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Fubini-Tonelli Theorems on the Basis of Inner and Outer Premeasures

Heinz König
Universität des Saarlandes,
Fakultät für Mathematik und Informatik, 66123 Saarbrücken, Germany
hkoenig@math.uni-sb.de

Dedicated to Stephen Simons on the occasion of his 70th birthday.

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The present article obtains comprehensive Fubini-Tonelli type theorems on the basis of the author's work in measure and integration. The basic tools are the product theory and the complemental pairs of inner and outer premeasures.

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The theorem of Fubini-Tonelli is one of the most important and widespread tool theorems in measure and integration. It is a deep theorem, manifested in the basic requirement that the functions in it must be measurable with respect to the relevant *product* formation. There are additional obstacles which impede its formulation in both simple and universal manner: on the one hand the relevant iterated integrals can be delicate as to their kind of existence, and on the other hand the assertions require some sort of σ finiteness of the data.

The present article wants to invoke the author's work in measure and integration, developed in his 1997 book [4] and subsequent papers and summarized in [7] and [8], in order to obtain a transparent version of the field. In this work the basic concepts are the inner and outer premeasures and their maximal inner and outer extensions, and the basic devices are new inner and outer envelopes of set functions. We combine these envelopes with the concept of the Choquet integral, and thus ensure the existence of all relevant integrals, at least for functions with values in $[0, \infty]$. This limitation is no loss, because the Choquet integral has powerful additive properties, and it also removes the distinction between Fubini and Tonelli type theorems.

The present article then is based on two central points in our previous development: Section 1 uses the *product theory* for set functions to obtain the basic lower and upper estimations of the relevant iterated integrals. The upper estimation has been known before in particular situations, but in place of the Choquet integral with the less estimable conventional *upper integral*. Section 2 then uses the method of *complemental pairs* of

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inner and outer premeasures in order to pass from the fundamental inequalities to equalities of the Fubini-Tonelli type theorem. This method has its roots in the presentation of the Radon measure theory in Schwartz [11], Part I. The fact that both these procedures work in the present frame can be viewed as a remarkable support for the basic concepts of our new development. After this then Section 3 will be devoted to the connection with the traditional Fubini-Tonelli theorems.

1. The Fundamental Inequalities

We refer to the previous work of the author in measure and integration cited above. As a rule we shall make free use of the fundamentals of this development.

1.1. Preliminaries on Set Functions

Let X be a nonvoid set and $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$ in X. We assume $\bullet = \sigma\tau$.

Lemma 1.1. *Let $\varphi : \mathfrak{S} \to [0, \infty[$ be isotone with $\varphi(\varnothing) = 0$ and downward $\bullet$ continuous. Define $\xi := \varphi_\bullet|\mathfrak{S}_\bullet$. Then*

0) $\xi : \mathfrak{S}_\bullet \to [0, \infty]$ *is isotone with* $\xi(\varnothing) = 0$ *and downward* $\bullet$ *continuous.*
i) $\xi = \varphi^\star|\mathfrak{S}_\bullet$ *and* $\xi_\star = \xi_\bullet = \varphi_\bullet$.
ii) φ *is modular* $\Longleftrightarrow$ ξ *is modular.*
iii) φ *is an inner* $\bullet$ *premeasure* $\Longleftrightarrow$ ξ *is an inner* $\bullet$ *premeasure.*

Proof. *0)i)* are from [7], 2.2.3)4), and *ii)* is from [7], 2.8.1). *iii)* If φ is an inner $\bullet$ premeasure then $\xi_\bullet|\mathfrak{C}(\xi_\bullet) = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is an extension of $\varphi_\bullet|\mathfrak{S}_\bullet = \xi$, so that ξ is an inner $\bullet$ premeasure. If ξ is an inner $\bullet$ premeasure then $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet) = \xi_\bullet|\mathfrak{C}(\xi_\bullet)$ is an extension of ξ and hence of φ, so that φ is an inner $\bullet$ premeasure. $\square$

Lemma 1.2. *Let $\varphi : \mathfrak{S} \to [0, \infty]$ be isotone with $\varphi(\varnothing) = 0$ and upward $\bullet$ continuous. Define $\xi := \varphi^\bullet|\mathfrak{S}^\bullet$. Then*

0) $\xi : \mathfrak{S}^\bullet \to [0, \infty]$ *is isotone with* $\xi(\varnothing) = 0$ *and upward* $\bullet$ *continuous.*
i) $\xi = \varphi_\star|\mathfrak{S}^\bullet$ *and* $\xi^\star = \xi^\bullet = \varphi^\bullet$.
ii) φ *is modular* $\Longleftrightarrow$ ξ *is modular.*
iii) φ *is an outer* $\bullet$ *premeasure* $\Longleftrightarrow$ ξ *is an outer* $\bullet$ *premeasure.*

The proof is like the previous one. The proof of the next lemma is routine.

