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Maximal Monotone Operators with a Unique Extension to the Bidual

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We study a sufficient condition under which a maximal monotone operator $T : X \rightrightarrows X^*$ admits a unique maximal monotone extension to the bidual $\widetilde{T} : X^{**} \rightrightarrows X^*$. We will prove that for non-linear operators this condition is equivalent to uniqueness of the extension. The central tool in our approach is the $\mathcal{S}$-function defined and studied by Burachik and Svaiter in 2000 [9](submission date: July 2000). For a generic operator, this function is the supremum of all convex lower semicontinuous functions which are majorized by the duality product in the graph of the operator.

We also prove in this work that if the graph of a maximal monotone operator is convex, then this graph is an affine linear subspace.

Keywords: Maximal monotone operators, extension, bidual, Banach spaces, Brøndsted-Rockafellar property, $\mathcal{S}$-function

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1. Introduction

Let X be a real Banach space. We use the notation X^* for the topological dual of X and $\pi_{X\times X^*}$, $\langle \cdot, \cdot \rangle_{X\times X^*}$ for the duality product

$$\pi_{X\times X^*}(x, x^*) = \langle x, x^* \rangle_{X\times X^*} = x^*(x).$$

Whenever the underlying domain of the duality product is clear, we will use the notations π and $\langle \cdot, \cdot \rangle$. In $X \times X^*$ we shall use the strong topology.

A point-to-set operator $T : X \rightrightarrows X^*$ (respectively $T : X^{**} \rightrightarrows X^*$) is a relation on X to X^* (respectively on X^{**} to X^*):

$$T \subset X \times X^* \quad \text{(respectively } T \subset X^{**} \times X^*\text{)},$$

and $r \in T(q)$ means $(q, r) \in T$. An operator $T : X \rightrightarrows X^*$, or $T : X^{**} \rightrightarrows X^*$, is *monotone* if

$$\langle q - q', r - r' \rangle \geq 0, \quad \forall (q, r), (q', r') \in T.$$

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An operator $T : X \rightrightarrows X^*$ is *maximal monotone* (in $X \times X^*$) if it is monotone and maximal (whit respect to the inclusion) in the family of monotone operators of X in X^*. An operator $T : X^{**} \rightrightarrows X^*$ is maximal monotone (in $X^{**} \times X^*$) if it is monotone and maximal (with respect to the inclusion) in the family of monotone operators of X^{**} in X^*.

The canonical injection of X into X^{**} allows one to identity X with a subset of X^{**}. Therefore, any maximal monotone operator $T : X \rightrightarrows X^*$ is also a monotone operator $T : X^{**} \rightrightarrows X^*$ and admits one (or more) maximal monotone extension in $X^{**} \times X^*$. In general this maximal monotone extension will not be unique. We are concerned with the problem:

> Under which conditions a maximal monotone operator $T : X \rightrightarrows X^*$ has a unique extension to the bidual, $X^{**} \rightrightarrows X^*$?

The problem of unicity of maximal extension of a generic monotone operator was studied in details by Martínez-Legaz and Svaiter in [22]. That paper will be an important reference for the present work.

The specific problem above mentioned, of uniqueness of extension of a *maximal monotone* operator to the bidual, has been previously addressed by Gossez [14, 15, 16, 17]. He found a condition under which uniqueness of the extension is guaranteed [17]. Latter the condition NI, was studied in [25]. This condition guarantees the uniqueness of the extension to the bidual and encompasses Gossez type D class. An open question is whether condition NI implies the Brøndsted-Rockafellar property.

Now we will discuss the Brøndsted-Rockafellar property. Let $T \subset X \times X^*$ be maximal monotone. Burachik, Iusem and Svaiter [5] defined the T^ε enlargement of T for $\varepsilon \geq 0$, as $T^\varepsilon : X \rightrightarrows X^*$

$$T^\varepsilon(x) = \{x^* \in X^* \mid \langle x - y, x^* - y^*\rangle \geq -\varepsilon \ \forall (y, y^*) \in T\}. \tag{1}$$

It is trivial to verify that $T \subset T^\varepsilon$. The T^ε enlargement is a generalization of the ε-subdifferential. As the ε-subdifferential, the T^ε has also practical uses [26, 27, 12, 18, 19, 20]. Brøndsted and Rockafellar proved that the ε-subdifferential may be seen as an approximation of the *exact* subdifferential at a nearby point. This property, may be extended to the context of maximal monotone operators. A maximal monotone operator has the *Brøndsted-Rockafellar property* if, for any $\varepsilon, \lambda > 0$,

$$x^* \in T^\varepsilon(x) \Rightarrow \forall \lambda > 0, \ \exists (\bar{x}, \bar{x}^*) \in T, \qquad \|x - \bar{x}\| \leq \lambda, \qquad \|\bar{x}^* - x^*\| \leq \varepsilon/\lambda.$$

