Journal of Convex Analysis
Volume 16 (2009), No. 3&4, 1035–1038

On Two Open Problems in Convex Analysis

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Dedicated to Stephen Simons on the occasion of his 70th birthday.

Received: March 5, 2008
Revised manuscript received: October 6, 2008

In his recent book "From Hahn–Banach to Monotonicity" S. Simons formulated several open problems. In this short note we give the answer to Problem 11.6 and a short proof to the answer given by R. I. Boţ and R. Csetnek to Problem 11.5.

In his recent book [3] S. Simons formulated the following two open problems (see [3, Problems 11.5, 11.6]); the notation is compatible with that in [5].

Problem 1. *Let C be a bounded closed convex subset of a Banach space E, x_0 be an extreme point of C, $y^* \in E^*$ and $\varepsilon > 0$. Then does there always exist $M \geq 0$ such that, for all $u, v \in C$, $M \|u + v - 2x_0\| \geq \langle v - x_0, y^*\rangle - \varepsilon$?*

Problem 2. *Do there exist a nonzero finite dimensional Banach space E and $f, g \in \mathcal{PC}(E)$ such that the pair (f, g) is totally Fenchel unstable?*

The answer to Problem 2 is NO, as seen in Proposition 3 below.

Let E be a nonzero Banach space and $f, g \in \mathcal{PC}(E) = \Lambda(E)$, that is, f, g are proper convex functions. One says (see [3]) that (f, g) satisfies *Fenchel duality* if there exists $z^* \in E^*$ such that $f^*(-z^*) + g^*(z^*) = (f + g)^*(0)$; (f, g) is *Fenchel stable* if

$$(f + g)^*(x^*) = \min \{f^*(y^*) + g^*(z^*) \mid y^* + z^* = x^*\} \quad \forall x^* \in E^*. \tag{1}$$

As in [3] one says that the pair (f, g) is *totally Fenchel unstable* if (f, g) satisfies Fenchel duality and

$$y^*, z^* \in E^* \text{ and } f^*(y^*) + g^*(z^*) = (f + g)^*(y^* + z^*) \Rightarrow y^* + z^* = 0.$$

Notice that for $f, g \in \Lambda(E)$ one has

$$(f, g) \text{ Fenchel stable } \Rightarrow (f, g) \text{ satisfies Fenchel duality } \Rightarrow \operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset. \tag{2}$$

The first implication is obvious; just take $x^* = 0$ in (1). If $\operatorname{dom} f \cap \operatorname{dom} g = \emptyset$ then $f + g = \infty$ and so $(f + g)^* = -\infty$; because f^*, g^* don't take the value $-\infty$, the pair (f, g) does not satisfy the Fenchel duality. Moreover, if (f, g) is Fenchel stable then (f, g)

ISSN 0944-6532 / $ 2.50

is not totally Fenchel unstable. Indeed, taking $\overline{x}^* \in X^* \setminus \{0\}$, from (1) we get $\overline{y}^*, \overline{z}^* \in X^*$ with $\overline{y}^* + \overline{z}^* = \overline{x}^* \neq 0$ and $(f+g)^*(\overline{x}^*) = f^*(\overline{y}^*) + g^*(\overline{z}^*)$.

Recall that $\Gamma(E) := \{f \in \Lambda(E) \mid f \text{ is lower semicontinuous}\}$.

Proposition 3. *Let $E \neq \{0\}$ be a finite dimensional normed vector space and $f, g \in \Lambda(E)$. Then the pair (f, g) is not totally Fenchel unstable.*

Proof. In the sequel we assume that (f, g) satisfies Fenchel duality because in the contrary case clearly (f, g) is not totally Fenchel unstable; hence, by (2), $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. We have that

$$\inf(f+g) = \inf \overline{f+g} \geq \inf(\overline{f} + \overline{g}) \geq \sup\{-f^*(-u^*) - g^*(u^*) \mid u^* \in E^*\}, \qquad (3)$$

where $\overline{k}$ is the lsc hull of k. Since $\overline{k}^* = k^*$ we get

$$(f+g)^*(0) \leq (\overline{f} + \overline{g})^*(0) \leq \inf\{f^*(-u^*) + g^*(u^*) \mid u^* \in E^*\}.$$