Lemma 1.3. *Let $\varphi : \mathfrak{S} \to [0, \infty]$ be isotone with $\varphi(\varnothing) = 0$, and assume that $[\varphi < \infty]$ is a lattice* (which is true when φ is submodular). *Then $\varphi^\bullet = (\varphi|[\varphi < \infty])^\bullet$.*

1.2. Preliminaries on the Choquet Integral

We consider the *Choquet integral* from [4], Section 11 (at that place called the *horizontal* integral) and [7], Section 5. We first recall [4], Theorem 11.16.

Proposition 1.4. *Let $\varphi : \mathfrak{S} \to [0, \infty]$ be isotone with $\varphi(\varnothing) = 0$. For $f : X \to [0, \infty]$ then*

$$⨍ f d\varphi_\star = \sup\left\{ ⨍ u d\varphi : u \in S(\mathfrak{S}) \text{ with } u \leqq f \right\}.$$

We turn to the outer counterpart of the assertion, which is more involved. We present two versions, of which the second one will be used later on.

Proposition 1.5. *Let $\varphi : \mathfrak{S} \to [0,\infty]$ be isotone with $\varphi(\varnothing) = 0$. If $f : X \to [0,\infty[$ is bounded above and has $\varphi^\star([f > 0]) < \infty$ then*

$$⨍ f d\varphi^\star = \inf\left\{ ⨍ u d\varphi : u \in S(\mathfrak{S}) \text{ with } u \geqq f \right\}.$$

Proof. 1) Fix $0 < b < \infty$ with $f < b$, and $0 < C < \infty$ with $\varphi^\star([f > 0]) < C$. Then

$$⨍ f d\varphi^\star = \int_{0\leftarrow}^{\to\infty} \varphi^\star([f \geqq t])dt = \int_{0\leftarrow}^{b} \varphi^\star([f \geqq t])dt \leqq Cb < \infty.$$

Next fix $0 < c < \infty$ with $⨍ f d\varphi^\star < c$, and then $\varepsilon > 0$ with $⨍ f d\varphi^\star < c - \varepsilon$. At last fix $0 < a < b$ with $Ca < \varepsilon$.

2) From $\int_a^b \varphi^\star([f \geqq t])dt \leqq ⨍ f d\varphi^\star < c - \varepsilon$ and the definition of the Riemann integral there exists a decomposition $a = t(0) < t(1) < \cdots < t(r) = b$ such that

$$\sum_{l=1}^{r} \varphi^\star\big([f \geqq t(l-1)]\big)\big(t(l) - t(l-1)\big) < c - \varepsilon.$$

Then fix $S(l) \in \mathfrak{S}$ $(l = 1, \cdots)$ with $S(l) \supset [f \geqq t(l-1)]$ and

$$\sum_{l=1}^{r} \varphi(S(l))(t(l) - t(l-1)) < c - \varepsilon.$$

Also fix $S(0) \in \mathfrak{S}$ with $S(0) \supset [f > 0]$ and $\varphi(S(0)) < C$. Because of $[f > 0] \supset [f \geqq t(0)] \supset \cdots \supset [f \geqq t(r-1)]$ we can replace $S(p)$ $(p = 1, \cdots, r)$ by $S(0) \cap \bigcap_{l=1}^{p} S(l)$, and hence assume that $S(0) \supset S(1) \supset \cdots \supset S(r)$. Now define

$$u := a\chi_{S(0)} + \sum_{l=1}^{r} (t(l) - t(l-1))\chi_{S(l)} \in S(\mathfrak{S}).$$

From [4], 11.8.1) we obtain

$$⨍ u d\varphi = a\varphi(S(0)) + \sum_{l=1}^{r} (t(l) - t(l-1))\varphi(S(l)) < Ca + c - \varepsilon < c.$$

3) It remains to show that $u(x) \geqq f(x)$ for all $x \in X$. This is clear when $x \in [f = 0]$, so we can assume that $x \in [f > 0] \subset S(0)$. Since $x \in [f \geqq b] = [f \geqq t(r)]$ does not happen, we are left with the two cases

i) $0 < f(x) < t(0) = a$ and hence $u(x) \geqq a > f(x)$, and

ii) $t(p-1) \leqq f(x) < t(p)$ for some $1 \leqq p \leqq r$.

