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The Cosserat Vector in Membrane Theory: A Variational Approach

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In a previous article (see [2]) the authors studied a model of nonlinear membrane where the external surface loading induces a density of bending moment. Due to the special form of the applied surface forces, the emerging Cosserat vector, resulting from the 3D-2D dimension reduction, was restricted to a class of two dimensional functions. In this paper the full 3D dependence of the Cosserat vector is analyzed via Γ-convergence techniques.

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1. Introduction

In a previous article (see [2]) the authors studied a model of nonlinear membrane where the external surface loading induces a density of bending moment. Due to the special form of the applied surface forces, the emerging Cosserat vector, result of the 3D-2D dimension reduction, was restricted to a class of two dimensional functions. In this paper we analyze the more general case where the Cosserat vector depends also on the thickness variable. In order to detail our main result, relating it with the one in [2], we will use the same notations.

Let ω be an open bounded subset of $\mathbb{R}^2$ and let I be the interval $(-1/2, 1/2)$. Define $\Omega := \omega \times I$, $\Sigma^{\pm} := \omega \times \{\pm 1/2\}$, $\Gamma := \partial\omega \times I$ and, for each $\varepsilon > 0$, $\Omega_\varepsilon := \omega \times \varepsilon I$,

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$\Sigma_\varepsilon^\pm := \omega \times \{\pm\varepsilon/2\}$, $\Gamma_\varepsilon := \partial\omega \times \varepsilon I$.

In what follows $\mathcal{L}^N$ stands for the N-dimensional Lebesgue measure in $\mathbb{R}^N$, $N = 2, 3$, and $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure in $\mathbb{R}^3$. Greek indexes will be used to distinguish the first two components of a tensor, for instance (x_α) and (x_α, x_3), designates (x_1, x_2) and (x_1, x_2, x_3), respectively.

We write $\mathbb{R}^{3\times 2}$ to denote the vector space of 3×2 real-valued matrices, and for $\overline{F} \in \mathbb{R}^{3\times 2}$ and $b \in \mathbb{R}^3$, let $(\overline{F}|b)$ denote the 3×3 matrix whose first two columns are those of $\overline{F}$ and the last one is b.

Consider the rescaled total energy of a deformation $U : \tilde{x} \in \Omega_\varepsilon \mapsto U(\tilde{x}) \in \mathbb{R}^3$,

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} W(DU)\, d\tilde{x} - < F_\varepsilon\,, U >,$$

where $DU = (D_\alpha U|D_3 U)$ is the strain of the deformation $U \in W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3)$, W satisfies some suitable growth hypotheses (see (H)) and F_ε represents the external loading. The key point in [2] is that we used an external surface loading of the kind (for simplicity we will not consider bulk loads)

$$F_\varepsilon := \frac{1}{\varepsilon}\, g(\mathcal{H}^2\lfloor\Sigma_\varepsilon^+ - \mathcal{H}^2\lfloor\Sigma_\varepsilon^-), \tag{1}$$

with $g \in L^{p'}(\omega; \mathbb{R}^3)$ and $p' = p/(p-1)$ for a fixed p such that $1 < p < +\infty$, and where the scaling factor ε^{-1} enhances the role of the Cosserat vector field as described below.

Let $W^{1,p}_{\Gamma_\varepsilon}(\Omega_\varepsilon; \mathbb{R}^3)$ (respectively $W^{1,p}_{\Gamma}(\Omega; \mathbb{R}^3)$) denote the space of functions in $W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3)$ (respectively $W^{1,p}(\Omega; \mathbb{R}^3)$) that vanish on Γ_ε (respectively on Γ). Assuming that the deformations of the body satisfy a boundary condition of place on Γ_ε, the equilibrium problem under the load F_ε given in (1) can be formulated as the minimization problem

$$\inf_{U-\tilde{x}\in W^{1,p}_{\Gamma_\varepsilon}(\Omega_\varepsilon;\mathbb{R}^3)} \left\{\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} W(DU)\, d\tilde{x} - < F_\varepsilon, U >\right\}. \tag{2}$$

In the sequel we will assume that the potential W is a Borel function satisfying the following p-growth and coercivity conditions

$$(H) \qquad \frac{1}{C}|\xi|^p - C \le W(\xi) \le C(1 + |\xi|^p),$$

for some $C > 0$ and for all $\xi \in \mathbb{R}^{3\times 3}$.

The existence of a solution for problem (2) may be obtained via the Direct Method of the Calculus of Variations under the additional hypothesis that W is *quasiconvex*, i.e. it satisfies

$$W(\xi) \le \frac{1}{\mathcal{L}^N(D)}\int_D W(\xi + D\psi)\, dx,$$

for all $\xi \in \mathbb{R}^{3\times 3}$ and for all $\psi \in W^{1,\infty}_0(D; \mathbb{R}^3)$, where D is any open bounded domain of $\mathbb{R}^3$ such that $\mathcal{L}^3(\partial D) = 0$.

In order to transform the problem (2) from the thin, varying domain Ω_ε, into the fixed domain Ω, we perform the usual change of variables that to each $\tilde{x} = (\tilde{x}_\alpha, \tilde{x}_3) \in \Omega_\varepsilon$ associates $x = (x_\alpha, x_3) = \left(\tilde{x}_\alpha, \frac{1}{\varepsilon}\tilde{x}_3\right) \in \Omega$, and define u, $u^\pm$ and $u_{0,\varepsilon}$ by

$$u\,(x_\alpha, x_3) := U(\tilde{x}_\alpha, \tilde{x}_3), \qquad u^\pm(x_\alpha) := u\left(x_\alpha, \pm\frac{1}{2}\right), \qquad u_{0,\varepsilon}(x_\alpha, x_3) := (x_\alpha, \varepsilon x_3).$$

Taking into account (1), we may rewrite (2) as

$$(\mathcal{P}_\varepsilon) \qquad \inf_{u-u_{0,\varepsilon}\in W^{1,p}_\Gamma(\Omega;\mathbb{R}^3)} \left\{ \int_\Omega W\left(D_\alpha u \;\Big|\; \frac{1}{\varepsilon} D_3 u\right) dx \;-\; L_\varepsilon(u) \right\},$$

where now the work $L_\varepsilon(u)$ of the external surface loads is given by

$$L_\varepsilon(u) := \int_\omega g\,\left(\frac{u^+ - u^-}{\varepsilon}\right) dx_\alpha = \int_\omega g\,\left(\int_I \frac{1}{\varepsilon} D_3 u_\varepsilon\right) dx_\alpha.$$

Defining $b_\varepsilon := \frac{1}{\varepsilon} D_3 u_\varepsilon$, one easily sees that, due to the loading forces, only the weak limit of the mean $\bar{b}_\varepsilon := \int_I b_\varepsilon$ plays a role in the limit problem.