Torralba [30] and Burachik and Svaiter [6] proved independently that in a *reflexive* Banach space, all maximal monotone operators satisfy this property. The operator T satisfies the *restricted Brøndsted-Rockafellar property* [21] if

$$\begin{gathered} x^* \in T^\varepsilon(x), \ \tilde{\varepsilon} > \varepsilon \Rightarrow \forall \lambda > 0, \ \exists (\bar{x}, \bar{x}^*) \in T, \\ \|x - \bar{x}\| < \lambda, \qquad \|\bar{x}^* - x^*\| < \tilde{\varepsilon}/\lambda. \end{gathered} \tag{2}$$

In a recent work [21] the authors defined a general class of maximal monotone operators in non-reflexive Banach spaces which satisfies the above property.

In this paper we will prove that operators type NI satisfies the restricted Brøndsted-Rockafellar property and we will study the properties of Fitzpatrick families of these operators. We will also prove that, for non-linear operators, the condition NI is equivalent to the unicity of maximal monotone extension to the bidual. For proving this equivalence we will show that if $T \subset X \times X^*$ is maximal monotone and convex then T is an affine subspace of $X \times X^*$.

As we will study the relation of condition NI with general properties of Fitzpatrick family (defined below), the main properties of this families, systematically studied in the series [28, 9, 29, 10, 21] will also be used. Since this field has been the subject of intense research, to clarify previous issues, we will include submission date of some works where these properties were obtained.

Given a maximal monotone operator $T : X \rightrightarrows X^*$, Fitzpatrick defined [13] the family $\mathcal{F}_T$ as those convex, lower semicontinuous functions in $X \times X^*$ which are bounded below by the duality product and coincides with it at T:

$$\mathcal{F}_T = \left\{ h \in \bar{\mathbb{R}}^{X\times X^*} \;\middle|\; \begin{array}{l} h \text{ is convex and lower semicontinuous} \\ \langle x, x^*\rangle \le h(x,x^*),\ \forall (x,x^*) \in X \times X^* \\ (x,x^*) \in T \Rightarrow h(x,x^*) = \langle x, x^*\rangle \end{array} \right\}. \tag{3}$$

Fitzpatrick found an explicit formula for the minimal element of $\mathcal{F}_T$, from now on the *Fitzpatrick function* of T:

$$\varphi_T(x,x^*) = \sup_{(y,y^*)\in T} \langle x, y^*\rangle + \langle y, x^*\rangle - \langle y^*, y\rangle. \tag{4}$$

Note that in the above definition, T may be a generic subset of $X \times X^*$.

The *conjugate* of a function $f : X \to \bar{\mathbb{R}}$ is defined as $f^* : X^* \to \bar{\mathbb{R}}$,

$$f^*(x^*) = \sup_{x\in X} \langle x, x^*\rangle - f(x),$$

and the *convex closure* of f is $\operatorname{cl\,conv} f : X \to \bar{\mathbb{R}}$, the largest convex lower semicontinuous function majorized by f:

$$\operatorname{cl\,conv} f(x) := \sup\{h(x) \mid h \text{ convex, lower semicontinuous, } h \le f\}.$$

The *effective domain* of $f : X \to \bar{\mathbb{R}}$ is

$$\operatorname{ed}(f) = \{x \in X \mid f(x) < \infty\}.$$

The indicator function of $A \subset X$ is $\delta_{A,X} : X \to \bar{\mathbb{R}}$,

$$\delta_{A,X}(x) = \begin{cases} 0 & x \in A \\ \infty & \text{otherwise.} \end{cases}$$

Whenever the set X is implicitly defined, we use the notation δ_A.

Definition 1.1 ([9, Eq. 35], [8, Eq. 29]).
The $\mathcal{S}$-function (original notation $\Lambda_{\overline{ST}}$) associated with a maximal monotone operator $T : X \rightrightarrows X^*$ is $\mathcal{S}_T : X \times X^* \to \bar{\mathbb{R}}$

$$\mathcal{S}_T = \operatorname{cl\,conv}(\pi + \delta_T). \tag{5}$$

The $\mathcal{S}$-function will be central to this article. Hence, a discussion of it origin is appropriate: This function was defined in Burachik Svaiter paper [9], which was submitted for publication in July 2000, and was published online in 2001 [8] at IMPA preprint server http://www.preprint.impa.br . Some authors [3, 31, 4] attribute the $\mathcal{S}$-function to [23] although [23] was submitted after the publication of [9]. Moreover, the content of [9], and specifically the $\mathcal{S}$-function, was presented on Erice workshop on July 2001, by R. S. Burachik [7]. A list of the talks of this congress, which includes [24], is available on the www[1].