Then for $1 \leqq l \leqq p$ we have $f(x) \geqq t(l-1)$ and hence $x \in S(l)$. It follows that $u(x) \geqq t(p) > f(x)$. $\square$

Now define $\mathrm{Inn}(\mathfrak{S})$ and $\mathrm{Out}(\mathfrak{S})$ to consist of the functions $f : X \to [0, \infty]$ with $[f \geqq t] \in \mathfrak{S}$ and $[f > t] \in \mathfrak{S}$ respectively for all $0 < t < \infty$, the former $\mathrm{UM}(\mathfrak{S})$ and $\mathrm{LM}(\mathfrak{S})$ of [4], Section 11. Thus $S(\mathfrak{S}) \subset \mathrm{Inn}(\mathfrak{S}) \cap \mathrm{Out}(\mathfrak{S})$ from [4], 11.4.

Proposition 1.6. *Assume that* $\mathfrak{S} = \mathfrak{S}^\sigma$. *Let* $\varphi : \mathfrak{S} \to [0, \infty]$ *be isotone with* $\varphi(\varnothing) = 0$, *and modular and upward* σ *continuous. For* $f : X \to [0, \infty]$ *then*

$$⨍ f d\varphi^\star = \inf\left\{ ⨍ u d\varphi : u \in (S(\mathfrak{S}))^\sigma = \mathrm{Out}(\mathfrak{S}) \textit{ with } u \geqq f \right\}.$$

Note that in the present case $\mathfrak{S} = \mathfrak{S}^\sigma$ we have $(S(\mathfrak{S}))^\sigma \subset \mathrm{Out}(\mathfrak{S}^\sigma) = \mathrm{Out}(\mathfrak{S})$, which after [4], 22.1 is $\subset (S(\mathfrak{S}))^\sigma$, so that in fact $(S(\mathfrak{S}))^\sigma = \mathrm{Out}(\mathfrak{S})$.

Proof. 1) We can assume that $⨍ f d\varphi^\star = \int_{0\leftarrow}^{\to\infty} \varphi^\star([f \geqq t])dt < \infty$, so that $\varphi^\star([f \geqq t]) < \infty$ for $0 < t < \infty$. Fix $0 < c < \infty$ with $⨍ f d\varphi^\star < c$. For each two-sided sequence $(t(l))_{l\in\mathbb{Z}}$ in $]0, \infty[$ with $t(l-1) < t(l)$ and with $t(l) \downarrow 0$ for $l \downarrow -\infty$ and $t(l) \uparrow \infty$ for $l \uparrow \infty$ we have

$$⨍ f d\varphi^\star = \sum_{l\in\mathbb{Z}} \int_{t(l-1)}^{t(l)} \varphi^\star([f \geqq t])dt < c.$$

We can pass to appropriate subdivisions of the individual intervals $[t(l-1), t(l)]$, while we retain the notation, in order to achieve that

$$\sum_{l\in\mathbb{Z}} \varphi^\star\big([f \geqq t(l-1)]\big)(t(l) - t(l-1)) < c.$$

Then we fix $S(l) \in \mathfrak{S}$ with $S(l) \supset [f \geqq t(l-1)]$ such that

$$\sum_{l\in\mathbb{Z}} \varphi(S(l))(t(l) - t(l-1)) < c.$$

2) Now define $u : X \to [0, \infty]$ to be

$$u := \sum_{l\in\mathbb{Z}} (t(l) - t(l-1))\chi_{S(l)}.$$

We have $u(x) \geqq f(x)$ for all $x \in X$: This is obvious when $f(x) = 0$ and when $f(x) = \infty$. In case $f(x) \in]0, \infty[$ we have $t(p-1) \leqq f(x) < t(p)$ for some $p \in \mathbb{Z}$. For $l \leqq p$ then $t(l-1) \leqq f(x)$ and hence $x \in S(l)$, so that

$$u(x) \geqq \sum_{l\leqq p} (t(l) - t(l-1)) = t(p) > f(x).$$

3) After this we define

$$u_n := \sum_{l=-n}^{n} (t(l) - t(l-1))\chi_{S(l)} \in S(\mathfrak{S}) \quad \text{for } n \in \mathbb{N}.$$

Thus $u_n \uparrow u$ for $n \to \infty$ and hence $u \in (S(\mathfrak{S}))^\sigma = \mathrm{Out}(\mathfrak{S})$. From [7], 5.6 we have

$$⨍ u_n d\varphi = \sum_{l=-n}^{n} (t(l) - t(l-1))\varphi(S(l)),$$

and hence from [4], 11.18 that

$$\int\!\!\!\!- u d\varphi = \sup_{n\in\mathbb{N}} \int\!\!\!\!- u_n d\varphi = \sum_{l\in\mathbb{Z}} (t(l) - t(l-1))\varphi(S(l)) < c.$$