In [2], to describe the limit problem we proved that the Γ-limit with respect to the weak topology of the corresponding stored energy

$$E_\varepsilon(u, \bar{b}) := \begin{cases} \displaystyle\int_\Omega W\left(D_\alpha u \,\Big|\, \frac{1}{\varepsilon} D_3 u\right) dx & \text{if } \displaystyle\frac{1}{\varepsilon}\int_I D_3 u(x_\alpha, x_3)\; dx_3 = \bar{b}(x_\alpha), \\ +\infty & \text{otherwise,} \end{cases}$$

with $(u, \bar{b}) \in W^{1,p}(\Omega;\mathbb{R}^3) \times L^p(\omega;\mathbb{R}^3)$, has the form

$$E(u, \bar{b}) := \begin{cases} \displaystyle\int_\omega \mathcal{Q}^* W(D_\alpha u \mid \bar{b}) dx_\alpha\,, & \text{if } (u, \bar{b}) \in \mathcal{V} \times L^p(\omega;\mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathcal{V} := \{u \in W^{1,p}(\Omega;\mathbb{R}^3) \mid D_3 u(x) = 0 \text{ a.e. in } x \in \Omega\}$, and where $\mathcal{Q}^* W$ introduced in [2] is given by (46). It coincides with the cross-quasiconvex envelop of W, used in [5] (see also [7]; the detailed argument may be found in the Appendix), precisely

$$\begin{aligned} &\mathcal{Q}^* W(F|b) \\ &= \inf_{(\varphi,\psi)} \left\{ \int_{Q'} W(F + D_\alpha\varphi | b + \psi)\; dx_\alpha : \varphi \in W^{1,p}_0(Q';\mathbb{R}^3),\; \psi \in L^p_0(Q';\mathbb{R}^3) \right\}, \end{aligned} \tag{3}$$

for $F \in \mathbb{R}^{3\times 2}$ and $b \in \mathbb{R}^3$, where $Q := (-1/2, 1/2)^3, Q' := (-1/2, 1/2)^2$, $L^p_0(Q';\mathbb{R}^3)$ is the subspace of $L^p(Q';\mathbb{R}^3)$ of functions with null mean. In view of the upper bound in (H), it can be shown that (3) remains unchanged if the condition $\varphi \in W^{1,p}_0(Q';\mathbb{R}^3)$ is replaced by $\varphi \in W^{1,p}_\#(Q';\mathbb{R}^3)$, the subscript $\#$ in $W^{1,p}_\#(Q';\mathbb{R}^3)$ indicating the subspace of Q'-periodic functions in $W^{1,p}(Q';\mathbb{R}^3)$.

We remark that the description of the limit energy in terms of the 2D deformation $u(x_\alpha)$ and adittionally of the mean Cosserat vector $\bar{b}(x_\alpha)$, the bending moment, given in [2] is more precise than the one given in the usual membrane models. However, this still does not give insight into the limit energy in the case where the Cosserat vector field b may also depend on the x_3 variable. In this note we seek to characterize the Γ-limit of the sequence of internal energy functionals independently of the applied forces. We study the asymptotic behavior of the sequence with respect to u and to the Cosserat vector b, instead of its mean with respect to the thickness of the membrane, the bending moment $\bar{b}$. Precisely, in Theorem 2.3 we present an integral representation of the Γ-limit, with respect to the weak topology, of the functional $\mathcal{I}_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^3) \times L^p(\Omega;\mathbb{R}^3) \to \overline{\mathbb{R}}$ defined by

$$\mathcal{I}_\varepsilon(u,b) := \begin{cases} \displaystyle\int_\Omega W\left(D_\alpha u \,\middle|\, \frac{1}{\varepsilon} D_3 u\right) dx & \text{if } \dfrac{1}{\varepsilon} D_3 u = b, \\ +\infty & \text{otherwise.} \end{cases} \tag{4}$$

In spite of the particular case analyzed in Proposition 2.4 (see also Remark 2.5), we conjecture that, in general, the limit functional is non local. This is an interesting open problem.

In Section 2 we state the main result whose proof is developed in Section 3.

2. Main result.

As it is usual, we localize the functionals $\mathcal{I}_\varepsilon$ introduced in (4). Representing by $\mathcal{A}(\omega)$ the family of all open subsets of ω, define $\mathcal{I}_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^3) \times L^p(\Omega;\mathbb{R}^3) \times \mathcal{A}(\omega) \to \overline{\mathbb{R}}$ by

$$\mathcal{I}_\varepsilon(u,b,A) := \begin{cases} \displaystyle\int_{A\times I} W\left(D_\alpha u \,\middle|\, \frac{1}{\varepsilon} D_3 u\right) dx & \text{if } \dfrac{1}{\varepsilon} D_3 u = b \text{ on } A \times I, \\ +\infty & \text{otherwise.} \end{cases}$$

We are interested in the integral representation of the following functional, defined for $(u,b,A) \in \mathcal{V} \times L^p(\Omega;\mathbb{R}^3) \times \mathcal{A}(\omega)$ by

$$\begin{aligned} &\mathcal{I}(u,b,A) \\ &:= \inf\Big\{ \liminf_n \int_{A\times I} W\left(D_\alpha u_n \,|\lambda_n\, D_3 u_n\right) dx \mid u_n \rightharpoonup u \text{ in w - } W^{1,p}(A\times I;\mathbb{R}^3), \\ &\qquad \lambda_n \to +\infty,\ \lambda_n D_3 u_n \rightharpoonup b \ \text{ in w - } L^p(A\times I;\mathbb{R}^3)\Big\}. \end{aligned} \tag{5}$$

Finding an integral representation of $\mathcal{I}$ independent of the sequence $\{\lambda_n\}$ corresponds to determining the Γ-limit of the sequence $\{\mathcal{I}_\varepsilon\}$ introduced above, with respect to the weak topology of $W^{1,p}(\Omega;\mathbb{R}^3) \times L^p(\Omega;\mathbb{R}^3)$.

Fix a countable dense family $\{\theta_i\}_{i\in\mathbb{N}}$ in $L^{p'}(I;\mathbb{R}^3)$, where p' is the conjugate exponent of p. For every $k \in \mathbb{N}$ and $(F,b) \in \mathbb{R}^{3\times 2} \times L^p(I;\mathbb{R}^3)$ define $Q := (-1/2,1/2)^3$, $Q' := (-1/2,1/2)^2$,

$$\mathcal{Q}_\infty W(F|b) := \sup_k \mathcal{Q}_k W(F|b) \tag{6}$$

where

$$\mathcal{Q}_k W(F|b)$$
$$:= \inf_{(\varphi,\lambda)} \Big\{ \int_Q W(F + D_\alpha\varphi|\ \lambda\ D_3\varphi)\ dx \mid \lambda > 0\ ,\ \varphi \in W^{1,p}(Q;\mathbb{R}^3),\ \varphi(\cdot, x_3) \text{ is} \tag{7}$$
$$Q' \text{ periodic a.e. } x_3 \in I\ ,\ \left| \int_Q \lambda D_3\varphi\ \ \theta_i\ dx - \int_I b\ \theta_i\ dx_3 \right| < \frac{1}{k},\ \forall\, i = 1, \cdots, k \Big\}.$$

Remark 2.1. Let us emphasize that $Q_\infty W(F, \cdot)$ is a functional defined in $L^p(I, \mathbb{R}^3)$. We conjecture that, in general, this functional is non local in the sense that it does not exist any integrand $\tilde{W}$ so that

$$Q_\infty W(F, b) = \int_I \tilde{W}(F, b(x_3))\ dx_3. \tag{8}$$

Notice that (8) would imply that $Q_\infty W(F, b(\cdot))$ is completely determined by its restriction to constant functions. In fact, this is the case if the initial energy density W is cross-quasiconvex (see Proposition 2.4 bellow). Finding an explicit counter example to (8) is a challenging problem.

