ be an enumeration of $\mathbb{Z}^2$ such that $|x_1| \le |x_2| \le \dots$. First we extend f to the set $\{x_1, \dots, x_9\} = \{-1, 0, 1\} \times \{-1, 0, 1\}$ as follows: we define $f(0,1) = f(1,0) = f(-1,-1) = 2$ and $f(-1,1) = f(1,-1) = 4$. It is easy to see that f is convex on $\{x_1, \dots, x_9\}$. Then we define $f(x_n) = 2^n$ for every $n \ge 10$. We claim that f is convex on $\mathbb{Z}^2$. Let $x, h \in \mathbb{Z}^2$; we prove (1). We may assume $h \ne 0$. Let $x = x_i$, $x + h = x_j$ and $x - h = x_k$. Since $\max\{|x+h|, |x-h|\} > |x|$, we have $\max\{j,k\} > i$. By symmetry we may assume $j \le k$ and thus $i < k$. If $k \le 9$ then (1) follows from the fact that f is convex on $\{x_1, \dots, x_9\}$. If $k \ge 10$ then

$$\begin{aligned} 2f(x) = 2f(x_i) &\le \max\{8, 2 \cdot 2^{k-1}\} = 2^k = f(x_k) \\ &= f(x-h) \le f(x+h) + f(x-h), \end{aligned}$$

which again gives (1).

Now take f such that $f(0,0) = 1$ and $f(1,1) = f(-1,0) = f(0,-1) = 0$. Then f violates C_3, since $3(0,0) = (1,1) + (-1,0) + (0,-1)$.

Corollary 1.9. *Let A be a convex subset of an Abelian group G, and suppose that $A + A = 2A$. If $f : A \to \mathbb{R}$ is a convex function, then inequality* (3) *holds for every $x, y, u \in A$ and $\lambda \in \mathbb{Q} \cap [0,1]$ satisfying $u \sim \lambda x + (1-\lambda)y$.*

Proof. Let k, n be coprime integers with $\lambda = k/n$ and $nu = kx + (n-k)y$. By Theorem 1.5 the function f satisfies C_n, so $nf(u) \le f(x) + (n-k)f(y)$, which gives (3). □

Corollary 1.10. *Let A be a convex subset of a linear space over the rationals, and let $f : A \to \mathbb{R}$ be a convex function. Then inequality* (2) *holds for every $x, y \in A$ and $\lambda \in \mathbb{Q} \cap [0,1]$ such that $\lambda x + (1-\lambda)y \in A$.* □

Corollary 1.11. *Let A be a convex subset of a linear topological space and let $f : A \to \mathbb{R}$ be a continuous convex function. Then inequality* (2) *holds for every $x, y \in A$ and $\lambda \in [0,1]$ such that $\lambda x + (1-\lambda)y \in A$.*

Proof. Let $x, y \in A$ and $\lambda \in [0,1]$ be arbitrary. Since the set $\mathbb{D} = \{k/2^n : n \in \mathbb{N}, k = 0, \dots, 2^n\}$ is dense in $[0,1]$, there is a sequence $(\lambda_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{D}$ converging to λ. As A is convex, we have $\lambda_n x + (1-\lambda_n)y \in A$ for every $n \in \mathbb{N}$. To complete the proof it is enough to make use of Corollary 1.10 and the continuity of f. □

2. Extension of convex functions

We start with another consequence of Theorem 1.5.

Corollary 2.1. *Let A be a convex subset of an Abelian group G such that $A+A=2A$. In order that a real function f, defined on a subset of A, be extendible to A as a convex function, it is necessary that f satisfies condition C_n for every $n \in \mathbb{N}$.* $\square$

In Example 1.8 we defined a function $f : H \to \mathbb{R}$, where $H = \{(0,0), (1,1), (-1,0), (0,-1)\}$, which admits a convex extension to $\mathbb{Z}^2$ but does not satisfy C_3. This shows that the assumption $A+A=2A$ is essential in Corollary 2.1.

Theorem 2.2. *There exists a convex function defined on a subgroup of the reals which cannot be extended to $\mathbb{R}$ as a convex function.*

Proof. Let a, $b \in \mathbb{R}$ be linearly independent over the rationals. Then the group H generated by a and b is isomorphic to $\mathbb{Z}^2$. Making use of the function constructed in Example 1.8 we come to a convex function $f : H \to \mathbb{R}$ which does not satisfy C_3. Since $\mathbb{R}$ is divisible, it satisfies $\mathbb{R} + \mathbb{R} = 2\mathbb{R}$, and then it follows from Corollary 2.1 that f cannot be extended to $\mathbb{R}$ as a convex function. $\square$

Our main aim in this section is to prove the following extension theorem.

Theorem 2.3. *Let H be a subgroup of an Abelian group G. If $f : H \to \mathbb{R}$ satisfies C_n for every $n \in \mathbb{N}$, then f can be extended to G as a function satisfying C_n for every $n \in \mathbb{N}$.*

Theorem 2.2 shows that in Theorem 2.3 it is not sufficient to assume that f is convex. Also, the assumption that H is a subgroup of G is essential, as the following simple example shows. Let $-\infty < a < b < \infty$ and let $f : (a,b) \to \mathbb{R}$ be a convex function such that $f(a+) = f(b-) = \infty$. Put $A = (a,b)$. By Theorem 1.5, the function $f_{|A}$ satisfies C_n for every $n \in \mathbb{N}$. On the other hand it does not admit a convex extension to $\mathbb{R}$.