The assertion follows. $\square$

In the special case that $\varphi : \mathfrak{S} \to [0,\infty]$ is a measure on a σ algebra the assertion 1.6 says for $f : X \to [0,\infty]$ that

$$\int\!\!\!\!- f d\varphi^\star = \inf\left\{ \int u d\varphi : u \in [0,\infty]^X \text{ measurable } \mathfrak{S} \text{ with } u \geqq f \right\},$$

where the second member is called the *upper integral* $\overline{\int} f d\varphi$ of f under φ. This assertion is for example in Fremlin [3], Exercise 252Yi. But one notes as in [4], Section 3, p. 27 that there is a huge gap between the measures and the set functions $\varphi : \mathfrak{S} \to [0,\infty]$ admitted in 1.6.

1.3. Preliminaries on the Product of Set Functions

We recall the product formation developed in [4], Chapter VII and summarized in [7], Section 6. Let X and Y be nonvoid sets. For nonvoid set systems $\mathfrak{S}$ in X and $\mathfrak{T}$ in Y we have the usual *product* set system $\mathfrak{S} \times \mathfrak{T} := \{S \times T : S \in \mathfrak{S} \text{ and } T \in \mathfrak{T}\}$ in $X \times Y$. For *lattices* $\mathfrak{S}$ and $\mathfrak{T}$ with $\varnothing$ then $\mathfrak{R} := (\mathfrak{S} \times \mathfrak{T})^\star$ is a lattice with $\varnothing$ as well, and the same for rings and algebras. Now let $\varphi : \mathfrak{S} \to [0,\infty]$ and $\psi : \mathfrak{T} \to [0,\infty]$ be isotone set functions with $\varphi(\varnothing) = \psi(\varnothing) = 0$. One proves that for $E \in \mathfrak{R}$ the function $x \mapsto \psi(E(x))$, where $E(x) := \{y \in Y : (x,y) \in E\} \in \mathfrak{T}$ is the vertical section of E at $x \in X$, is in $\mathrm{Inn}(\mathfrak{S}) \cap \mathrm{Out}(\mathfrak{S})$. We define the *product set function*

$$\vartheta = \varphi \times \psi : \mathfrak{R} \to [0,\infty] \quad \text{to be } \vartheta(E) = \int\!\!\!\!- \psi(E(\cdot))d\varphi.$$

Its basic properties are listed in [7], 6.2: ϑ is isotone with $\vartheta(\varnothing) = 0$ and fulfils $\vartheta(S \times T) = \varphi(S)\psi(T)$ for $S \in \mathfrak{S}$ and $T \in \mathfrak{T}$ (with $0\infty = 0$ as usual), and inherits from φ and ψ the properties to be *modular* and to be *finite*.

We want to add a note on the question of symmetry: In [7], 6.2.5) we assert that in the frame of *modular* φ and ψ the present $\vartheta = \varphi \times \psi$ is the *unique natural* product formation. In particular it coincides with the opposite formation $\theta : \theta(E) = \int\!\!\!\!- \varphi(E[\cdot])d\psi$ for $E \in \mathfrak{R}$, where $E[y] := \{x \in X : (x,y) \in E\} \in \mathfrak{S}$ is the horizontal section of E at $y \in Y$. However, we want to present a simple example that ϑ and θ can be different when φ and ψ are not both modular.

Example 1.7. Assume that $\varphi : \mathfrak{S} \to [0,\infty[$ has a pair $P, Q \in \mathfrak{S}$ with $P \cap Q = \varnothing$ and $\varphi(P) = \varphi(Q) = 1$, and that $\psi : \mathfrak{T} \to [0,\infty[$ has a pair $U, V \in \mathfrak{T}$ with $U \cap V = \varnothing$ such that $\psi(U) = \psi(V) = 0$ and $\psi(U \cup V) = 1$. Thus ψ is not modular. For $E := (P \times U) \cup (Q \times V) \in \mathfrak{R}$ then on the one hand $\psi(E(x)) = 0$ for all $x \in X$ and hence $\vartheta(E) = 0$. On the other hand $\varphi(E[y]) = 1$ for $y \in U \cup V$ and $\varphi(E[y]) = 0$ for the other $y \in Y$, so that $\theta(E) = 1$.

After this the development splits into the inner and the other ones, both for $\bullet = \sigma\tau$. In the inner situation we recall the basic properties obtained in [4], 21.4–7 and [5], 1.4 and summarized in [7], 6.3.