The main theorem of this paper is

Theorem 2.2. *Let W be a Borel function satisfying hypothesis (H). Then*

$$\mathcal{I}(u, b, A) = \int_A \mathcal{Q}_\infty W(D_\alpha u(x_\alpha)|b(x_\alpha, \cdot))\ dx_\alpha$$

for every $(u, b) \in \mathcal{V} \times L^p(\Omega; \mathbb{R}^3)$.

Remark 2.3. We observe that, in view of (H), we may assume, without loss of generality, that W is quasiconvex. As we will see in Proposition 2.6, denoting the quasiconvex envelop of W by QW (see [3] for the definition), we get

$$\mathcal{Q}_\infty W(F|b) = \mathcal{Q}_\infty(QW)(F|b).$$

Also in (6) the definition of $\mathcal{I}(u, b, A)$ remains unchanged if we replace the integrand W by QW (see Proposition 2.7). Therefore, since a quasiconvex function with p-growth is p-Lipschitz (see [8]), in the sequel we may assume that

$$|W(\xi) - W(\xi')| \le C\ (1 + |\xi|^{p-1} + |\xi'|^{p-1})|\xi - \xi'| \tag{9}$$

for some $C > 0$ and for all $\xi, \xi' \in M^{3\times 3}$.

Proposition 2.4. *The following inequality holds*

$$\int_I \mathcal{Q}^* W(F|b(x_3))\ dx_3 \le \mathcal{Q}_\infty W(F|b) \le \int_I W(F|b(x_3))\ dx_3,$$

for $(F, b) \in \mathbb{R}^{3\times 2} \times L^p(I; \mathbb{R}^3)$. Consequently, if W is cross-quasiconvex ($\mathcal{Q}^* W = W$) then

$$\mathcal{Q}_\infty W(F|b) = \int_I W(F|b(x_3))\ dx_3.$$

Proof. To see that $\mathcal{Q}_\infty W(F|b) \le \int_I W(F|b(x_3))\, dx_3$, it suffices to take $\varphi(x_3) := \frac{1}{\lambda} \int_0^{x_3} b(s)\, ds$ as test function in the definition (7).

To prove the other inequality, let $k \in \mathbb{N}$ and let (φ, λ) denote an arbitrary admissible pair for the infimum in (7). Since $\mathcal{Q}^*W$ is given by (3), we have

$$\begin{aligned}
\mathcal{Q}_k W(F|b) &= \inf_{(\varphi,\lambda)} \int_Q W(F + D_\alpha\varphi|\ \lambda\ D_3\varphi)dx \\
&= \inf_{(\varphi,\lambda)} \int_I \left[\int_{Q'} W \left(F + D_\alpha\varphi(x_\alpha, x_3) \middle|\ \lambda \int_{Q'} D_3\varphi\ dy_\alpha \right.\right. \\
&\qquad\qquad \left.\left. + \left(\lambda D_3\varphi - \lambda \int_{Q'} D_3\varphi\ dy_\alpha \right) \right) dx_\alpha \right] dx_3 \qquad (10) \\
&\ge \inf_{(\varphi,\lambda)} \int_I \mathcal{Q}^*W \left(F \middle|\ \lambda \int_{Q'} D_3\varphi \right) dx_3 \\
&\ge \inf_{c \in L^p(I;\mathbb{R}^3)} \int_I \mathcal{Q}^*W(F|\ c)\ dx_3,
\end{aligned}$$

where c satisfies

$$\left| \int_I c\ \theta_i\ dx_3 - \int_I b\ \theta_i\ dx_3 \right| \le \frac{1}{k}, \quad \forall\ i = 1 \cdots k. \qquad (11)$$

Using (10) and (11), we associate to each k a function $c_k \in L^p(I;\mathbb{R}^3)$ satisfying

$$\mathcal{Q}_k W(F|b) \ge \int_I \mathcal{Q}^*W(F|\ c_k)\ dx_3 - \frac{1}{k} \qquad (12)$$

and

$$\left| \int_I c_k\ \theta_i\ dx_3 - \int_I b\ \theta_i\ dx_3 \right| \le \frac{1}{k}, \quad \forall\ i = 1 \cdots k. \qquad (13)$$

In view of hypothesis (H) the cross-quasiconvex envelope of W, $\mathcal{Q}^*W$, is also coercive (see [2] or [7]) and therefore $\{c_k\}$ is a bounded in $L^p(\Omega;\mathbb{R}^3)$ and, in view of (13), it converges weakly in $L^p(\Omega;\mathbb{R}^3)$ to b. From the definition of $\mathcal{Q}_\infty W$, from the convexity of $\mathcal{Q}^*W$ with respect to its second variable and from the lower semicontinuity of convex functionals, one obtains, from (12)

$$\mathcal{Q}_\infty W(F|b) \ge \liminf_k \int_I \mathcal{Q}^*W(F|\ c_k)\ dx_3 \ge \int_I \mathcal{Q}^*W(F|\ b)\ dx_3,$$

and this completes the proof. □

Remark 2.5. If W is cross-quasiconvex, then we conclude, from Theorem 2.2 and Proposition 2.4, that

$$\mathcal{I}(u, b, A) = \int_{A\times I} W(D_\alpha u(x_\alpha)|b(x))\ dx.$$

We end this section by proving the two properties mentioned in Remark 2.3 and related to the invariance of the asymptotic energy with respect to the quasiconvexification of the bulk energy.

Proposition 2.6. *Let QW represent the quasiconvex envelop of W. Then*

$$\mathcal{Q}_\infty W(F|b) = \mathcal{Q}_\infty(\mathcal{Q}W)(F|b). \tag{14}$$

Proof. In order to obtain (14) it is enough to prove that for each $k \in \mathbb{N}$

$$\mathcal{Q}_k W(F|b) = \mathcal{Q}_k(\mathcal{Q}W)(F|b).$$

Since $W \geq \mathcal{Q}W$ it follows that $\mathcal{Q}_k W(F|b) \geq \mathcal{Q}_k(\mathcal{Q}W)(F|b)$. To obtain the opposite inequality we use the Relaxation Theorem (see [1]) to guarantee, for a fixed pair (φ, λ) admissible for $\mathcal{Q}_k(\mathcal{Q}W)(F|b))$, the existence of a sequence $\{\varphi_n\}$ weakly converging in $W^{1,p}(Q;\mathbb{R}^3)$ to φ and satisfying

$$\int_Q \mathcal{Q}W(F + D_\alpha\varphi|\ \lambda\ D_3\varphi)\ dx = \lim_n \int_Q W(F + D_\alpha\varphi_n|\ \lambda\ D_3\varphi_n)\ dx.$$