As an immediate consequence of Theorems 1.5 and 2.3 we obtain the following.

Corollary 2.4. *Let H be a subgroup of the Abelian group G. If H is divisible by 2, then every convex function defined on H can be extended to G as a convex function.* $\square$

Proof of Theorem 2.3. *I.* First we shall assume that $G = \mathbb{R}^k$ and $H = \mathbb{Z}^k$ for some $k \in \mathbb{N}$. Assume that $f : \mathbb{Z}^k \to \mathbb{R}$ satisfies C_n for every $n \in \mathbb{N}$. Let A denote the convex hull of the graph of f, and put $\varphi(x) = \inf\{y \in \mathbb{R} : (x,y) \in A\}$ for every $x \in \mathbb{R}^k$. We prove that φ is an extension of f and satisfies C_n for every $n \in \mathbb{N}$.

Let $x \in \mathbb{Z}^k$. It is clear that $\varphi(x) \le f(x)$. Suppose $\varphi(x) < f(x)$. Then there is a $y \in \mathbb{R}$ such that $\varphi(x) < y < f(x)$ and $(x,y) \in A$. Consequently, there are points $x_1, \dots, x_m \in \mathbb{Z}^k$ and positive numbers $t_1, \dots, t_m$ such that $\sum_{i=1}^m t_i = 1$, $\sum_{i=1}^m t_i x_i = x$, and $\sum_{i=1}^m t_i f(x_i) = y$. Slightly changing the numbers t_i and y we may assume that $t_1, \dots, t_m \in \mathbb{Q}$. If n is the common denominator of $t_1, \dots, t_m$, then it follows from condition C_n that $f(x) \le \sum_{i=1}^m t_i f(x_i) = y$, which contradicts $y < f(x)$. This proves that φ is an extension of f. It is clear from the definition of φ that it is a convex function. Therefore, since $\mathbb{R}^n$ is divisible, φ satisfies C_n for every $n \in \mathbb{N}$.

II. The element $x \in G$ is said to be *independent of the subgroup H* of G if $px \notin H$ for every $p \in \mathbb{N}$. We prove that if $f : H \to \mathbb{R}$ satisfies C_n for every $n \in \mathbb{N}$ and if $x \in G$ is independent of H, then f can be extended with the same property to the group $\langle H, x\rangle$ generated by H and x. Indeed, every element of $\langle H, x\rangle$ has a unique representation $h+qx$, where $h \in H$ and $q \in \mathbb{Z}$. We define $f(h+qx) = f(h)$ for every $h \in H$ and $q \in \mathbb{Z}$. It is easy to check that this gives a suitable extension of f to $\langle H, x\rangle$.

III. Let H be a finitely generated subgroup of G and let $x \in G \setminus H$. We prove that if $f : H \to \mathbb{R}$ satisfies C_n for every $n \in \mathbb{N}$, then f can be extended with the same property to the group $\langle H, x\rangle$.

If x is independent of H, then this was proved in *II*. Therefore, we may assume that $px \in H$ for some $p \in \mathbb{N}$. Let s be the smallest positive integer with this property and let $c = sx$. Then every element of $\langle H, x\rangle$ has a unique representation $h + qx$, where $h \in H$ and $0 \le q < s$. By the fundamental theorem of finitely generated Abelian groups [10, Theorem 10.26]) there exists a nonnegative integer k such that H is isomorphic to the direct product of $\mathbb{Z}^k$ with a finite group. Thus there are subgroups K, F of H such that K is isomorphic to $\mathbb{Z}^k$, F is finite, and every element of H can be written uniquely as $y + z$, where $y \in K$ and $z \in F$. Choose $a \in K$ and $b \in F$ such that $c = a + b$.

Let $i : \mathbb{Z}^k \to K$ be an isomorphism. Since f satisfies C_n for every $n \in \mathbb{N}$, the function $f \circ i : \mathbb{Z}^k \to \mathbb{R}$ has the same property. Therefore, by *I*, $f \circ i$ can be extended to $\mathbb{R}^k$ satisfying C_n for every $n \in \mathbb{N}$. Let φ be such an extension. Then we put

$$g(y+z+qx) = \varphi\left(i^{-1}(y) + \frac{q}{s} \cdot i^{-1}(a)\right)$$

for every $y \in K,\ z \in F$ and $q \in \mathbb{Z}$. We claim that the equality above defines a function g on $\langle H, x\rangle$. Let $y_1 + z_1 + q_1 x = y_2 + z_2 + q_2 x$, where $y_j \in K,\ z_j \in F$ and $q_j \in \mathbb{Z}$ for $j = 1, 2$. Then $(q_2 - q_1)x \in H$ and thus $q_2 = q_1 + ms$ with a suitable integer m. Therefore

$$y_1 + z_1 = y_2 + z_2 + msx = y_2 + z_2 + mc = (y_2 + ma) + (z_2 + mb),$$

and hence $y_1 = y_2 + ma$. Then we have

$$\begin{aligned} i^{-1}(y_1) + \frac{q_1}{s} \cdot i^{-1}(a) &= i^{-1}(y_2) + m \cdot i^{-1}(a) + \frac{q_1}{s} \cdot i^{-1}(a) \\ &= i^{-1}(y_2) + \frac{q_1 + ms}{s} \cdot i^{-1}(a) = i^{-1}(y_2) + \frac{q_2}{s} \cdot i^{-1}(a), \end{aligned}$$

which was claimed. If $y \in K$ and $z \in F$ then

$$g(y+z) = \varphi\left(i^{-1}(y)\right) = f(y) = f(y+z)$$

by Lemma 1.3. Therefore g is an extension of f. It is easy to check that g has property C_n for every $n \in \mathbb{N}$.