In [9, 8] it is proved that the $\mathcal{S}$-function is the *supremum* of the family of Fitzpatrick function. The epigraphical structure of the $\mathcal{S}$-function was previously studied in [28] (submission date: September 1999). This function will be central for the new characterization of the class NI and for proving the main result of this work.

The $\mathcal{S}$-function and Fitzpatrick function are still well defined for arbitrary sets (or operators) $T \subset X \times X^*$:

$$\mathcal{S}_T : X \times X^* \to \bar{\mathbb{R}}, \qquad \mathcal{S}_T = \operatorname{cl}\operatorname{conv}(\pi + \delta_T), \tag{6}$$

$$\varphi_T : X \times X^* \to \bar{\mathbb{R}}, \qquad \varphi_T(x, x^*) = \sup_{(y,y^*)\in T} \langle x, y^*\rangle + \langle y, x^*\rangle - \langle y^*, y\rangle. \tag{7}$$

Martínez-Legaz and Svaiter also studied in [22] generic properties of $\mathcal{S}$ (with the notation σ_T) and φ_T for arbitrary sets and its relation with monotonicity and maximal monotonicity.

First let us recall the definition of operators of type NI:

Definition 1.2 ([25]). Let X be a real Banach space. A maximal monotone operator $T : X \rightrightarrows X^*$ is of type *NI* if

$$\inf_{(y,y^*)\in T} \langle y^* - x^*, y - x^{**}\rangle \le 0, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

Now we use the $\mathcal{S}$-function to give a new characterization of operators the type NI.

Proposition 1.3. *Let X be a real Banach space and $T : X \rightrightarrows X^*$ be maximal monotone. Then T is of type* NI *if, and only if,*

$$(\mathcal{S}_T)^*(x^*, x^{**}) \ge \langle x^*, x^{**}\rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

To simplify the notation, define

$$\Lambda : X^{**} \times X^* \to X^* \times X^{**}, \quad \Lambda(x^{**}, x^*) = (x^*, x^{**}).$$

Note that $\Lambda(X \times X^*) = X^* \times X$. We will prove three main results in this paper:

Theorem 1.4. *Let X be a generic Banach space and $T : X \rightrightarrows X^*$ a maximal monotone operator of type* NI*, which is equivalent to*

$$(\mathcal{S}_T)^*(x^*, x^{**}) \ge \langle x^*, x^{**}\rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}. \tag{8}$$

Then

[1] http://www.polyu.edu.hk/∼ama/events/conference/EriceItaly-OCA2001/Abstract.html

1. *T admits a unique maximal monotone extension $\widetilde{T} : X^{**} \rightrightarrows X^*$,*
2. *$(\mathcal{S}_T)^* = \varphi_{\Lambda\widetilde{T}}$*
3. *for all $h \in \mathcal{F}_T$,*

$$h^*(x^*, x^{**}) \geq \langle x^*, x^{**}\rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}; \qquad h^* \in \mathcal{F}_{\Lambda\widetilde{T}}\,.$$

4. *T satisfies the restricted Brøndsted-Rockafellar property.*

Item 1 on the above theorem was proved in [25]. The last statement of the above theorem is a particular case of a more general result proved in [21]. In that paper, it is proved that if a convex lower semicontinuous function in $X \times X^*$ and its conjugate majorizes the duality product in $X \times X^*$ and $X^* \times X^{**}$, respectively, then this function is in the Fitzpatrick family of a maximal monotone operator and this maximal monotone operator satisfies the restricted Brøndsted-Rockafellar condition. This result, in a reflexive Banach space was previously obtained in [10].