Making use of hypothesis (H) and of the Decomposition Lemma (see [6]), up to a subsequence (not relabeled) we may write $\varphi_n = v_n + w_n$, where $v_n \rightharpoonup \varphi$ weakly in $W^{1,p}(Q;\mathbb{R}^3)$, $\{|\nabla v_n|^p\}$ is equi-integrable and the Lebesgue measure of $\{w_n \neq 0\}$ converges to zero. It follows that

$$\int_Q \mathcal{Q}W(F + D_\alpha\varphi|\ \lambda\ D_3\varphi)\ dx \geq \limsup_n \int_Q W(F + D_\alpha v_n|\ \lambda\ D_3 v_n)\ dx$$

For each $j \in \mathbb{N}$, let $\psi_j \in C_c^\infty(Q', [0,1])$ be a cutt-off function such that $\psi_j \to 1$ in $L^p(Q';\mathbb{R}^3)$ and define $v_{j,n} := \psi_j v_n + (1-\psi_j)\varphi$. We have $v_{j,n}(\cdot, x_3)$ Q'periodic and, due to the equi-integrability of $\{|\nabla v_n|^p\}$ and hypothesis (H), we easily obtain that

$$\limsup_j \limsup_n \int_Q W(F + D_\alpha v_{j,n}|\ \lambda\ D_3 v_{j,n})\ dx \leq \int_Q \mathcal{Q}W(F + D_\alpha\varphi|\ \lambda\ D_3\varphi)\ dx \tag{15}$$

and

$$\begin{aligned} &\lim_j \lim_n \left| \int_Q \lambda D_3 v_{j,n}\ \theta_i\ dx - \int_I b\ \theta_i\ dx_3 \right| \\ &= \left| \int_Q \lambda D_3\varphi\ \theta_i\ dx - \int_I b\ \theta_i\ dx_3 \right| < \frac{1}{k}, \quad \forall\ i = 1, \cdots, k. \end{aligned} \tag{16}$$

In view of (15) and (16) we may find a sequence $n = n(j)$ such that

$$\limsup_j \int_Q W(F + D_\alpha v_{j,n(j)}|\ \lambda\ D_3 v_{j,n(j)})\ dx \leq \int_Q \mathcal{Q}W(F + D_\alpha\varphi|\ \lambda\ D_3\varphi)\ dx \tag{17}$$

and

$$\left| \int_Q \lambda D_3 v_{j,n(j)}\ \theta_i\ dx - \int_I b\ \theta_i\ dx_3 \right| < \frac{1}{k}, \quad \forall\ i = 1, \cdots, k.$$

Since all $v_{j,n(j)}$ are admissible for $\mathcal{Q}_k W(F|b)$, we deduce from (17) that

$$\int_Q \mathcal{Q}W(F + D_\alpha\varphi|\ \lambda\ D_3\varphi)\ dx \geq Q_k W(F|b) \tag{18}$$

and, taking the infimum in all the admissible pairs (φ, λ) on the left hand side of (18), we obtain $Q_k(\mathcal{Q}W)(F|b) \geq \mathcal{Q}_k W(F|b)$, and this completes the proof. $\square$

Proposition 2.7. *The infimum in* (6) *remains unchanged if* W *is replaced by its quasi-convex envelope* $\mathcal{Q}W$.

Proof. Fix $(u, b, A) \in \mathcal{V} \times L^p(A \times I; \mathbb{R}^3) \times \mathcal{A}(\omega)$ and define

$$\begin{aligned}
&\tilde{\mathcal{I}}(u, b, A) \\
&:= \inf \left\{ \liminf_n \int_{A\times I} \mathcal{Q}W\left(D_\alpha u_n \,|\lambda_n\; D_3 u_n\right) dx \mid u_n \rightharpoonup u \text{ in w - } W^{1,p}(A \times I; \mathbb{R}^3), \right. \\
&\qquad \left. \lambda_n \to +\infty,\; \lambda_n D_3 u_n \rightharpoonup b, \text{ in w - } L^p(A \times I; \mathbb{R}^3) \right\}.
\end{aligned}$$

We show that $\tilde{\mathcal{I}}(u, b, A) = \mathcal{I}(u, b, A)$.

Since $\mathcal{Q}W \le W$ it follows that $\mathcal{I}(u, b, A) \ge \tilde{\mathcal{I}}(u, b, A)$.

We prove the opposite inequality. For fixed $\delta > 0$, let $u_n \rightharpoonup u$ weakly in $W^{1,p}(A \times I; \mathbb{R}^3)$ and $\lambda_n\; D_3 u_n \rightharpoonup b$ weakly in $L^p(A \times I; \mathbb{R}^3)$ be such that

$$\tilde{\mathcal{I}}(u, b, A) \ge \lim_n \int_{A\times I} \mathcal{Q}W\left(D_\alpha u_n \,\Big|\lambda_n D_3 u_n\right) dx - \delta. \tag{19}$$

Using, as in Proposition 2.6, the Relaxation Theorem (see [1]), for each n there exists a sequence $\{u_{n,k}\}$ converging to u_n weakly in $W^{1,p}(A \times I; \mathbb{R}^3)$ such that

$$\int_{A\times I} \mathcal{Q}W\left(D_\alpha u_n \,\Big|\lambda_n D_3 u_n\right)\, dx = \lim_k \int_{A\times I} W\left(D_\alpha u_{n,k} \,\Big|\lambda_n D_3 u_{n,k}\right)\, dx. \tag{20}$$

From (19) and (20) we have

$$\tilde{\mathcal{I}}(u, b, A) \ge \lim_n \lim_k \int_{A\times I} W\left(D_\alpha u_{n,k} \,\Big|\lambda_n D_3 u_{n,k}\right) dx - \delta \tag{21}$$

with

$$\lim_n \lim_k \|u_{n,k} - u\|_{L^p(A\times I;\mathbb{R}^3)} = 0 \tag{22}$$

and, for the weak topology of $L^p(A \times I; \mathbb{R}^3)$,

$$\lim_n \lim_k \;\lambda_n D_3 u_{n,k} = b. \tag{23}$$

In view of hypothesis (H) we have

$$\sup_{n,k} \left(\|\lambda_n\; D_3 u_{n,k}\|_{L^p(A\times I;\mathbb{R}^3)} + \|u_{n,k}\|_{W^{1,p}(A\times I;\mathbb{R}^3)}\right) < +\infty. \tag{24}$$

Since the weak topology is metrizable in bounded sets of $L^p(A \times I; \mathbb{R}^3)$, (21), (22), (23) and (24) yield the existence of a diagonal sequence $\{u_{n,k_n}\}$ satisfying $u_{n,k_n} \to u$ in $L^p(A \times I; \mathbb{R}^3)$ (and weakly in $W^{1,p}(A \times I; \mathbb{R}^3)$), $\lambda_n\; D_3 u_{n,k_n} \rightharpoonup b$ weakly in $L^p(A \times I; \mathbb{R}^3)$, and realizing the double limit in the right hand side of (21). Consequently we have

$$\begin{aligned}
\tilde{\mathcal{I}}(u, b, A) &\ge \lim_n \int_{A\times I} W\left(D_\alpha u_{n,k_n} \,|\lambda_n\; D_3 u_{n,k_n}\right)\, dx - \delta \\
&\ge \mathcal{I}(u, b, A) - \delta.
\end{aligned}$$

Letting δ go to zero, the conclusion follows. □

3. Proof of Theorem 2.3.

The following three lemmas are simple adaptations of Lemma 2.1 and Lemma 2.2 presented in [2] and we will omit the proof.