IV. Let H be an arbitrary subgroup of G and assume that $f : H \to \mathbb{R}$ satisfies C_n for every $n \in \mathbb{N}$. We prove that for every $x \in G \setminus H$ we can extend f to the group $\langle H, x\rangle$ as a function satisfying C_n for every $n \in \mathbb{N}$. If x is independent of H, then we already proved this in *II*. Therefore, we may assume that $px \in H$ for some $p \in \mathbb{N}$. Let s be the smallest positive integer with $sx \in H$, and let $a = sx$.

Let F be a finitely generated subgroup of H containing a. Then, by *III*, the restriction of f to F has an extension g_F to the group $\langle F, x\rangle$ satisfying C_n for every $n \in \mathbb{N}$. Consider the family $\mathcal{G}$ of all functions g_F, where F is an arbitrary finitely generated subgroup of H containing a. We prove that the family $\mathcal{G}$ has the following properties:

(i) for every finite subset A of $\langle H, x\rangle$ there is an element of $\mathcal{G}$ which is defined at each element of A;

(ii) for every $y \in \langle H, x\rangle$ there exists a compact interval I_y such that $g(y) \in I_y$ whenever a function $g \in \mathcal{G}$ is defined at y.

Indeed, if $A = \{h_1 + m_1x, \dots, h_p + m_px\}$ for some $h_1, \dots, h_p \in H$ and $m_1, \dots, m_p \in \mathbb{Z}$, then g_F will be defined on A for $F = \langle h_1, \dots, h_p, a\rangle$. This proves (i).

In order to prove (ii) let $y = h+mx$ for some $h \in H$ and $m \in \{0, \dots, s-1\}$, and suppose that g_F is defined at y. Then $y \in \langle F, x\rangle$, $h = y-mx \in \langle F, x\rangle$, and thus $h \in H\cap\langle F, x\rangle = F$. This implies $h + a, h + 2a \in F$.

Since $sy = sh + ma = (s - m)h + m(h + a)$, the C_s property of g_F gives

$$sg_F(y) \le (s - m)g_F(h) + mg_F(h + a) = (s - m)f(h) + mf(h + a), \tag{7}$$

since g_F is an extension of $f|_F$. Also,

$$(2s - m)(h + a) = sh + ma + (s - m)(h + 2a) = sy + (s - m)(h + 2a).$$

Then, by the property C_{2s-m} of g_F, we have

$$\begin{aligned}(2s - m)f(h + a) &= (2s - m)g_F(h + a) \le sg_F(y) + (s - m)g_F(h + 2a)\\ &= sg_F(y) + (s - m)f(h + 2a).\end{aligned} \tag{8}$$

Therefore, if we put

$$I_y = \left[\frac{(2s - m)f(h + a) - (s - m)f(h + 2a)}{s}, \frac{(s - m)f(h) + mf(h + a)}{s}\right],$$

then, by (7) and (8), we have $g_F(y) \in I_y$ whenever $g_F \in \mathcal{G}$ is defined at y.

Now we prove, using the compactness of the product $T = \prod_{y\in\langle H,x\rangle} I_y$, that there exists a function $t : \langle H, x\rangle \to \mathbb{R}$ with the following property: for every finite subset A of $\langle H, x\rangle$ and for every $\varepsilon > 0$ there exists an element g of $\mathcal{G}$ such that g is defined on A, and $|t(x) - g(x)| < \varepsilon$ for every $x \in A$. Suppose there is no such t. Then for every $t \in T$ there are an $\varepsilon_t > 0$ and a finite set $A_t \subset \langle H, x\rangle$ such that $\max\{|t(x) - g(x)| : x \in A_t\} \ge \varepsilon_t$ whenever $g \in \mathcal{G}$ is defined on A_t.

The set $U_t(A_t, \varepsilon_t) = \{\gamma \in T : |\gamma(x) - t(x)| < \varepsilon_t$ for every $x \in A_t\}$ is a neighbourhood of t in T. Therefore, by the compactness of T, there are functions $t_1, \dots, t_N \in T$ such that $\bigcup_{i=1}^N U_{t_i}(A_{t_i}, \varepsilon_{t_i}) = T$. Since the set $A = \bigcup_{i=1}^N A_{t_i}$ is finite, it follows from (i) that there is a $g \in \mathcal{G}$ that is defined on A. Then $g(x) \in I_y$ for every $x \in A$, so there is an extension $t \in T$ of g to $\langle H, x\rangle$. Then $t \in U_{t_i}(A_{t_i}, \varepsilon_{t_i})$ for some $i = 1, \dots, N$. But then $|g(x) - t_i(x)| < \varepsilon_{t_i}$ for every $x \in A_{t_i}$, which contradicts the definition of A_{t_i} and ε_{t_i}. This proves that there is a function $t : \langle H, x\rangle \to \mathbb{R}$ with the properties described above.