A natural question is whether the converse of Item 1 of Theorem 1.4 holds. To give a partial answer to this question, first recall that a linear (affine) subspace of a real linear space Z is a set $A \subset Z$ such that there exists V, subspace of Z, and a point z_0 such that

$$A = V + \{z_0\} = \{z + z_0 \mid z \in V\}.$$

We will need an auxiliary result, which is the second main result of this paper. It states that a convex maximal monotone operator is "essentially" linear:

Lemma 1.5. *If $T : X \rightrightarrows X^*$ is maximal monotone and convex, then T is affine linear.*

This lemma generalizes a result of Burachik and Iusem [11, Lemma 2.14], which states that if a point-to-point maximal monotone operator is convex and its domain has a non-empty interior, then the operator is affine. Burachik and Iusem also proved that under these assumptions, the operator is defined in the whole space. After the submission of the first version of this work, Bauschke, Wang and Yao published, in the arXiv.org preprint server, a preprint [1] with the same result of Lemma 1.5.

The partial converse of Theorem 1.4 is the third main result of this paper.

Theorem 1.6. *Suppose that $T : X \rightrightarrows X^*$ is maximal monotone and has a unique maximal monotone extension to $X^{**} \times X^*$. Then either*

$$(\mathcal{S}_T)^*(x^*, x^{**}) \geq \langle x^*, x^{**}\rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}, \tag{9}$$

that is, T is of type NI*, or T is affine linear and $T = \mathrm{ed}(\varphi_T)$.*

According to the above theorems, for non-linear maximal monotone operators, condition (8) is equivalent to unicity of maximal monotone extension to the bidual.

2. Convexity and maximal monotonicity

In this section we will prove Lemma 1.5.

Proof of Lemma 1.5. Take an arbitrary $(x_0, x_0^*) \in T$ and define

$$T_0 = T - \{(x_0, x_0^*)\}\,.$$

Note that T_0 is maximal monotone and convex. So, it suffices to prove that T_0 is a linear subspace of $X \times X^*$. Take an arbitrary $(x, x^*) \in T_0$. First we claim that

$$t(x, x^*) \in T_0, \quad \forall t \geq 0. \tag{10}$$

For $0 \leq t \leq 1$ the above inclusion holds because $(0, 0) \in T_0$ and T_0 is convex. For the case $t \geq 1$ let $(y, y^*) \in T$. Then, $t^{-1}(y, y^*) \in T_0$ and so

$$\langle x - t^{-1}y, x^* - t^{-1}y^* \rangle \geq 0.$$

Multiplying this inequality by t we conclude that $\langle tx - y, tx^* - y^* \rangle \geq 0$. As (y, y^*) is a generic element of T_0, which is maximal monotone, we conclude that $t(x, x^*) \in T_0$ and the claim (10) holds.

We have just proved that T_0 is a convex cone. Now take an arbitrary pair

$$(x, x^*), (y, y^*) \in T_0.$$

Then

$$(x + y, x^* + y^*) = 2 \left[\frac{1}{2}(x, x^*) + \frac{1}{2}(y, y^*) \right] \in T_0. \tag{11}$$

As $(0, 0) \in T_0$, we have

$$\langle y - (-x), y^* - (-x^*) \rangle = \langle (y + x) - 0, (y^* + x^*) - 0 \rangle \geq 0.$$

Since T_0 is maximal monotone, we conclude that $-(x, x^*) \in T_0$. Therefore, using again (10) we conclude that T_0 is closed under scalar multiplication. To end the proof, combine this result with (11) to conclude that T_0 is a linear subspace. □

3. Preliminary results

As mentioned before, Fitzpatrick proved that the family $\mathcal{F}_T$ is non-empty by producing its smallest element φ_T. Fitzpatrick also proved that *any* function in this family fully characterizes the maximal monotone operator which defines the family:

Theorem 3.1 ([13]). *Let $T : X \rightrightarrows X^*$ be maximal monotone. Then, for any $h \in \mathcal{F}_T$,*

$$h(x, x^*) = \langle x, x^* \rangle \iff (x, x^*) \in T, \quad \forall (x, x^*) \in X \times X^*.$$

Moreover, φ_T is the smallest function of $\mathcal{F}_T$.

In [9](submission date: July 2000), Burachik and Svaiter proved

Theorem 3.2 ([9, Eqns. 32, 37, 39]). *Let $T : X \rightrightarrows X^*$ be maximal monotone and $\mathcal{S}_T$ be the $\mathcal{S}$-function associated with T, as defined in* (5). *Then, $\mathcal{S}_T \in \mathcal{F}_T$ and*

$$\varphi_T \leq h \leq \mathcal{S}_T, \quad \forall h \in \mathcal{F}_T.$$

Moreover, φ_T and $\mathcal{S}_T$ are related as follows:

$$\varphi_T(x, x^*) = (\mathcal{S}_T)^*(x^*, x), \quad \forall (x, x^*) \in X \times X^*.$$