Lemma 3.1. *Let W be a Borel function satisfying hypothesis (H). Then the functional defined in* (5) *satisfies*

$$\begin{aligned}
&\mathcal{I}(u,b,A)\\
&= \inf\Big\{\liminf_n \int_{A\times I} W\left(D_\alpha u_n\,|\lambda_n\ D_3 u_n\right)dx \mid u_n \rightharpoonup u \text{ in } w\text{ - } W^{1,p}(A\times I;\mathbb{R}^3),\\
&\qquad \lambda_n D_3 u_n \rightharpoonup b, \text{ in } w\text{ - } L^p(A\times I;\mathbb{R}^3),\quad u_n = u \text{ on } \partial A\times I\Big\},
\end{aligned} \tag{25}$$

for all $(u,b,A)\in\mathcal{V}\times L^p(\Omega;\mathbb{R}^3)\times\mathcal{A}(\omega)$.

Lemma 3.2. *Let W be a Borel function satisfying hypothesis (H). Then the following inequality holds*

$$\mathcal{I}(u,b,A)\le C\left(\mathcal{L}^2(A)+\int_A |D_\alpha u|^p\ dx_\alpha+\int_{A\times I}|b|^p\ dx\right), \tag{26}$$

for some constant $C>0$ and for all $(u,b,A)\in\mathcal{V}\times L^p(\Omega;\mathbb{R}^3)\times\mathcal{A}(\omega)$.

Lemma 3.3. *Let W be a Borel function satisfying hypothesis (H). Then there exists a subsequence of $\{\lambda_n\}$ (not relabeled), such that for $(u,b)\in\mathcal{V}\times L^p(\Omega;\mathbb{R}^3)$, the set function $\mathcal{I}(u,b,\cdot)$ defined in* (5) *is the trace on $\mathcal{A}(\omega)$ of a measure, absolutely continuous with respect to the two dimensional Lebesgue measure $\mathcal{L}^2$.*

The proof of Theorem 2.3 is a consequence of the two propositions below.

Proposition 3.4. *Let W be a Borel function satisfying hypotheses $(H1)$. Consider the functional defined in* (5)*. Then*

$$\mathcal{I}(u,b,A)\ge\int_A \mathcal{Q}_\infty W(D_\alpha u(x_\alpha)|b(x_\alpha,\cdot))\ dx_\alpha, \tag{27}$$

for each $(u,b)\in\mathcal{V}\times L^p(\Omega;\mathbb{R}^3)$.

Proof. *Step 1.* We prove that for $k\in\mathbb{N}$, $u(x_\alpha):=Fx_\alpha+u_0$ with $F\in\mathbb{R}^{3\times 2}$, $u_0\in\mathbb{R}$, $b\in L^p(I;\mathbb{R}^3)$, and for any two sequences $\lambda_n\to+\infty$ and $\varphi_n\rightharpoonup 0$ in $W^{1,p}(Q;\mathbb{R}^3)$, such that $\lambda_n D_3\varphi_n \rightharpoonup b$ in $L^p(Q'\times I;\mathbb{R}^3)$ for $i=1,\cdots,k$, then

$$\liminf_n\int_Q W(F+D_\alpha\varphi_n|\ \lambda_n\ D_3\varphi_n)\ dx\ge\mathcal{Q}_k W(F,b).$$

Fix $n\in\mathbb{N}$. By Lemma 3.1 we may assume that $\varphi_n=0$ on $\partial Q'\times I$ (see (25)). Since $\int_I\lambda_n D_3\varphi_n\ \theta_i\ dx_3\rightharpoonup\bar{b}_i$ in $L^p(Q';\mathbb{R}^3)$ for $i=1,\cdots,k$, there exists $n_k\in\mathbb{N}$ such that for $n\ge n_k$ implies

$$\left|\int_Q\lambda_n D_3\varphi_n\ \ \theta_i\ dx-\bar{b}_i\ \right|\le\frac{1}{k},\ \ \text{for all } i=1\cdots k.$$

Then, for $n \geq n_k$, λ_n and φ_n are admissible with respect to the infimum in the right hand side of (5), and so

$$\liminf_n \int_Q W(F + D_\alpha \varphi_n |\ \lambda_n\ D_3 \varphi_n)\ dx \geq \mathcal{Q}_k W(F|b).$$

Taking the supremum in k we get

$$\mathcal{I}(u, b; Q') \geq \mathcal{Q}_\infty W(F|b).$$

Step 2. Now we establish (27) in the general case.

Fix $(u, b, A) \in \mathcal{V} \times L^p(\Omega, \mathbb{R}^3) \times \mathcal{A}(\omega)$. Consider $\{u_n\}$ and $\{\lambda_n\}$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(A \times I; \mathbb{R}^3)$, $\lambda_n\ D_3 u_n \rightharpoonup b$ weakly in $L^p(A \times I; \mathbb{R}^3)$, and upon the extraction of a subsequence (not relabeled) we may assume that

$$\liminf_n \int_{A \times I} W(D_\alpha u_n\ |\lambda_n\ D_3 u_n)\ dx = \lim_n \int_{A \times I} W(D_\alpha u_n\ |\lambda_n\ D_3 u_n)\ dx.$$

Define the sequence of measures $\mu_n := \left(\int_I W(D_\alpha u_n\ |\lambda_n\ D_3 u_n)\ dx_3\right) \mathcal{L}^2 \lfloor A$. Since $\{\mu_n\}$ is bounded, up to a further subsequence (not relabeled) it converges weakly-$\star$ to some measure μ. Represent by ρ the absolutely continuous part of μ with respect to the 2-dimensional Lebesgue measure. To prove (27) it suffices to show that, for a.e. $x_0 \in A$ and for an arbitrary fixed $k \in \mathbb{N}$,

$$\rho(x_0) \geq \mathcal{Q}_k W(D_\alpha u(x_0), b(x_0, \cdot)) \tag{28}$$

Let $\bar{b}_i(x_\alpha) := \int_I b(x_\alpha, x_3)\ \theta_i(x_3)\ dx_3$, $i = 1, \cdots, k$. It is known that, for a.e. $x_0 \in A$,

$$\rho(x_0) = \lim_{\varepsilon \to 0} \frac{\mu(x_0 + \varepsilon Q')}{\varepsilon^2} \quad \text{exists and is finite,} \tag{29}$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{p+2}} \int_{x_0 + \varepsilon Q'} |u(x_\alpha) - u(x_0) - \nabla u(x_0)(x_\alpha - x_0)|^p\ dx_\alpha = 0, \tag{30}$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{x_0 + \varepsilon Q'} |\bar{b}_i(x_\alpha) - \bar{b}_i(x_0)|^p\ dx_\alpha = 0, \quad i = 1, \cdots, k. \tag{31}$$