To prove that t extends f, fix any $y \in H$ and $\varepsilon > 0$. Put $A = \{y\}$. Since $g_F(y) = f(y)$ whenever $g_F \in \mathcal{G}$ is defined on A, we have $|t(y) - f(y)| < \varepsilon$. As ε was an arbitrary positive number, we obtain $t(y) = f(y)$. In a similar way one can prove that t satisfies property C_n for every $n \in \mathbb{N}$.

V. Now we turn to the proof of the theorem. Let H be an arbitrary subgroup of G and assume that $f : H \to \mathbb{R}$ satisfies C_n for every $n \in \mathbb{N}$. By Zorn's lemma there exists a maximal subgroup M of G such that $H \subset M$ and f can be extended to M as a function satisfying C_n for every $n \in \mathbb{N}$. By IV we must have $M = G$, which completes the proof. □

3. Convex functions on Abelian topological groups

By a well-known theorem of F. Bernstein and G. Doetsch, if D is a convex open subset of a normed linear space, $f : D \to \mathbb{R}$ is convex and f is locally bounded from above at a point of D, then f is continuous (see [5, Chap. VI, Sec. 4, Theorem 2] or [9, Chap. VII, Sec. 71, Theorem C and Chap. IV, Sec. 41, Theorem C], also [1]). In this section our aim is to find the possible generalizations of this theorem to topological Abelian groups.

The statement of the Bernstein-Doetsch theorem can be split as follows.

(BD_1) *If D is a convex open subset of a normed linear space, $f : D \to \mathbb{R}$ is a convex function, bounded from above, then f is continuous.*

(BD_2) *If D is a convex open subset of a normed linear space, $f : D \to \mathbb{R}$ is a convex function, locally bounded from above at a point of D, then f is locally bounded from above at each point of D.*

We shall prove that (BD_1) is valid in every topological Abelian group. Moreover, the convexity of D plays no part in the statement, and even the local uniform continuity of f follows. In addition, in *metric groups*, that is, in groups endowed with an invariant metric, the local Lipschitz property of f can be proved.

Theorem 3.1. *Let U be an open subset of an Abelian topological [metric] group G, and let $f : U \to \mathbb{R}$ be a convex function. If f is bounded from above, then it is locally uniformly continuous [locally Lipschitz] in U.*

Lemma 3.2. *Let A be a subset of an Abelian topological group G, and let $f : A \to \mathbb{R}$ be a convex function. Assume that f is bounded from above on the set $x_0 + V \subset A$, where $x_0 \in A$ and $V \subset G$ is a symmetric neighbourhood of 0. Then f is bounded on $x_0 + V$.*

Proof. We may assume $x_0 = 0$. Suppose that $f(x) \le M$ for every $x \in V$. If $x \in V$ then $-x \in V$, whence, by the convexity of f, $2f(0) \le f(x) + f(-x) \le f(x) + M$. Thus $|f(x)| \le K$ for every $x \in V$, where $K = \max\{|M|, |2f(0) - M|\}$. □

Lemma 3.3. *Let A be a subset of an Abelian topological group G, and let $f : A \to \mathbb{R}$ be a convex function. Assume that f is bounded from above on the set $x_0 + V + V \subset A$, where $x_0 \in A$ and $V \subset G$ is a symmetric neighbourhood of 0. Then f is uniformly continuous on $x_0 + V$.*

Proof. We may assume $x_0 = 0$. By Lemma 3.2, there is a real number K such that $|f(x)| \le K$ for every $x \in V + V$. Let $\varepsilon > 0$ be given, choose a positive integer $n > 2K/\varepsilon$, and let $W \subset G$ be a symmetric neighbourhood of 0 such that the n-fold sum $W + \ldots + W$ is a subset of V. We prove that if $x, y \in V$ and $y - x \in W$, then $|f(y) - f(x)| < \varepsilon$. Let $h = y - x$. Then $ih \in V$ and $x + ih \in V + V$ for every $i = 0, \ldots, n$. In particular, $x, x + nh \in V + V$, and thus $|f(x)| \le K$, $|f(x + nh)| \le K$. It follows from Lemma 1.2

that

$$nf(y) = nf(x+h) \le (n-1)f(x) + f(x+nh) \le (n-1)f(x) + K \le nf(x) + 2K,$$

and $f(y) - f(x) \le 2K/n < \varepsilon$. Exchanging the roles of x and y we obtain $|f(y) - f(x)| < \varepsilon$. Thus f is uniformly continuous on V. □

Proof of Theorem 3.1. For every $x_0 \in U$ there is a symmetric neighbourhood V of 0 such that $x_0 + V + V \subset U$, so the local uniform continuity of f follows from Lemma 3.3.