Since (f, g) satisfies Fenchel duality we get some $\overline{z}^* \in E^*$ such that $(f+g)^*(0) = (\overline{f} + \overline{g})^*(0) = f^*(\overline{z}^*) + g^*(-\overline{z}^*)$. In particular $\inf(f+g) = \inf(\overline{f} + \overline{g})$. Replacing f by $f_0 := f - \overline{z}^*$ and g by $g_0 := f + \overline{z}^*$ if necessary, we may (and do) assume that $\overline{z}^* = 0$. Hence

$$(f+g)^*(0) = (\overline{f} + \overline{g})^*(0) = f^*(0) + g^*(0). \qquad (4)$$

We consider the following 3 cases: (a) $\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri}(\operatorname{dom} g) \neq \emptyset$, (b) $0 \notin \operatorname{ri}(\operatorname{dom}(f+g)^*)$, (c) $\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri}(\operatorname{dom} g) = \emptyset$ and $0 \in \operatorname{ri}(\operatorname{dom}(f+g)^*)$, where $\operatorname{ri} A$ means the interior of A with respect to the affine hull of A.

Case (a). It is known (see e.g. [4, Th. 16.4] or [5, Th. 2.8.7 (viii)]) that (1) holds, that is, (f, g) is Fenchel stable. As seen above, the pair (f, g) is not totally Fenchel unstable.

Case (b). It is obvious that $\operatorname{dom} f^* + \operatorname{dom} g^* \subset \operatorname{dom}(f+g)^*$. Take $-\overline{x}^* \in \operatorname{ri}(\operatorname{dom}(f+g)^*)$ and $\overline{z}^* \in \operatorname{dom} g^*$. Of course, $\overline{x}^* \neq 0$, $\overline{x}^* \notin \operatorname{dom}(f+g)^*$ and $\overline{y}^* := \overline{x}^* - \overline{z}^* \notin \operatorname{dom} f^*$. Hence $f^*(\overline{y}^*) + g^*(\overline{z}^*) = \infty = (f+g)^*(\overline{y}^* + \overline{z}^*)$ and $\overline{y}^* + \overline{z}^* = \overline{x}^* \neq 0$. Therefore, (f, g) is not totally Fenchel unstable.

Recall that if $h \in \Gamma(\mathbb{R}^n)$ and $0 \in \operatorname{ri}(\operatorname{dom} h^*)$ then h attains its infimum.

Case (c). Set $h = \overline{f+g} \in \Gamma(E)$. Since $0 \in \operatorname{ri}(\operatorname{dom} h^*)$, there exists $\overline{x} \in E$ such that

$$f(x) + g(x) \geq \overline{f+g}(x) \geq \overline{f+g}(\overline{x}) = \inf \overline{f+g} = \inf(f+g) = -(f+g)^*(0) \in \mathbb{R}$$

for every $x \in E$. Hence $f^*(0), g^*(0) \in \mathbb{R}$. From (4) we have that $\inf(f+g) = \inf(\overline{f} + \overline{g}) = \inf h$. Therefore,

$$\inf(\overline{f} + \overline{g}) = \inf(f+g) = \overline{f+g}(\overline{x}) \geq \overline{f}(\overline{x}) + \overline{g}(\overline{x})$$

and so $\overline{x} \in \operatorname{dom} \overline{f} \cap \operatorname{dom} \overline{g} \subset \operatorname{cl}(\operatorname{dom} f) \cap \operatorname{cl}(\operatorname{dom} g)$. Since $\operatorname{ri}(\operatorname{dom} \overline{f}) \cap \operatorname{ri}(\operatorname{dom} \overline{g}) = \operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri}(\operatorname{dom} g) = \emptyset$ and $\overline{x} \in \operatorname{dom} \overline{f} \cap \operatorname{dom} \overline{g}$, there exists $\overline{u}^* \in E^* \setminus \{0\}$ such that

$$\langle y, \overline{u}^* \rangle \leq \langle \overline{x}, \overline{u}^* \rangle \leq \langle z, \overline{u}^* \rangle \quad \text{for all } y \in \operatorname{dom} \overline{f},\ z \in \operatorname{dom} \overline{g}.$$