Proposition 1.8. *Assume that* $\varphi : \mathfrak{S} \to [0, \infty[$ *and* $\psi : \mathfrak{T} \to [0, \infty[$ *are isotone with* $\varphi(\varnothing) = \psi(\varnothing) = 0$ *and downward* $\bullet$ *continuous. Then*

1) $\vartheta = \varphi \times \psi$ *is downward* $\bullet$ *continuous* (the same implication holds true for downward $\bullet$ continuous at $\varnothing$).
2) *For* $E \in \mathfrak{R}_\bullet$ *one has* $E(x) \in \mathfrak{T}_\bullet$ *for all* $x \in X$. *Moreover the function* $\psi_\bullet(E(\cdot)) : X \to [0, \infty[$ *is in* $\mathrm{Inn}(\mathfrak{S}_\bullet)$, *and* $\vartheta_\bullet(E) = ⨍ \psi_\bullet(E(\cdot))d\varphi_\bullet$.
3) $\vartheta_\bullet(A \times B) = \varphi_\bullet(A)\psi_\bullet(B)$ *for all* $A \subset X$ *and* $B \subset Y$.

In the inner situation one then has the fundamental product theorem [4], 21.9 = [7], 6.4 which follows.

Theorem 1.9. *Assume that* $\varphi : \mathfrak{S} \to [0, \infty[$ *and* $\psi : \mathfrak{T} \to [0, \infty[$ *are inner* $\bullet$ *premeasures. Then* $\vartheta = \varphi \times \psi : \mathfrak{R} \to [0, \infty[$ *is an inner* $\bullet$ *premeasure. Moreover* $\Theta = \vartheta_\bullet | \mathfrak{C}(\vartheta_\bullet)$ *is an extension of the product* $\Phi \times \Psi$ *of* $\Phi = \varphi_\bullet | \mathfrak{C}(\varphi_\bullet)$ *and* $\Psi = \psi_\bullet | \mathfrak{C}(\psi_\bullet)$.

In the outer situation we repeat the basic properties summarized in [7], 6.5 and add their proofs, which had been left out so far. The example [7], 6.6 shows an imperfection compared with the inner situation, and in particular that there is no full counterpart of the product Theorem 1.9.

Proposition 1.10. *Assume that* $\varphi : \mathfrak{S} \to [0, \infty]$ *and* $\psi : \mathfrak{T} \to [0, \infty]$ *are isotone with* $\varphi(\varnothing) = \psi(\varnothing) = 0$ *and upward* $\bullet$ *continuous. Then*

1) $\vartheta = \varphi \times \psi$ *is upward* $\bullet$ *continuous.*
2) *For* $E \in \mathfrak{R}^\bullet$ *one has* $E(x) \in \mathfrak{T}^\bullet$ *for all* $x \in X$. *Moreover the function* $\psi^\bullet(E(\cdot)) : X \to [0, \infty]$ *is in* $\mathrm{Out}(\mathfrak{S}^\bullet)$, *and* $\vartheta^\bullet(E) = ⨍ \psi^\bullet(E(\cdot))d\varphi^\bullet$.
3) $\vartheta^\bullet(A \times B) = \varphi^\bullet(A)\psi^\bullet(B)$ *for* $A \subset X$ *and* $B \subset Y$, *except perhaps when the latter product is* 0∞ *or* $\infty 0$.

Proof. *1*) Let $\mathfrak{M} \subset \mathfrak{R}$ be nonvoid $\bullet$ with $\mathfrak{M} \uparrow E \in \mathfrak{R}$. For $x \in X$ we have $\{M(x) : M \in \mathfrak{M}\} \subset \mathfrak{T}$ nonvoid $\bullet$ with $\uparrow E(x) \in \mathfrak{T}$ from [4], 20.1.3), and hence $\sup_{M \in \mathfrak{M}} \psi(M(x)) = \psi(E(x))$. Thus from [4], 11.18 we obtain $\sup_{M \in \mathfrak{M}} ⨍ \psi(M(\cdot))d\varphi = ⨍ \psi(E(\cdot))d\varphi$, that is $\sup_{M \in \mathfrak{M}} \vartheta(M) = \vartheta(E)$.

2) For $E \in \mathfrak{R}^\bullet$ we have $E(x) \in \mathfrak{T}^\bullet\ \forall x \in X$ from [4], 20.3.1) and 20.1.3). Define $\mathfrak{K} \subset \mathfrak{R}^\bullet$ to consist of all those $E \in \mathfrak{R}^\bullet$ which are as claimed in the second sentence in *2*). Thus $\mathfrak{R} \subset \mathfrak{K}$. It remains to show that $\mathfrak{K}^\bullet \subset \mathfrak{K}$, because then $\mathfrak{R}^\bullet \subset \mathfrak{K}^\bullet \subset \mathfrak{K}$ and hence $\mathfrak{K} = \mathfrak{R}^\bullet$.