Next let G be a metric group with an invariant metric d, and denote by $B(x_0, r)$ the open ball centred at $x_0 \in G$ with radius r. We prove that f is Lipschitz in a neighbourhood of an arbitrary $x_0 \in U$. We may assume $x_0 = 0$. Choose a positive r such that $B(0, 2r) \subset U$. By Lemma 3.2, there is a real number K such that $|f(x)| \le K$ for every $x \in B(0, 2r)$. It is enough to prove that

$$|f(x) - f(y)| \le \frac{4K}{r}\, d(x, y) \tag{9}$$

holds for every $x, y \in B(0, r)$. This is clear if $x = y$ or $d(x, y) \ge r$, therefore we may assume that $0 < d(x, y) < r$. Then there is a nonnegative integer n such that

$$\frac{r}{2^{n+1}} \le d(x, y) < \frac{r}{2^n}.$$

Put $h = y - x$. Then, for every $m = 0, \dots, 2^n$ we have

$$\begin{aligned} d(x, x+mh) &\le \sum_{i=1}^{m} d(x + (i-1)h, x + ih) = m \cdot d(x, x+h) \\ &= m \cdot d(x, y) \le 2^n d(x, y) < r. \end{aligned}$$

Thus $d(0, x+mh) < d(0, x) + r < 2r$, $x + mh \in B(0, 2r)$ and $|f(x+mh)| \le K$ for every $m = 0, \dots, 2^n$. Then it follows from Lemma 1.2 that

$$2^n f(y) = 2^n f(x+h) \le (2^n - 1) f(x) + f(x + 2^n h) \le 2^n f(x) + 2K.$$

Thus

$$f(y) - f(x) \le \frac{2K}{2^n} = \frac{4K}{r} \frac{r}{2^{n+1}} \le \frac{4K}{r} d(x, y).$$

Exchanging the roles of x and y we obtain (9), which completes the proof. □

Now we consider the possible generalization of the statement (BD_2). As we shall see in Theorem 3.6, the statement is valid in every Abelian group G in which every convex neighbourhood $U \subset G$ of 0 is *absorbing*; i.e., for every $x \in G$ there is an $n \in \mathbb{N}$ such that $x \in 2^n U$. First we show that this condition cannot be omitted.

Example 3.4. Let $G = \mathbb{R} \times \mathbb{Z}$ and put $d((x_1, n_1), (x_2, n_2)) = |x_1 - x_2| + |n_1 - n_2|$ for every $(x_1, n_1), (x_2, n_2) \in G$. It is clear that d is an invariant metric on G, and the topology induced by d is the product topology when the factors $\mathbb{R}$ and $\mathbb{Z}$ are equipped with the Euclidean topology and the discrete topology, respectively.

Then $D = \mathbb{R} \times \{0, 1\}$ is convex and open. Let $a : \mathbb{R} \to \mathbb{R}$ be a non-continuous additive function, and let $f : D \to \mathbb{R}$ be given by $f(x, n) = n \cdot a(x)$. Then f is convex. Indeed, if $(x,n) \in D$ and $(x+h,n+k) \in D$, $(x-h,n-k) \in D$, then k must be zero, and $2f(x,n) = 2n \cdot a(x) = 2n \cdot a(x+h) + 2n \cdot a(x-h) = f(x+h,n) + f(x-h,n)$. The set $U = \mathbb{R} \times \{0\}$ is an open subset of D, and f vanishes on U. Still, f is not locally bounded at any point of $\mathbb{R} \times \{1\}$, as $f(x,1) = a(x)$ for every $x \in \mathbb{R}$.

Example 3.5. Let d_0 be the discrete metric on $\mathbb{R}$, that is $d_0(x,y) = 1$ if x, y are distinct real numbers and $d_0(x,x) = 0$ for every $x \in \mathbb{R}$. Let $G = \mathbb{R} \times \mathbb{R}$. Put $d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + d_0(y_1,y_2)$ for every $(x_1,y_1),(x_2,y_2) \in G$. It is clear that d is a metric on G, and the topology induced by d is the product topology when the first $\mathbb{R}$ factor is equipped with the Euclidean topology and the second $\mathbb{R}$ factor is equipped with the discrete topology. Note that G is divisible.

Then $D = \mathbb{R} \times [0,1]$ is convex and open. Let $a : \mathbb{R} \to \mathbb{R}$ be a non-continuous additive function, and put $f(x,y) = 0$ for every $x \in \mathbb{R}$, $y \in [0,1)$ and $f(x,1) = a(x)^2$ for every $x \in \mathbb{R}$. Then f is convex on D. Indeed, let (x,y), $(x+h,y+k)$, $(x-h,y-k) \in D$; we prove

$$2f(x,y) \leq f(x+h,y+k) + f(x-h,y-k). \tag{10}$$

If $y < 1$ then $f(x,y) = 0$ and then (10) is obvious. If $y = 1$ then k must be zero, and then (10) follows from $2a(x)^2 \leq a(x+h)^2 + a(x-h)^2$.

Now $U = \mathbb{R} \times \{0\}$ is an open subset of D and f vanishes on U. Still, f is not locally bounded at any point of $\mathbb{R} \times \{1\}$, as $f(x,1) = a(x)^2$ for every $x \in \mathbb{R}$.

In the next theorem we shall prove that, in every topological Abelian group, both (BD_2) and the statement of the Bernstein-Doetsch theorem are true if, among others, $D = G$ or D is connected. Note that statement (*iii*) below generalizes the Bernstein-Doetsch theorem even in the case $G = \mathbb{R}^n$, as our notion of convexity of sets is more general than the classical one. Our statement (*iii*) is closely related to [3, Theorem 1], where a similar result is proved without the assumption of the commutativity of G. However, in [3] it is assumed that G is "root-approximable", which is slightly more restrictive than our condition, and the local uniform continuity of f is not stated.