In particular $\langle x, \overline{u}^* \rangle = \langle \overline{x}, \overline{u}^* \rangle$ for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$. We have that $f(x) \geq -f^*(0)$ and $g(x) \geq -g^*(0)$ for all $x \in E$. Then

$$f^*(\overline{u}^*) = \sup\{\langle x, \overline{u}^* \rangle - f(x) \mid x \in \operatorname{dom} f\} \leq \langle \overline{x}, \overline{u}^* \rangle + f^*(0),$$

$$\begin{aligned}(f+g)^*(\overline{u}^*) &= \sup\{\langle x, \overline{u}^*\rangle - f(x) - g(x) \mid x \in \operatorname{dom} f \cap \operatorname{dom} g\} \\ &= \sup\{\langle \overline{x}, \overline{u}^*\rangle - f(x) - g(x) \mid x \in \operatorname{dom} f \cap \operatorname{dom} g\} \\ &= \langle \overline{x}, \overline{u}^*\rangle + (f+g)^*(0) = \langle \overline{x}, \overline{u}^*\rangle + f^*(0) + g^*(0),\end{aligned}$$

whence

$$f^*(\overline{u}^*) + g^*(0) \le \langle \overline{x}, \overline{u}^*\rangle + f^*(0) + g^*(0) = (f+g)^*(\overline{u}^*).$$

Since clearly $(f+g)^*(y^*+z^*) \le f^*(y^*)+g^*(z^*)$, we have that $(f+g)^*(\overline{u}^*) = f^*(\overline{u}^*)+g^*(0)$ with $\overline{u}^* \neq 0$, and so the pair (f,g) is not totally Fenchel unstable in this case, too. □

As S. Simons informed us, R. I. Boţ and R. Csetnek [1] gave the answer to Problem 1. This answer is formulated in the next proposition in the case $x_0 = 0$ (assumption that we do without loss of generality); we give a short proof of this statement.

Proposition 4. *Let C be a bounded closed convex subset of a Banach space E and 0 be an extreme point of C. Then*

$$\forall y^* \in E^*,\ \forall \varepsilon > 0,\ \exists M \ge 0,\ \forall u, v \in C:\ M\|u+v\| \ge \langle v, y^*\rangle - \varepsilon \tag{5}$$

if and only if 0 is an extreme point of $\mathrm{cl}_{w^*} J(C)$, where $J : E \to E^{**}$ is the canonical injection mapping.

Proof. The conclusion of the proposition is an immediate consequence of Facts 1, 2, 3 below.

Fact 1. Consider $C \subset E$ a convex set with $0 \in C$, where E is a real linear space, and set $f = \iota_C$ and $g := \iota_{-C}$, ι_C representing the indicator function of C. Then 0 is an extreme point of C iff $C \cap (-C) = \{0\}$ iff $f + g = \iota_{\{0\}}$.

Fact 2. Consider $C \subset (E, \|\cdot\|)$ a convex set with $0 \in C$; then $h := f^* \ge 0$, $k := g^* \ge 0$ and (5) is equivalent to $h \Box k = 0$.

The fact that $h, k \ge 0$ is obvious (because $0 \in C$); hence $h \Box k \ge 0$.

If $h \Box k = f^* \Box g^* = 0$ and $y^* \in E^*$, $\varepsilon > 0$, then there exists $v^* \in E^*$ such that $f^*(-v^*) + g^*(-y^* + v^*) \le \varepsilon$, whence $\langle u, -v^*\rangle + \langle -v, -y^* + v^*\rangle \le \varepsilon$ for all $u, v \in C$. Hence $\langle v, y^*\rangle - \varepsilon \le \langle u+v, v^*\rangle \le M\|u+v\|$ for all $u, v \in C$ with $M := \|v^*\|$. Therefore (5) holds.