To see this fix $E \subset \mathfrak{K}^\bullet \subset \mathfrak{R}^\bullet$, and then $\mathfrak{M} \subset \mathfrak{K} \subset \mathfrak{R}^\bullet$ nonvoid $\bullet$ with $\uparrow E$. As above we have $\{M(x) : M \in \mathfrak{M}\} \subset \mathfrak{T}^\bullet$ nonvoid $\bullet$ with $\uparrow E(x) \in \mathfrak{T}^\bullet$ for $x \in X$. Since $\psi^\bullet | \mathfrak{T}^\bullet$ is upward $\bullet$ continuous by 1.2.*0*), it follows that $\sup_{M \in \mathfrak{M}} \psi^\bullet(M(x)) = \psi^\bullet(E(x))$ for $x \in X$. Now $\psi^\bullet(M(\cdot)) \in \mathrm{Out}(\mathfrak{S}^\bullet)$ because $\mathfrak{M} \subset \mathfrak{K}$, and this implies at once that $\psi^\bullet(E(\cdot)) \in \mathrm{Out}(\mathfrak{S}^\bullet)$. Since $\varphi^\bullet | \mathfrak{S}^\bullet$ is upward $\bullet$ continuous by 1.2.*0*), it follows from [4], 11.18 that $\sup_{M \in \mathfrak{M}} ⨍ \psi^\bullet(M(\cdot))d\varphi^\bullet = ⨍ \psi^\bullet(E(\cdot))d\varphi^\bullet$. In view of $\mathfrak{M} \subset \mathfrak{K}$ this says that $\sup_{M \in \mathfrak{M}} \vartheta^\bullet(M) = ⨍ \psi^\bullet(E(\cdot))d\varphi^\bullet$. Now $\vartheta^\bullet | \mathfrak{R}^\bullet$ is upward $\bullet$ continuous by 1) and 1.2.*0*), so that $\mathfrak{M} \subset \mathfrak{R}^\bullet$ and $E \in \mathfrak{R}^\bullet$ implies that $\sup_{M \in \mathfrak{M}} \vartheta^\bullet(M) = \vartheta^\bullet(E)$. Thus

$\vartheta^\bullet(E) = ⨍\psi^\bullet(E(\cdot))d\varphi^\bullet$ and hence $E \in \mathfrak{K}$.

3) We first show that $⨍\psi^\bullet\big((A\times B)(\cdot)\big)d\varphi^\bullet = \varphi^\bullet(A)\psi^\bullet(B)$ for all $A \subset X$ and $B \subset Y$ (with $0\infty = 0$ as usual). In fact, we have $(A \times B)(x) = B$ and hence $\psi^\bullet((A \times B)(x)) = \psi^\bullet(B)$ for $x \in A$, and $(A \times B)(x) = \varnothing$ and hence $\psi^\bullet((A \times B)(x)) = 0$ for $x \in X \setminus A$. Thus

$$⨍\psi^\bullet\big((A \times B)(\cdot)\big)d\varphi^\bullet = \int_{0\leftarrow}^{\rightarrow\infty} \varphi^\bullet\Big([\psi^\bullet((A \times B)(\cdot)) \geqq t]\Big)dt$$

is on the one hand $= 0$ in the two particular cases $\varphi^\bullet(A) = 0$ and $\psi^\bullet(B) = 0$, and on the other hand for $\varphi^\bullet(A) > 0$ and $\psi^\bullet(B) > 0$ is

$$= \int_{0\leftarrow}^{\rightarrow\infty} \varphi^\bullet\Big([\psi^\bullet(B)\chi_A \geqq t]\Big)dt = \int_{0\leftarrow}^{\rightarrow\psi^\bullet(B)} \varphi^\bullet(A)dt = \varphi^\bullet(A)\psi^\bullet(B) \text{ in all cases.}$$

Now fix $A \subset X$ and $B \subset Y$. To prove $\geqq$ we note for $E \in \mathfrak{R}^\bullet$ with $E \supset A \times B$ from *2)* that

$$\vartheta^\bullet(E) = ⨍\psi^\bullet(E(\cdot))d\varphi^\bullet \geqq ⨍\psi^\bullet\big((A \times B)(\cdot)\big)d\varphi^\bullet = \varphi^\bullet(A)\psi^\bullet(B),$$

and obtain $\vartheta^\bullet(A \times B) \geqq \varphi^\bullet(A)\psi^\bullet(B)$, since $\vartheta^\bullet$ is outer regular $\mathfrak{R}^\bullet$. To prove $\leqq$ we can assume that $\varphi^\bullet(A) < \infty$ and $\psi^\bullet(B) < \infty$ in view of the admitted exceptions. For all $P \in \mathfrak{S}^\bullet$ with $A \subset P$ and $\varphi^\bullet(P) < \infty$ and all $Q \in \mathfrak{T}^\bullet$ with $B \subset Q$ and $\psi^\bullet(Q) < \infty$ then $P \times Q \in \mathfrak{R}^\bullet$ and hence from *2)*

$$\vartheta^\bullet(A \times B) \leqq \vartheta^\bullet(P \times Q) = ⨍\psi^\bullet\big((P \times Q)(\cdot)\big)d\varphi^\bullet = \varphi^\bullet(P)\psi^\bullet(Q).$$