Theorem 3.6. *Let D be an open subset of an Abelian topological [metric] group G, and assume that D and G satisfy one of the following conditions.*

(*i*) $D = G$;

(*ii*) *D is connected;*

(*iii*) *D is convex, and every convex neighbourhood of 0 is absorbing.*

If a convex function $f : D \to \mathbb{R}$ is locally bounded from above at a point, then f is locally uniformly continuous [locally Lipschitz] in D.

Lemma 3.7. *Let G be an Abelian topological group G and let $f : G \to \mathbb{R}$ be a convex function. If f is locally bounded from above at a point, then there is a neighbourhood V of 0 such that f is bounded from above on $x + V$ for every $x \in G$.*

Proof. Let $f(x) \leq M$ for every $x \in U$, where U is a nonempty open subset of D. We may assume $0 \in U$. Let V be a symmetric neighbourhood of 0 such that $2V \subset U$. If $x \in G$ and $v \in V$, then $2v \in U$ and, by the convexity of f, we have $2f(x+v) \leq f(2x) + f(2v) \leq f(2x) + M$. Thus f is bounded from above on $x + V$. $\square$

Proof of Theorem 3.6. By Theorem 3.1, it is enough to show that f is locally bounded from above at each point of D.

(i) Apply Lemma 3.7.

(ii) Let $E = \{x \in D : f$ is locally bounded from above at $x\}$. Then E is an open subset of D. In order to prove $E = D$, it is enough to show that E is relatively closed in D. Let $x_0 \in (\operatorname{cl} E) \cap D$ be arbitrary; we prove $x_0 \in E$.

Let V be a symmetric neighbourhood of zero such that $x_0 + V \subset D$. Since $x_0 \in \operatorname{cl} E$, it follows that $(x_0 + V) \cap E \neq \emptyset$. Let $h_0 \in V$ be such that $x_0 + h_0 \in E$. Then $x_0 - h_0 \in x_0 + V \subset D$ by the symmetry of V.

Since $x_0 + h_0 \in E$, there are a neighbourhood W of 0 and a real number K such that $x_0 + h_0 + W \subset D$ and $f(x_0 + h_0 + x) \leq K$ for every $x \in W$. Now let Z be a symmetric neighbourhood of 0 such that $Z \subset V$ and $2Z \subset W$. If $h \in Z$ then $2h \in W$, $f(x_0 + h_0 + 2h) \leq K$, and

$$2f(x_0 + h) \leq f(x_0 - h_0) + f(x_0 + h_0 + 2h) \leq f(x_0 - h_0) + K$$

by the convexity of f. Thus f is bounded from above in $x_0 + Z$, which proves $x_0 \in E$.

(iii) We may assume that f is bounded from above in a neighbourhood $U \subset D$ of 0. Assume that $f(x) \leq M$ for every $x \in U$. Let $x_0 \in D$ be arbitrary. Since $D - x_0$ is a convex neighbourhood of 0, it is absorbing, and thus there is an $n \in \mathbb{N}$ such that $x_0 \in 2^n(D - x_0)$. Select an $a \in D$ satisfying $x_0 = 2^n(a - x_0)$, and a neighbourhood V of 0 such that $x_0 + V \subset D$ and $(2^n + 1)V \subset U$. We prove that f is bounded from above on $x_0 + V$. Let $x \in V$ be arbitrary. Then the point $b = (2^n + 1)x$ is in U, whence $f(b) \leq M$. Put $h = x - a + x_0$. Then $(2^n + 1)h = (2^n + 1)x - (2^n + 1)(a - x_0) = b - a$. As $a + h = x_0 + x \in D$ and $b \in D$, it follows from the convexity of D that $a + kh \in D$ for every $k = 0, \dots, 2^n + 1$. Therefore, by Lemma 1.2,

$$(2^n + 1)f(a + h) \leq 2^n f(a) + f(a + (2^n + 1)h),$$

that is,

$$(2^n + 1)f(x_0 + x) \leq 2^n f(a) + f(b) \leq 2^n f(a) + M.$$

Thus $f(x_0 + x) \leq (2^n f(a) + M)/(2^n + 1)$ for every $x \in V$. □

As an application of Theorem 3.6 we obtain the following result about extensions of convex functions.

Corollary 3.8. *Let H be a dense subgroup of an Abelian topological group G, and let $f : H \to \mathbb{R}$ be a convex function. If f is locally bounded from above at a point, then f can be extended to G as a continuous convex function.*

Proof. By Lemmas 3.7 and 3.3 there exists a relative neighbourhood U of 0 in H such that f is uniformly continuous in $x + U$ for every $x \in H$. Let V be a neighbourhood of zero in G such that $U = V \cap H$, and choose a neighbourhood W of zero in G such that $2W \subset V$. Let $x \in G$ be arbitrary. Then f is uniformly continuous in $(x + W) \cap H$. Indeed, as H is dense in G, there is an element $y \in (x - W) \cap H$. Thus $(x + W) \cap H \subset (y + 2W) \cap H \subset (y + V) \cap H = y + U$, and f is uniformly continuous in $y + U$ by the choice of U. This implies that the limit $\lim_{t \to x} f(t)$ exists. (In fact, the limit equals c, where