Let x_0 satisfy (29), (30) and (31). Let $\{\varepsilon\}$ represent a sequence converging to zero such that, for all ε,

$$\mu(\partial(x_0 + \varepsilon Q')) = 0. \tag{32}$$

Using (29), the definition of μ and (32), we have

$$\begin{aligned} \rho(x_0) &= \lim_{\varepsilon \to 0} \lim_n \frac{1}{\varepsilon^2} \int_{(x_0 + \varepsilon Q') \times I} W(D_\alpha u_n\ |\lambda_n\ D_3 u_n)\ dx \\ &= \lim_{\varepsilon \to 0} \lim_n \int_Q W(D_\alpha u_n(x_0 + \varepsilon y_\alpha, y_3)\ |\lambda_n\ D_3 u_n(x_0 + \varepsilon y_\alpha, y_3))\ dy \qquad (33) \\ &= \lim_{\varepsilon \to 0} \lim_n \int_Q W(D_\alpha u_{n,\varepsilon}\ |\varepsilon \lambda_n\ D_3 u_{n,\varepsilon})\ dy, \end{aligned}$$

where $u_{n,\varepsilon}(y) := \dfrac{u_n(x_0+\varepsilon y_\alpha, y_3) - u(x_0)}{\varepsilon}$.

Since $u_n \to u$ in $L^p(A \times I; \mathbb{R}^3)$, (30) yields

$$\lim_{\varepsilon\to 0}\lim_n \|u_{n,\varepsilon}(\cdot) - \nabla u(x_0)\cdot\|_{L^p(Q;\mathbb{R}^3)} = 0. \tag{34}$$

We also have for all $\varphi \in L^{p'}(Q';\mathbb{R}^3)$, and as $\lambda_n \int_I D_3 u_n\, \theta_i\, dx_3 \rightharpoonup \bar{b}_i,\ i = 1,\cdots,k$, weakly in $L^p(A;\mathbb{R}^3)$,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_n \int_Q \varepsilon\lambda_n\ D_3 u_{n,\varepsilon}(y)\ \varphi(y_\alpha)\ \theta_i(y_3) dy \\
&= \lim_{\varepsilon\to 0}\lim_n \frac{1}{\varepsilon^2}\int_{(x_0+\varepsilon Q')\times I} \lambda_n D_3 u_n(x)\varphi\left(\frac{x_\alpha - x_0}{\varepsilon}\right)\theta_i(x_3) dx \\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{x_0+\varepsilon Q'} \bar{b}_i(x_\alpha)\ \varphi\left(\frac{x_\alpha - x_0}{\varepsilon}\right)\ dx_\alpha \\
&= \bar{b}_i(x_0)\int_{Q'}\varphi(y_\alpha)\ dy_\alpha,
\end{aligned} \tag{35}$$

where we have used (31).

By means of a standard diagonalization process, from (33), (34) and (35), we construct $\tilde{u}_j := u_{\varepsilon_j,n_j}$ and $\tilde{\lambda}_j := \varepsilon_j\lambda_{n_j}$ such that

$$\begin{aligned}
&\tilde{\lambda}_j \to +\infty, \qquad \tilde{u}_j(y) \to D_\alpha u(x_0)y \ \text{ in } L^p(Q;\mathbb{R}^3), \\
&\tilde{\lambda}_j \int_I D_3\tilde{u}_j\ \theta_i\ dy_3 \rightharpoonup \bar{b}_i(x_0) \ \text{ weakly in } L^p(Q';\mathbb{R}^3)
\end{aligned}$$

and

$$\rho(x_0) = \lim_k \int_Q W(D_\alpha\tilde{u}_j\,|\tilde{\lambda}_j\ D_3\tilde{u}_j)\ dy. \tag{36}$$

Since by *Step 1* we have

$$\lim_j \int_Q W(D_\alpha\tilde{u}_j\,|\tilde{\lambda}_j\ D_3\tilde{u}_j)\ dy \ge \mathcal{Q}_k W(D_\alpha u(x_0)|\bar{b}(x_0)),$$

(28) follows from (36) and from the arbitrariness of $\{u_n\}$ and $\{\lambda_n\}$. □

Proposition 3.5. *Let W be a Borel function satisfying hypothesis (H). Consider the functional defined in* (5)*. Then*

$$\mathcal{I}(u,b,A) \le \int_A \mathcal{Q}_\infty W(D_\alpha u(x_\alpha)|b(x_\alpha,\cdot))\ dx_\alpha,$$

for each $(u,b,A) \in \mathcal{V}\times L^p(\Omega,\mathbb{R}^3)\times\mathcal{A}(\omega)$

Proof. *Step 1.* First we consider the case where $u(x_\alpha) := Fx_\alpha + u_0$, with $F \in \mathbb{R}^{3\times 2}$ and $u_0 \in \mathbb{R}^3$, and $b \in L^p(I;\mathbb{R}^3)$. Clearly it suffices to consider the case where

$\sup_k \mathcal{Q}_k W(F|b) < +\infty$. Since $\mathcal{Q}_k W(F|b)$ is nondecreasing in k, we have that $\sup_k \mathcal{Q}_k W(F|b) = \lim_k \mathcal{Q}_k W(F|b)$. Using the definition of $\mathcal{Q}_k W(F|b)$ there exist $\{t_k\}$ and $\{\varphi^k\}$, satisfying $\varphi^k \in W^{1,p}(Q;\mathbb{R}^3)$, $\varphi^k(\cdot, x_3)$ is Q' periodic a.e. in $x_3 \in I$, $\left|\int_Q t_k D_3\varphi^k \;\; \theta_i \; dx - \int_I b \; \theta_i \; dx_3\right| \leq \frac{1}{k}$, for all $i = 1 \cdots k$, and

$$\mathcal{Q}_k W(F,b) \leq \int_Q W(F + D_\alpha \varphi^k \mid t_k D_3 \varphi^k) \; dx < \mathcal{Q}_k W(F,b) + \frac{1}{k}. \tag{37}$$

Let $\lambda_n \to +\infty$. Using the Q'-periodicity of φ^k, we define $\varphi_n^k : \mathbb{R}^2 \times I \to \mathbb{R}^3$ by $\varphi_n^k(x) := \frac{t_k}{\lambda_n} \; \varphi^k \; \left(\frac{\lambda_n}{t_k} \; x_\alpha, x_3\right)$.