This implies that $\vartheta^\bullet(A\times B) \leqq \varphi^\bullet(A)\psi^\bullet(B)$, since $\varphi^\bullet$ is outer regular $\mathfrak{S}^\bullet$ and $\psi^\bullet$ is outer regular $\mathfrak{T}^\bullet$. □

1.4. The Fundamental Inequalities

We continue to assume lattices $\mathfrak{S}$ in X and $\mathfrak{T}$ in Y with $\varnothing$ and their product lattice $\mathfrak{R} = (\mathfrak{S} \times \mathfrak{T})^\star$ in $X \times Y$. Let $\bullet = \sigma\tau$.

Inner Theorem 1.11. *Assume that $\varphi : \mathfrak{S} \to [0,\infty[$ and $\psi : \mathfrak{T} \to [0,\infty[$ are isotone with $\varphi(\varnothing) = \psi(\varnothing) = 0$, and modular and downward $\bullet$ continuous, and let $\vartheta = \varphi \times \psi$. For all $f : X \times Y \to [0,\infty]$ then*

$$⨍fd\vartheta_\bullet \leqq ⨍\Big(⨍f(x,y)d\psi_\bullet(y)\Big)d\varphi_\bullet(x).$$

Inner Lemma 1.12. *Assume φ and ψ as in* 1.11*. For $f \in S(\mathfrak{R}_\bullet)$ then $f(x,\cdot) \in S(\mathfrak{T}_\bullet)$ for all $x \in X$, and the function $F : F(x) = ⨍f(x,\cdot)d\psi_\bullet$ for $x \in X$ is in* $\mathrm{Inn}(\mathfrak{S}_\bullet)$ *with $⨍Fd\varphi_\bullet = ⨍fd\vartheta_\bullet$.*

For the subsequent proofs we recall two formal rules from [4], p. 226: 1) For $f : X\times Y \to [0,\infty]$ and all $0 < t < \infty$ and $x \in X$ one has

$$[f(x,\cdot) \geqq t] = [f \geqq t](x) \quad \text{and} \quad [f(x,\cdot) > t] = [f > t](x).$$

2) For $E \subset X \times Y$ and all $x \in X$ one has $\chi_E(x,\cdot) = \chi_{E(x)}$.

Proof of 1.12. We have

$$f=\sum_{l=1}^{r}t_l\chi_{E(l)}\quad\text{with } t_1,\cdots,t_r>0 \text{ and } E(1)\supset\cdots\supset E(r) \text{ in } \mathfrak{R}_\bullet.$$

From the formal rule 2) for $x\in X$ hence $f(x,\cdot)=\sum_{l=1}^{r}t_l\chi_{E(l)(x)}$, and 1.8.*2*) implies that $E(l)(x)\in\mathfrak{T}_\bullet$ and hence $f(x,\cdot)\in S(\mathfrak{T}_\bullet)$. Next [4], 11.8 shows that

$$F(x)=\sum_{l=1}^{r}t_l\psi_\bullet\big(E(l)(x)\big)\ \text{ for } x\in X,\ \text{ that is } F=\sum_{l=1}^{r}t_l\psi_\bullet\big(E(l)(\cdot)\big).$$

We have $\psi_\bullet\big(E(l)(\cdot)\big)\in\operatorname{Inn}(\mathfrak{S}_\bullet)$ $(l=1,\cdots,r)$ from 1.8.*2*) and hence $F\in\operatorname{Inn}(\mathfrak{S}_\bullet)$ from [4], 11.1.3). Now [7], 5.6 can be applied to $\varphi_\bullet|\mathfrak{S}_\bullet$, which is modular by 1.1.*ii*). It follows that

$$\begin{aligned}⨍ Fd\varphi_\bullet &= ⨍ Fd(\varphi_\bullet|\mathfrak{S}_\bullet)=\sum_{l=1}^{r}t_l ⨍\psi_\bullet\big(E(l)(\cdot)\big)d(\varphi_\bullet|\mathfrak{S}_\bullet)\\ &=\sum_{l=1}^{r}t_l ⨍\psi_\bullet\big(E(l)(\cdot)\big)d\varphi_\bullet=\sum_{l=1}^{r}t_l\vartheta_\bullet(E(l))=⨍ fd\vartheta_\bullet,\end{aligned}$$