$\{c\} = \bigcap_Z \operatorname{cl} f(Z \cap H)$ and Z runs through all neighbourhoods of x.) Now, if we define $f(x)$ by this limit, then we obtain an extension having the properties required. $\square$

4. Convex functions on locally compact Abelian groups

A well-known theorem of Blumberg and Sierpiński states that every Lebesgue measurable convex function, defined on an open convex subset of $\mathbb{R}^k$, is continuous (see [2] and [11], also [5, Chap. IX, Sec. 4, Theorem 2]). The same is true if we replace Lebesgue measurability by the Baire property (see [5, Chap. IX, Sec. 3, Theorem 2] and [5, Exercise 7, p. 231]). The Blumberg–Sierpiński theorem was generalized to locally compact groups in [3, Corollary 2] with a sketch of proof (see [3, Proposition 4]). Below we propose another short argument in the commutative case.

Theorem 4.1. *Let U be an open subset of a locally compact Abelian topological* [*metric*] *group. If $f : U \to \mathbb{R}$ is a measurable convex function, then f is locally uniformly continuous* [*locally Lipschitz*] *in U.*

Proof. By Theorem 3.1 it is enough to show that f is locally bounded from above at any point $x_0 \in U$. We may assume that $x_0 = 0$. Let $V \subset U$ be a symmetric neighbourhood of 0 with compact closure, and put $A_k = \{x \in V : \max\{f(x), f(-x)\} \le k\}$, $k \in \mathbb{N}$. Since $\bigcup_{k=1}^{\infty} A_k = V$, there is a $k \in \mathbb{N}$ such that $0 < \mu(A_k) < \infty$, where μ is the Haar measure. The function φ given by $\varphi(x) = \mu(A_k \cap (x - A_k))$ is continuous (see [4, (20.17) Corollary]). Since $\varphi(0) = \mu(A_k \cap (-A_k)) = \mu(A_k) > 0$, it follows that there exists a neighbourhood V_1 of 0 such that $\varphi(x) > 0$ for every $x \in V_1$. Let V_2 be a neighbourhood of 0 such that $2V_2 \subset V_1$. We prove that f is bounded from above in V_2. Let $x \in V_2$ be arbitrary. Then $2x \in V_1$, and $\mu(A_k \cap (2x - A_k)) = \varphi(2x) > 0$, and thus the set $A_k \cap (2x - A_k)$ is nonempty. If y is an element of this set, then $y \in A_k$ and $2x - y \in A_k$, and thus $f(y) \le k$ and $f(2x - y) \le k$, whence, by the convexity of f, we obtain $2f(x) \le f(y) + f(2x - y) \le 2k$, and $f(x) \le k$. $\square$

The next result is the category analogue of Theorem 4.1.

Theorem 4.2. *Let U be an open subset of a locally compact Abelian topological* [*metric*] *group. If $f : U \to \mathbb{R}$ is a convex function with the Baire property, then f is locally uniformly continuous* [*locally Lipschitz*] *in U.*

Proof. By Theorem 3.1 it is enough to show that f is locally bounded from above at any point $x_0 \in U$. We may assume that $x_0 = 0$. Let $V \subset U$ be a symmetric neighbourhood of 0, and put $A_k = \{x \in V : \max\{f(x), f(-x)\} \le k\}$, $k \in \mathbb{N}$. Since $\bigcup_{k=1}^{\infty} A_k = V$, there is a $k \in \mathbb{N}$ such that A_k is of second category. Let $W \subset V$ be an open set such that $W \Delta A_k$ is meager (that is, of first category). Since A_k is symmetric, the set $(-W)\Delta A_k = -[W \Delta A_k]$ is also meager, and thus $W \cap (-W) \ne \emptyset$. Take a point $w \in W \cap (-W)$; then $0 \in w + W$. Let V_1 be a neighbourhood of 0 such that $2V_1 \subset w + W$. We prove that f is bounded from above in V_1. Let $x \in V_1$ be arbitrary. Then $2x \in w + W$, and $w \in W \cap (2x - W)$. Thus $W \cap (2x - W)$ is a nonempty open set. Since $[W \cap (2x - W)]\Delta[A_k \cap (2x - A_k)]$ is a subset of $[W \Delta A_k] \cup [2x - (W \Delta A_k)]$ which is meager, it follows that $A_k \cap (2x - A_k)$ is nonempty. If y is an element of this set, then $y \in A_k$ and $2x - y \in A_k$, and thus $f(y) \le k$ and $f(2x - y) \le k$. By the convexity of f, we obtain $2f(x) \le f(y) + f(2x - y) \le 2k$, and $f(x) \le k$. $\square$

Remark 4.3. As the proof shows, Theorem 4.2 is valid in every topological Abelian group G in which every nonempty open set is of second category. It is easy to see that this happens if and only if G is of the second category. Indeed, if there is a nonempty open set which is of the first category then, as its translations cover G, it follows that G is the union of a family of open sets of the first category. By Banach's Category Theorem [8, Theorem 16.1], this implies that G is of the first category.

However, in Theorems 4.1 and 4.2 the condition of local compactness cannot be removed altogether, as the following example shows. Let α be an irrational number, and let $G = \{n + k\alpha : n, k \in \mathbb{Z}\}$; then G is a subgroup of $\mathbb{R}$. Let G be equipped with the subspace topology. Since G is countable, every function on G is Borel. However, there are convex functions on G which are not continuous. Indeed, let $f : G \to \mathbb{R}$ be given by $f(n + k\alpha) = k$. It is additive, therefore convex, but it is not continuous, even not bounded on any nonempty open set.