Define, for $h : X \times X^* \to \bar{\mathbb{R}}$,

$$\mathcal{J}h : X \times X^* \to \bar{\mathbb{R}}, \qquad \mathcal{J}h(x, x^*) = h^*(x^*, x). \tag{12}$$

According to the above theorem, $\mathcal{J}\mathcal{S}_T = \varphi_T \in \mathcal{F}_T$. So, it is natural to ask whether $\mathcal{J}$ maps $\mathcal{F}_T$ into itself. Burachik and Svaiter also proved that this happens in fact:

Theorem 3.3 ([9, Theorem 5.3]). *Suppose that T is maximal monotone. Then*

$$\mathcal{J}h \in \mathcal{F}_T, \quad \forall h \in \mathcal{F}_T,$$

that is, if $h \in \mathcal{F}_T$, and

$$g : X \times X^* \to \bar{\mathbb{R}}, \qquad g(x, x^*) = h^*(x^*, x),$$

then $g \in \mathcal{F}_T$.

In a reflexive Banach space $\mathcal{J}\varphi_T = \mathcal{S}_T$.

It is interesting to note that $\mathcal{J}$ is an order-reversing mapping of $\mathcal{F}_T$ into itself. This fact suggests that this mapping may have fixed points in $\mathcal{F}_T$. Svaiter proved [29](submission date: July 2002) that if T is maximal monotone, then $\mathcal{J}$ has always has a fixed point in $\mathcal{F}_T$.

Martínez-Legaz and Svaiter [22] observed that for a generic $T \subset X \times X^*$

$$\varphi_T(x, x^*) = (\pi + \delta_T)^*(x^*, x) = (\mathcal{S}_T)^*(x^*, x), \quad \forall (x, x^*) \in X \times X^*, \tag{13}$$

Therefore, also for an arbitrary T, one has $\mathcal{J}\mathcal{S}_T = \varphi_T$.

It will be useful to define a relation μ which characterizes monotonicity and study monotonicity in the framework of this relation and the classical notion of polarity [2]. Recall that a relation in a set V is a subset of $V \times V$.

Definition 3.4 ([22]). The *monotone relation* in $X \times X^*$, notation μ, is

$$\mu = \{((x, x^*), (y, y^*)) \in (X \times X^*)^2 \mid \langle x - y, x^* - y^* \rangle \geq 0\}.$$

Two points $(x, x^*), (y, y^*) \in X \times X^*$ are *monotone related* or *in monotone relation* if $(x, x^*)\ \mu\ (y, y^*)$, that is,

$$\langle x - y, x^* - y^* \rangle \geq 0.$$

Given $A \subset X \times X^*$, the *monotone polar* (in $X \times X^*$) of A is the set A^μ,

$$\begin{aligned} A^\mu &= \{(x, x^*) \in X \times X^* \mid (x, x^*)\ \mu\ (y, y^*),\ \forall (y, y^*) \in A\}, \\ &= \{(x, x^*) \in X \times X^* \mid \langle x - y, x^* - y^* \rangle \geq 0,\ \forall (y, y^*) \in A\}. \end{aligned} \tag{14}$$

We shall need some results of Martínez-Legaz and Svaiter which are scattered along [22] and which we expound in the next two theorems:

Theorem 3.5 ([22, Eq. 22, Prop. 2, Prop. 21]). *Let $A \subset X \times X^*$. Then*

$$A^\mu = \{(x, x^*) \in X \times X^* \mid \varphi_T(x, x^*) \leq \langle x, x^* \rangle\}, \tag{15}$$

and the following conditions are equivalent

1. *A is monotone,*
2. *$\varphi_A \leq (\pi + \delta_A)$.*
3. *$A \subset A^\mu$.*

Moreover, A is maximal monotone if and only if $A = A^\mu$.

Note in the above theorem and in the definition of Fitzpatrick's family, the convenience of defining as in [22, Eq. 12 and below], for $h : X \times X^* \to \bar{\mathbb{R}}$:

$$\begin{aligned} b(h) &:= \{(x, x^*) \in X \times X^* \mid h(x, x^*) \leq \langle x, x^* \rangle\}, \\ L(h) &:= \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle\}. \end{aligned} \tag{16}$$

Theorem 3.6 ([22, Prop. 36, Lemma 38]). *Suppose that $A \subset X \times X^*$ is monotone. Then the following conditions are equivalent*

1. *A has a unique maximal monotone extension (in $X \times X^*$),*
2. *A^μ is monotone*
3. *A^μ is maximal monotone,*

and if any of these conditions holds, then A^μ is the unique maximal monotone extension of A.