Assume that (5) holds and fix $y^* \in E^*$ and $\varepsilon > 0$. From our hypothesis, there exists $M \ge 0$ such that $M\|u-v\| + f(u) + g(v) + \varepsilon \ge \langle v, -y^*\rangle$ for all $u, v \in E$, that is, $(\phi + \psi)^*(0, -y^*) \le \varepsilon$, where $\phi(u,v) := M\|u-v\|$ and $\psi(u,v) := f(u) + g(v)$. Since ϕ is finite, convex and continuous and ψ is proper and convex, $(\phi+\psi)^* = \phi^* \Box \psi^*$ with exact convolution, and so there exists $u^*, v^* \in E^*$ such that $\phi^*(-u^*, v^*) + \psi^*(u^*, y^* - u^*) \le \varepsilon$, that is, there exists $x^* \in E^*$ with $\|x^*\| \le M$ (and $u^* = v^* = x^*$) such that $f^*(x^*) + g^*(y^* - x^*) \le \varepsilon$. Hence $(h \Box k)(y^*) \le \varepsilon$. As $y^* \in E^*$ and $\varepsilon > 0$ are arbitrary, we get $h \Box k \le 0$, and so $h \Box k = 0$.

Fact 3. Consider $C \subset (E, \|\cdot\|)$ a bounded convex set with $0 \in C$. Then h, k are finite and norm-continuous convex functions and $h \Box k = 0$ if and only if 0 is an extreme point of $\mathrm{cl}_{w^*} J(C)$ (in E^{**}).

The functions h, k are (convex) finite and norm-continuous because C is bounded. Since $h, k \geq 0$, we obtain that $h \square k$ is finite and continuous (being bounded from below by 0). Therefore, $h \square k = (h \square k)^{**} = (h^* + k^*)^*$ for the dual system (X^*, X^{**}). Moreover, $h^* = \iota_{\mathrm{cl}_{w^*} J(C)}$ and $k^* = \iota_{\mathrm{cl}_{w^*}(J(-C))}$. Hence, for the dual system (X^*, X^{**}), and taking into account that $\iota_{\mathrm{cl}_{w^*} J(C)}$, $\iota_{\mathrm{cl}_{w^*}(J(-C))}$ are proper w^*–lsc convex functions with proper sum, we have

$$\begin{aligned} h \square k = 0 &\iff \left(\iota_{\mathrm{cl}_{w^*} J(C)} + \iota_{\mathrm{cl}_{w^*} J(-C)}\right)^* = 0 \\ &\iff \iota_{\mathrm{cl}_{w^*} J(C)} + \iota_{\mathrm{cl}_{w^*} J(-C)} = \iota_{\{0\}} \\ &\iff [\mathrm{cl}_{w^*} J(C)] \cap [\mathrm{cl}_{w^*} J(-C)] = \{0\} \\ &\iff [\mathrm{cl}_{w^*} J(C)] \cap [-\mathrm{cl}_{w^*} J(C)] = \{0\}. \end{aligned}$$

Hence $h \square k = 0$ if and only if 0 is an extreme point of $\mathrm{cl}_{w^*} J(C)$. $\square$

Acknowledgements. R. I. Boţ and A. Löhne [2] gave another solution to [3, Problem 11.6].

References

[1] R. I. Boţ, R. Csetnek: On an open problem regarding totally Fenchel unstable functions, Proc. Amer. Math. Soc. 137 (2009) 1801–1805.

[2] R. I. Boţ, A. Löhne: On totally Fenchel unstable functions in finite dimensional spaces, Math. Program., to appear.

[3] S. Simons: From Hahn–Banach to Monotonicity, Springer, Berlin (2008).

[4] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton (1970).

[5] C. Zălinescu: Convex Analysis in General Vector Spaces, World Scientific, Singapore (2002).