Ostrowski proved in [7] that in the Blumberg–Sierpiński theorem the measurability of the function f can be replaced by the condition that f is bounded from above on a measurable set of positive measure. (See also [5, Chap. IX, Sec. 3, Theorem 1].) If we want to generalize this result to topological groups then we run into difficulties. First, as we saw in the Examples 3.4 and 3.5, the statement is not true for functions that are convex on arbitrary open and convex subsets of the group. Thus we have to impose some restrictions on the open set on which f is convex, as we did in Theorem 3.6. The second problem is that the statement may be false even if the function is defined on the whole group, as the following example shows.

Let G be as in Example 3.5, and let $a : \mathbb{R} \to \mathbb{R}$ be a non-continuous additive function. We define $f(x, y) = a(x)$ for every $(x, y) \in G$. Then f is convex on G, and is bounded on the set $F = \{0\} \times \mathbb{R}$, which is a closed set of positive (infinite) measure. Still, f is not continuous.

What makes this example possible is the fact that, although the set F is measurable and has positive measure, the interior of $F + F$ is empty (see the second footnote on page 296 of [4]). Therefore, in generalizing Ostrowski's theorem we have to assume that f is bounded from above on a measurable set of finite (or σ-finite) and positive measure.

Finally, the most natural proof seems to depend on the following statement: *if $A \subset G$ is a measurable set of finite and positive measure, then the interior of $A + A$ contains an element $2x$ for some $x \in G$.* If this is true then we shall say that G has property (P).

It is well-known that if G is a locally compact Abelian group and $A \subset G$ is a measurable set of finite and positive measure, then the interior of $A + A$ is nonempty [4, (20.17) Corollary]. Therefore, every locally compact Abelian group divisible by 2 has property (P).

Note that in general there can be nonempty open sets in G which do not contain elements of the form $2x$. For example, if G is compact and $2G \subsetneqq G$ (which happens in the Cantor group or in the group of 2-adic integers) then $G \setminus (2G)$ is such an open set. It is not clear, however, whether or not property (P) holds in every locally compact Abelian group.

Now we present our generalization of Ostrowski's theorem. It extends [3, Theorem 2] as well.

Theorem 4.4. *Let D be an open and convex subset of a locally compact Abelian topological [metric] group G, and assume that G and D satisfy at least one of the following conditions.*

(*i*) $D = G$ *and* G *has property* (P);
(*ii*) D *is connected;*
(*iii*) *every convex neighbourhood of* 0 *is absorbing.*

If a convex function $f : D \to \mathbb{R}$ is bounded from above on a measurable set of finite positive measure, then f is locally uniformly continuous [locally Lipschitz] in D.

Proof. By Theorem 3.6 it is enough to show that f is bounded from above on a nonempty open set. Let $f(x) \le M$ for every $x \in A$, where A is a measurable subset of D with finite positive measure. Suppose that G has property (P). Then we have $2x \in \operatorname{int}(A + A)$ for an $x \in G$. Thus $2x = a + b$ for some $a, b \in A \subset D$, and then $x \in D$ by the convexity of D.

Let U be a neighbourhood of zero such that $2x + U \subset A + A$ and $x + U \subset D$. Choose another neighbourhood V of zero with $V + V \subset U$. If $v \in V$, then $2x + 2v \in 2x + U \subset A + A$, and thus $2x + 2v = c + d$ for some $c, d \in A$. By the convexity of f we obtain $2f(x + v) \le 2M$, which proves that f is bounded from above in $x + V$. This proves the theorem in case (*i*).

Suppose (*ii*), and let H be the connected component of zero. We may assume that $0 \in D$. Then H is a connected locally compact Abelian group containing D. Then it follows from (24.19) Corollary and (24.23) Theorem of [4] that $2H$ is dense in H. This immediately implies that H has property (P). Therefore, as we proved above, f is bounded on a nonempty open subset of H.

Finally, if every convex neighbourhood of 0 is absorbing, then G is divisible by 2, and thus G has property (P). □

Our next result is the category analogue of Theorem 4.4; it generalizes Mehdi's theorem [6] (see also [5, Chap. IX, Sec. 3, Theorem 2]).

Theorem 4.5. *Let G be an Abelian topological [metric] group, let D be an open subset of G, and assume that D and G satisfy one of the conditions (i)–(iii) of Theorem 3.6. If a convex function $f : D \to \mathbb{R}$ is bounded from above on a set of second category with the Baire property, then f is locally uniformly continuous [locally Lipschitz] in D.*

Proof. Suppose that f is bounded from above on the set A, where A is residual in a nonempty open set U. We may assume that $A \subset U$. We prove that $A + A = U + U$. Indeed, if $z \in U + U$, then $V = U \cap (z - U)$ is a nonempty open subset of U. Since A and $z - A$ are both residual in V, it follows that $A \cap (z - A) \ne \emptyset$, and thus $z \in A + A$. This proves $A + A = U + U$.

Therefore, if $x \in A$, then $2x \in A + A = U + U = \operatorname{int}(A + A)$. From this we can complete the argument as in the proof of Theorem 4.4. □

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