Moreover, still assuming only A monotone,

$$\varphi_A \geq \pi \iff b(\varphi_A) = L(\varphi_A) \tag{17}$$

and if these conditions hold, then A has a unique maximal monotone extension, A^μ.

4. Proof of Theorems 1.4 and 1.6

From now on, $T : X \rightrightarrows X^*$ is a maximal monotone operator. The inverse of T is $T^{-1} : X^* \rightrightarrows X$,

$$T^{-1} = \{(x^*, x) \in X^* \times X \mid (x, x^*) \in T\}. \tag{18}$$

Note that $T^{-1} \subset X^* \times X \subset X^* \times X^{**}$. Fitzpatrick function of T^{-1}, regarded as a subset of $X^* \times X^{**}$ is, according to (7)

$$\begin{aligned} \varphi_{T^{-1}, X^* \times X^{**}}(x^*, x^{**}) &= \sup_{(y^*, y^{**}) \in T^{-1}} \langle x^*, y^{**} \rangle + \langle y^*, x^{**} \rangle - \langle y^*, y^{**} \rangle \\ &= \sup_{(y^*, y) \in T^{-1}} \langle x^*, y \rangle + \langle y^*, x^{**} \rangle - \langle y^*, y \rangle \\ &= (\pi + \delta_T)^* (x^*, x^{**}). \end{aligned}$$

where the last x^* is identified with its image under the canonical injection of X^* into X^{***}. Using the above equations, (6) and the fact that conjugation is invariant under the convex-closure operation we obtain

$$\varphi_{T^{-1}, X^* \times X^{**}} = (\pi + \delta_T)^* = (\mathcal{S}_T)^*, \tag{19}$$

where $\pi = \pi_{X \times X^*}$ and $\delta_T = \delta_{T, X \times X^*}$.

We will use the notation $(T^{-1})^{\mu,X^*\times X^{**}}$ for denoting the monotone polar of T^{-1} in $X^* \times X^{**}$. Combing the above equation with Theorem 3.5 we obtain a simple expression for this monotone polar:

$$\big(T^{-1}\big)^{\mu,X^*\times X^{**}} = \{(x^*,x^{**}) \in X^* \times X^{**} \,|\, (\mathcal{S}_T)^*(x^*,x^{**}) \le \langle x^*,x^{**}\rangle\}. \tag{20}$$

Proof of Theorem 1.4. Combining assumption (8) and (19) we have

$$\varphi_{T^{-1},X^*\times X^{**}}(x^*,x^{**}) = (\mathcal{S}_T)^*(x^*,x^{**}) \ge \langle x^*,x^{**}\rangle, \quad \forall (x^*,x^{**}) \in X^* \times X^{**}.$$

Therefore, using Theorem 3.6 and Theorem 3.5 for $A = T^{-1} \subset X^* \times X^{**}$ we conclude that $(T^{-1})^{\mu,X^*\times X^{**}}$, the monotone polar of T^{-1} in $X^* \times X^{**}$, is the *unique* maximal monotone extension of T^{-1} to $X^* \times X^{**}$ and

$$\big(T^{-1}\big)^{\mu,X^*\times X^{**}} = \{(x^*,x^{**}) \in X^* \times X^{**} \mid (\mathcal{S}_T)^*(x^*,x^{**}) = \langle x^*,x^{**}\rangle\}. \tag{21}$$

Using the above result and again (8), we conclude that

$$(\mathcal{S}_T)^* \in \mathcal{F}_{(T^{-1})^{\mu,X^*\times X^{**}}}.$$

Now, define

$$\widetilde{T} = \{(x^{**},x^*) \in X^{**} \times X^* \mid (x^*,x^{**}) \in \big(T^{-1}\big)^{\mu,X^*\times X^{**}}\}. \tag{22}$$

Note that $\Lambda T = T^{-1}$ and $\Lambda\,\widetilde{T} = (T^{-1})^{\mu,X^*\times X^{**}}$. Therefore

$$(\mathcal{S}_T)^* \in \mathcal{F}_{\Lambda\,\widetilde{T}}\,. \tag{23}$$

Moreover, as Λ is a bijection which preserves the duality product, we conclude that $\widetilde{T}$ is the unique maximal monotone extension of T in $X^{**} \times X^*$. It proves Item 1.