For fixed k we have $\varphi_n^k \in W^{1,p}(A\times I;\mathbb{R}^3)$ and, as n goes to $+\infty$, by the Riemann-Lebesgue Lemma we get

$$\begin{aligned}
&\varphi_n^k \rightharpoonup 0, \;\; \lambda_n \int_I D_3\varphi_n^k \theta_i \; dx_3 \\
&= t_k \int_I D_3\varphi^k\left(\frac{\lambda_n}{t_k} x_\alpha, x_3\right) \theta_i \; dx_3 \rightharpoonup t_k \int_I \int_{Q'} D_3 \varphi^k(y_\alpha, x_3) \; \theta_i \; dy_\alpha \; dx_3 =: \bar{b}_i + r_i^k,
\end{aligned} \tag{38}$$

weakly in $W^{1,p}(A \times I;\mathbb{R}^3)$ and weakly in $L^p(A;\mathbb{R}^3)$ respectively, with $|r_i^k| \leq 1/k$, for all $i = 1, \cdots, k$, and

$$\lim_n \int_{A\times I} W(F + D_\alpha \varphi_n^k \mid \lambda_n D_3 \varphi_n^k) = \mathcal{L}^2(A) \int_Q W(F + D_\alpha \varphi^k \mid t_k D_3 \varphi^k). \tag{39}$$

In view of the coercivity hypothesis (H) and since the weak topology is metrizable on bounded sets, using a diagonal argument, (38) and (39) allow us to construct a sequence $\{\lambda_{n_k}\}$ and $\{\varphi_{n_k}^k\}$, satisfying $\varphi_{n_k}^k \rightharpoonup 0$ in $W^{1,p}(A \times I;\mathbb{R}^3)$, $\lambda_{n_k} \int_I D_3 \varphi_{n_k}^k \theta_i \; dx_3 \rightharpoonup \int_I b \; \theta_i \; dx_3$ in $L^p(A;\mathbb{R}^3)$ for all $i \in \mathbb{N}$ (so that $\lambda_{n_k} D_3 \varphi_{n_k}^k \rightharpoonup b$ in $L^p(A \times I;\mathbb{R}^3)$) and $\lim_k \int_{A\times I} W(F + D_\alpha \varphi_{n_k}^k \mid \lambda_{n_k} D_3 \varphi_{n_k}^k) = \mathcal{L}^2(A) \sup_k \mathcal{Q}_k W(F,b)$. Consequently

$$\mathcal{I}(u,b,A) \leq \mathcal{L}^2(A) \sup_k \mathcal{Q}_k W(F,b) = \mathcal{L}^2(A) \mathcal{Q}_\infty W(F,b).$$

Step 2. We prove the claim for u and b for which there exists a finite and measurable partition $\{A_j\}_{j=1,\cdots,m}$ of A such that u is affine and b independent of x_α in each A_j. For each j we have, using *Step 1*,

$$\mathcal{I}(u,b,A_j) \leq \mathcal{L}^2(A_j) \;\; \mathcal{Q}_\infty W(D_\alpha u, b).$$

By Lemma 3.3 $\mathcal{I}(u,b,\cdot)$ is a measure, thus

$$\mathcal{I}(u,b,A) = \sum_{j=1}^m \mathcal{I}(u,b,A_j) \leq \sum_{j=1}^m \mathcal{L}^2(A_j) \;\; \mathcal{Q}_\infty W(D_\alpha u, b) = \int_A \mathcal{Q}_\infty W(D_\alpha u, b) \; dx_\alpha.$$

Step 3. We prove the claim for an arbitrary $(u,b,A) \in W^{1,p}(\omega;\mathbb{R}^3) \times L^p(\Omega;\mathbb{R}^3) \times \mathcal{A}(\omega)$.

For $(u,b) \in \mathcal{V} \times L^p(A \times I;\mathbb{R}^3)$ let $\{(u_n, b_n)\}$ be a sequence piecewisely defined like in the previous step and strongly converging in $W^{1,p}(A \times I;\mathbb{R}^3) \times L^p(A \times I;\mathbb{R}^3)$ to (u,b). For the construction of such a sequence we may assume, by a density argument, that u and b are $\mathcal{C}_0^\infty$ functions, so that we can apply, with minor adaptation the classical Approximation Theorem (see, for instance, [4]).

The lower semicontinuity of $(u,b) \in \mathcal{V} \times L^p(A \times I;\mathbb{R}^3) \mapsto \mathcal{I}(u,b,A)$ with respect to the weak topology yields, together with *Step 2*,

$$\mathcal{I}(u,b,A) \leq \liminf_n \mathcal{I}(u_n, b_n, A) \leq \liminf_n \int_A \mathcal{Q}_\infty W(D_\alpha u_n, b_n)\, dx_\alpha. \tag{40}$$

To complete the proof it is enough to remark that

$$\liminf_n \int_A \mathcal{Q}_\infty W(D_\alpha u_n, b_n)\, dx_\alpha \leq \int_A \mathcal{Q}_\infty W(D_\alpha u, b)\, dx_\alpha,$$

which is a consequence of the growth conditions (26) and of the continuity of

$$(F,b) \in \mathbb{R}^{3\times 2} \times L^p(I;\mathbb{R}^3) \mapsto \mathcal{Q}_\infty W(F,b). \tag{41}$$

Indeed, to prove the continuity of (41), let $\lambda \in \mathbb{R}$ and $k \in \mathbb{N}$ be fixed and define

$$\begin{aligned}
&\mathcal{Q}_k^\lambda W(F|b)\\
&:= \inf_\varphi \left\{ \int_Q W(F + D_\alpha\varphi|\ \lambda\ D_3\varphi)\ dx \ \Big|\ \ \varphi \in W^{1,p}(Q;\mathbb{R}^3),\ \varphi(\cdot, x_3) \text{ is}\right.\\
&\qquad\qquad \left. Q' \text{ periodic a.e. } x_3 \in I\ \left| \int_Q \lambda D_3\varphi\ \ \theta_i\ dx - \int_I b\ \theta_i\ dx_3 \right| \leq \frac{1}{k},\ \forall\ i = 1\cdots k \right\}.
\end{aligned}$$

For $(F,b), (F',b') \in \mathbb{R}^{3\times 2} \times L^p(I;\mathbb{R}^3)$, consider $\mathcal{Q}_k^\lambda W(F|b)$ and $Q_k^\lambda W(F'|b')$. For any infimizing sequence $\{\varphi_n\}$ in the definition of $\mathcal{Q}_k^\lambda W(F|b)$, consider the sequence $\psi_n := \varphi_n + \frac{\int_0^{x_3}(b'(s)-b(s))\ ds}{\lambda}$ of admissible functions in the definition of $\mathcal{Q}_k^\lambda W(F'|b')$, since

$$D_\alpha\psi_n = D_\alpha\varphi_n\ , \qquad D_3\psi_n = D_3\varphi_n + \frac{b'-b}{\lambda}, \tag{42}$$

we get

$$\begin{aligned}
&\left| \int_Q \lambda D_3\varphi_n\ \ \theta_i\ dx - \int_I b\ \theta_i\ dx_3 \right|\\
&= \left| \int_Q \lambda D_3\psi_n\ \ \theta_i\ dx - \int_I b'\ \theta_i\ dx_3 \right|, \quad \forall\ i = 1\cdots k, \quad \forall\ n \in \mathbb{N}.
\end{aligned}$$

From the p-Lipschitz condition (10) (see Remark 2.3) and Hölder inequality, we obtain

$$\begin{aligned}
&\left| \int_Q W(F' + D_\alpha\psi_n|\lambda\ D_3\psi_n)\ dx - \int_Q W(F + D_\alpha\varphi_n|\lambda\ D_3\varphi_n)\ dx \right|\\
&\leq C \Big(1 + \|(F' + D_\alpha\psi_n|\lambda D_3\psi_n)\|_{L^p(Q)}^{p-1}\\
&\qquad + \|(F + D_\alpha\varphi_n|\lambda D_3\varphi_n)\|_{L^p(Q)}^{p-1} \Big)\ \left(|F - F'| + \|b - b'\|_{L^p(I)} \right),
\end{aligned} \tag{43}$$

for a constant C independent of n.