Since $T \subset \widetilde{T}$,

$$\begin{aligned}
\varphi_{\Lambda\widetilde{T}}(x^*,x^{**}) &= \sup_{(y^*,y^{**})\in\Lambda\widetilde{T}} \langle x^*,y^{**}\rangle + \langle y^*,x^{**}\rangle - \langle y^*,y^{**}\rangle \\
&= \sup_{(y^{**},y^*)\in\widetilde{T}} \langle x^*,y^{**}\rangle + \langle y^*,x^{**}\rangle - \langle y^*,y^{**}\rangle \\
&\ge \sup_{(y,y^*)\in T} \langle y,x^*\rangle + \langle y^*,x^{**}\rangle - \langle y,y^*\rangle \quad = (\pi + \delta_T)^*(x^*,x^{**}).
\end{aligned}$$

Combining the above equation with the second equality in (19) we conclude that $\varphi_{\Lambda\widetilde{T}} \ge (\mathcal{S}_T)^*$. Using also the fact that $\varphi_{\Lambda\widetilde{T}}$ is minimal in $\mathcal{F}_{\Lambda\,\widetilde{T}}$ and (23) we obtain $\varphi_{\Lambda\widetilde{T}} = (\mathcal{S}_T)^*$. It proves Item 2.

By Theorem 3.2, $\varphi_T(x,x^*) = (\mathcal{S}_T)^*(x^*,x)$. Therefore,

$$\begin{aligned}
(\varphi_T)^*(x^*,x^{**}) &= \sup_{(y,y^*)\in X\times X^*} \langle y,x^*\rangle + \langle y^*,x^{**}\rangle - \varphi_T(y,y^*) \\
&= \sup_{(y,y^*)\in X\times X^*} \langle y,x^*\rangle + \langle y^*,x^{**}\rangle - (\mathcal{S}_T)^*(y^*,y) \\
&\le \sup_{(y^{**},y^*)\in X^{**}\times X^*} \langle y^{**},x^*\rangle + \langle y^*,x^{**}\rangle - (\mathcal{S}_T)^*(y^*,y^{**}) \\
&= (\mathcal{S}_T)^{**}(x^{**},x^*).
\end{aligned}$$

Take $h \in \mathcal{F}_T$. By Theorem 3.2 one has $\varphi_T \leq h \leq s_T$. Using also the fact that conjugation reverts the order, the above equation and assumption (8) we conclude that, for any (x^*, x^{**}),

$$\langle x^*, x^{**}\rangle \leq \mathcal{S}_T^*(x^*, x^{**}) \leq h^*(x^*, x^{**}) \leq \varphi_T^*(x^*, x^{**}) \leq (\mathcal{S}_T)^{**}(x^{**}, x^*). \tag{24}$$

Define $g : X^* \times X^{**}$ as $g = {}_{X^*\times X^{**}}\,(\mathcal{S}_T)^*$, that is,

$$g(x^*, x^{**}) = ((\mathcal{S}_T)^*)^*(x^{**}, x^*).$$

Using (23) and Theorem 3.3 we conclude that $g \in \mathcal{F}_{\Lambda\widetilde{T}}$. Therefore, using again the maximal monotonicity of $\Lambda\widetilde{T}$ in $X^* \times X^{**}$, we have

$$g(x^*, x^{**}) = (\mathcal{S}_T)^{**}(x^{**}, x^*) = \langle x^*, x^{**}\rangle, \quad \forall (x^*, x^{**}) \in \Lambda\widetilde{T}.$$

Combining the above equations with (24) we conclude that h^* majorizes the duality product in $X^* \times X^{**}$ and coincides with it in $\Lambda\widetilde{T}$. As h^* is also convex and closed, we have $h^* \in \mathcal{F}_{\Lambda\,\widetilde{T}}$. It proves Item 3.

The fact that T satisfies the restricted Brøndsted-Rockafellar property follows from the assumption on $\mathcal{S}_T$ and [21, Theorem 4.2]. □

Proof of Theorem 1.6. Suppose there exists only one $\widetilde{T} \subset X^{**} \times X^*$ maximal monotone extension of T to $X^{**} \times X^*$. If T is not of type NI, there exists $(x_0^*, x_0^{**}) \in X^* \times X^{**}$ such that

$$\mathcal{S}_T^*(x_0^*, x_0^{**}) < \langle x_0^*, x_0^{**}\rangle. \tag{25}$$

As Λ is a bijection that preserves the duality product and $\Lambda T = T^{-1}$, we conclude that $\Lambda\widetilde{T}$ is the unique maximal monotone extension of T^{-1} to $X^* \times X^{**}$. Using now Theorem 3.6, Theorem 3.5 and (19) we obtain

$$\begin{aligned}\Lambda\widetilde{T} &= \left(T^{-1}\right)^{\mu, X^*\times X^{**}}\\ &= \{(x^*, x^{**}) \in X^* \times X^{**} \mid \varphi_{T^{-1}, X^*\times X^{**}}(x^*, x^{**}) \leq \langle x^*, x^{**}\rangle\}\\ &= \{(x^*, x^{**}) \in X^* \times X^{**} \mid \mathcal{S}_T^*(x^*, x^{**}) \leq \langle x^*, x^{**}\rangle\}.\end{aligned} \tag{26}$$

Suppose that

$$(\mathcal{S}_T)^*(x^*, x^{**}) < \infty. \tag{27}$$

Define, for $t \in \mathbb{R}$,

$$p(t) := (x_0^*, x_0^{**}) + t(x^* - x_0^*, x^{**} - x_0^{**}) = (1-t)(x_0^*, x_0^{**}) + t(x^*, x^{**}).$$

As $(\mathcal{S}_T)^*$ is convex, we have the inequality

$$\begin{aligned}&(\mathcal{S}_T)^*(p(t)) - \pi_{X^*\times X^{**}}(p(t))\\ &\leq (1-t)(\mathcal{S}_T)^*(x_0^*, x_0^{**}) + t(\mathcal{S}_T)^*(x^*, x^{**}) - \pi_{X^*\times X^{**}}(p(t)), \quad \forall t \in [0,1].\end{aligned}$$

As the duality product is continuous, the limit of the right hand side of this inequality, for $t \to 0+$ is $(\mathcal{S}_T)^*(x_0^*, x_0^{**}) - \langle x_0^*, x_0^{**}\rangle < 0$. Combining this fact with (26) we conclude that for $t \geq 0$ and small enough,

$$(x_0^*, x_0^{**}) + t(x^* - x_0^*, x^{**} - x_0^{**}) \in \Lambda\widetilde{T}.$$

Altogether, we proved that

$$\begin{aligned}(\mathcal{S}_T)^*(x^*, x^{**}) < \infty \Rightarrow \exists \bar{t} > 0,\ \forall t \in [0, \bar{t}]\\ (x_0^*, x_0^{**}) + t(x^* - x_0^*, x^{**} - x_0^{**}) \in \Lambda\widetilde{T}.\end{aligned} \tag{28}$$

Now, suppose that

$$(\mathcal{S}_T)^*(x_1^*, x_1^{**}) < \infty, \qquad \mathcal{S}_T^*(x_2^*, x_2^{**}) < \infty.$$

Then, using (28), we conclude that there exists $t > 0$ such that

$$(x_0^*, x_0^{**}) + t(x_1^* - x_0^*, x_1^{**} - x_0^{**}) \in \Lambda\widetilde{T}, \qquad (x_0^*, x_0^{**}) + t(x_2^* - x_0^*, x_2^{**} - x_0^{**}) \in \Lambda\widetilde{T}\,.$$

As $\Lambda\widetilde{T}$ is (maximal) monotone, the above points are monotone related (in the sense of Definition 3.4) and

$$t^2\langle x_1^* - x_2^*, x_1^{**} - x_2^{**}\rangle \geq 0.$$

Hence, $\langle x_1^* - x_2^*, x_1^{**} - x_2^{**}\rangle \geq 0$. Therefore the set

$$W := \{(x^*, x^{**}) \in X^* \times X^{**} \mid (\mathcal{S}_T)^*(x^*, x^{**}) < \infty\},$$

is monotone. By (26), $\Lambda\widetilde{T} \subset W$. Hence $W = \Lambda\widetilde{T}$ and

$$\widetilde{T} = \{(x^{**}, x^*) \in X^{**} \times X^* \mid (\mathcal{S}_T)^*(x^*, x^{**}) < \infty\}\,.$$

As $(\mathcal{S}_T)^*$ is convex, W is also convex. Therefore, $\Lambda\widetilde{T}$ is convex and maximal monotone. Now, using Lemma 1.5 we conclude that $\Lambda\widetilde{T}$ is affine. This also implies that $\widetilde{T}$ is affine linear. As

$$T = \widetilde{T} \cap X \times X^*,$$

we conclude that T is affine and

$$\begin{aligned}T &= \{(x, x^*) \mid (\mathcal{S}_T)^*(x^*, x) < \infty\}\\ &= \{(x, x^*) \mid \varphi_T(x, x^*) < \infty\}\end{aligned}$$

where the last equality follow form Theorem 3.2. □

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