Since $\mathcal{Q}_k^\lambda W(F|b) \le \int_I W(F|b)\, dx_3$, using hypothesis (H) we conclude from (43) that

$$\begin{aligned} &\left| \int_Q W(F' + D_\alpha \psi_n | \lambda\, D_3 \psi_n)\, dx - \int_Q W(F + D_\alpha \varphi_n | \lambda\, D_3 \varphi_n)\, dx \right| \\ &\le C \left(1 + |F'|^{p-1} + |F|^{p-1} + \|b'\|^{p-1}_{L^p(I)} + \|b\|^{p-1}_{L^p(I)}\right) \left(|F - F'| + \|b - b'\|_{L^p(I)}\right). \end{aligned} \tag{44}$$

Letting $n \to +\infty$ in (44) we obtain

$$\begin{aligned} &\mathcal{Q}_k^\lambda W(F', b') - \mathcal{Q}_k^\lambda W(F, b) \\ &\le C \left(1 + |F'|^{p-1} + |F|^{p-1} + \|b'\|^{p-1}_{L^p(I)} + \|b\|^{p-1}_{L^p(I)}\right) \left(|F - F'| + \|b - b'\|_{L^p(I)}\right). \end{aligned}$$

Using the same argument for the pair (F', b') in place of (F, b), we get

$$\begin{aligned} &|\mathcal{Q}_k^\lambda W(F, b) - \mathcal{Q}_k^\lambda W(F', b')| \\ &\le C \left(1 + |F|^{p-1} + |F'|^{p-1} + \|b\|^{p-1}_{L^p(I)} + \|b'\|^{p-1}_{L^p(I)}\right) \left(|F - F'| + \|b - b'\|_{L^p(I)}\right). \end{aligned}$$

Again the independence of C with respect to λ and k allow us to conclude that

$$\begin{aligned} &|\mathcal{Q}_\infty W(F, b) - \mathcal{Q}_\infty W(F', b')| \\ &\le \left(1 + |F|^{p-1} + |F'|^{p-1} + \|b\|^{p-1}_{L^p(I)} + \|b'\|^{p-1}_{L^p(I)}\right) \left(|F - F'| + \|b - b'\|_{L^p(I)}\right). \end{aligned}$$

□

A. Appendix

We recall the potential $\mathcal{Q}^* W$, as defined in [2]. Consider, for every $F \in \mathbb{R}^{3\times 2}$ and $b \in \mathbb{R}^3$,

$$\begin{aligned} \mathcal{Q}^* W(F|b) := \inf_{(\varphi,\lambda)} \Bigg\{ &\int_Q W(F + D_\alpha \varphi | \lambda\, D_3 \varphi)\, dx : \lambda \in \mathbb{R}\,, \varphi \in W^{1,p}(Q; \mathbb{R}^3), \\ &\varphi(\cdot, x_3) \text{ is } Q' \text{-periodic } \mathcal{L}^1 \text{ a.e. } x_3 \in I,\ \lambda \int_Q D_3 \varphi\, dx = b \Bigg\}, \end{aligned} \tag{45}$$

with $Q := (-1/2, 1/2)^3,\ Q' := (-1/2, 1/2)^2$.

We prove here that $\mathcal{Q}^* W$ coincides with the cross-convex envelope of W, $\tilde{\mathcal{Q}} W$, defined by

$$\tilde{\mathcal{Q}} W(F, b) := \sup_{G \in \mathcal{F}} \{G(F, b) : G \le W\},$$

where $\mathcal{F}$ is the family of all $G : (F, b) \in R^{3\times 2} \times \mathbb{R}^3 \mapsto \mathbb{R}$ that are quasiconvex with respect to F, for fixed b, and convex with respect to b, for fixed F. The cross-convex envelope of W is also characterized as follows:

$$\begin{aligned} &\tilde{\mathcal{Q}} W(F|b) \\ &= \inf_{(\varphi,\psi)} \left\{ \int_{Q'} W(F + D_\alpha \varphi | b + \psi)\, dx_\alpha : \varphi \in W^{1,p}_{\#}(Q'; \mathbb{R}^3),\ \psi \in L^p_0(Q'; \mathbb{R}^3) \right\}, \end{aligned} \tag{46}$$

for $F \in \mathbb{R}^{3\times 2}$ and $b \in \mathbb{R}^3$, where the subscript # in $W^{1,p}_{\#}(Q';\mathbb{R}^3)$ indicates the subspace of Q'-periodic functions of $W^{1,p}(Q';\mathbb{R}^3)$ and $L^p_0(Q';\mathbb{R}^3)$ the subspace of $L^p(Q';\mathbb{R}^3)$ formed by the functions with null mean (see [5] and [7]).

Proposition A.1. *For all $F \in \mathbb{R}^{3\times 2}$ and $b \in \mathbb{R}^3$ it holds*

$$\mathcal{Q}^*W(F|b) = \tilde{\mathcal{Q}}W(F|b).$$

Proof. Since $\tilde{\mathcal{Q}}W$ is the cross-convex envelope of W and $\mathcal{Q}^*W$ is cross-convex (see [2, Remark 1.4]), one has $\mathcal{Q}^*W(F|b) \leq \tilde{\mathcal{Q}}W(F|b)$, for all $F \in R^{3\times 2}$ and $b \in \mathbb{R}^3$.

To obtain the converse inequality, we consider, for $F \in R^{3\times 2}$ and $b \in \mathbb{R}^3$, $\varphi \in W^{1,p}(Q;\mathbb{R}^3)$ and $\lambda \in \mathbb{R}$, satisfying $\varphi(\cdot, x_3)$ Q' -periodic $\mathcal{L}^1$ a.e. $x_3 \in I$, and $\lambda \int_Q D_3\varphi \; dx = b$. Define $\psi := \lambda \; D_3\varphi - \int_{Q'} \lambda \; D_3\varphi \; dx_\alpha$. Then, using (46) and the convexity of $\tilde{\mathcal{Q}}W(F,\cdot)$, we obtain

$$\begin{aligned}
&\int_I \int_{Q'} W(F + D_\alpha\varphi|\lambda \; D_3\varphi) \; dx_\alpha \; dx_3 \\
&= \int_I \int_{Q'} W(F + D_\alpha\varphi| \int_{Q'} \lambda \; D_3\varphi \; dx_\alpha + \psi) \; dx_\alpha \; dx_3 \\
&\geq \int_I \tilde{\mathcal{Q}}W(F| \int_{Q'} \lambda \; D_3\varphi \; dx_\alpha) \; dx_3 \\
&\geq \tilde{\mathcal{Q}}W(F|b).
\end{aligned} \tag{47}$$

Taking the infimum in the left hand side of (47), we get $\mathcal{Q}^*W(F|b) \geq \tilde{\mathcal{Q}}W(F|b)$. $\square$

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