where for the last two equalities 1.8.*2*) and [4], 11.8 have been used. □

Proof of 1.11. We know that ϑ is modular, and downward $\bullet$ continuous by 1.8.*1*). Thus from 1.1 and from 1.4 applied to $\xi:=\vartheta_\bullet|\mathfrak{R}_\bullet$ we obtain

$$⨍ fd\vartheta_\bullet=⨍ fd\xi_\star=\sup\{⨍ ud\xi: u\in S(\mathfrak{R}_\bullet)\text{ with } u\leqq f\}.$$

But for each such $u\in S(\mathfrak{R}_\bullet)$ we have from 1.12

$$⨍ ud\xi=⨍ ud\vartheta_\bullet=⨍\Big(⨍ u(x,y)d\psi_\bullet(y)\Big)d\varphi_\bullet(x)\leqq ⨍\Big(⨍ f(x,y)d\psi_\bullet(y)\Big)d\varphi_\bullet(x).$$

The assertion follows. □

Outer Theorem 1.13. *Assume that $\varphi:\mathfrak{S}\to[0,\infty]$ and $\psi:\mathfrak{T}\to[0,\infty]$ are isotone with $\varphi(\varnothing)=\psi(\varnothing)=0$, and modular and upward $\bullet$ continuous, and let $\vartheta=\varphi\times\psi$. For all $f:X\times Y\to[0,\infty]$ then*

$$⨍ fd\vartheta^\bullet\geqq ⨍\left(⨍ f(x,y)d\psi^\bullet(y)\right)d\varphi^\bullet(x).$$

Outer Lemma 1.14. *Assume φ and ψ as in* 1.13. *For $f\in S(\mathfrak{R}^\bullet)$ then $f(x,\cdot)\in S(\mathfrak{T}^\bullet)$ for all $x\in X$, and the function $F: F(x)=⨍ f(x,\cdot)d\psi^\bullet$ for $x\in X$ is in* $\operatorname{Out}(\mathfrak{S}^\bullet)$ *with $⨍ Fd\varphi^\bullet=⨍ fd\vartheta^\bullet$.*

The proof of 1.14 runs like that of 1.12, but with 1.10.*2*) instead of 1.8.*2*) and with 1.2.*ii*) instead of 1.1.*ii*).

Proof of 1.13. 0) We know that ϑ is modular, and upward $\bullet$ continuous by 1.10.*1*). Thus from 1.2 and from 1.6 applied to $\xi := \vartheta^\bullet|\mathfrak{R}^\bullet$ we obtain

$$\fint f d\vartheta^\bullet = \fint f d\xi^\star = \inf\left\{\fint u d\xi : u \in (S(\mathfrak{R}^\bullet))^\sigma = \mathrm{Out}(\mathfrak{R}^\bullet) \text{ with } u \geqq f\right\}.$$

1) Now fix $u \in (S(\mathfrak{R}^\bullet))^\sigma = \mathrm{Out}(\mathfrak{R}^\bullet)$, and then a sequence $(u_n)_n$ in $S(\mathfrak{R}^\bullet) \subset \mathrm{Out}(\mathfrak{R}^\bullet)$ such that $u_n \uparrow u$. We shall invoke [4], 11.18 for three times. 1.i) We have $\fint u_n d\vartheta^\bullet = \fint u_n d\xi \uparrow \fint u d\vartheta^\bullet = \fint u d\xi$. 1.ii) For $x \in X$ we have $[u(x,\cdot) > t] = [u > t](x) \in \mathfrak{T}^\bullet\ \forall\, 0 < t < \infty$ from the formal rule 1) and 1.10.*2*), so that $u(x,\cdot) \in \mathrm{Out}(\mathfrak{T}^\bullet)$. Likewise $u_n(x,\cdot) \in \mathrm{Out}(\mathfrak{T}^\bullet)$ for $n \in \mathbb{N}$, and of course $u_n(x,\cdot) \uparrow u(x,\cdot)$. Thus we have

$$U_n(x) := \fint u_n(x,\cdot) d\psi^\bullet \uparrow \fint u(x,\cdot) d\psi^\bullet =: U(x) \quad \text{for } x \in X.$$

1.iii) By 1.14 the functions $U_n : X \to [0,\infty]$ thus defined are in $\mathrm{Out}(\mathfrak{S}^\bullet)$ with $\fint U_n d\varphi^\bullet = \fint u_n d\vartheta^\bullet$. Then $U_n \uparrow U$ implies that $U \in \mathrm{Out}(\mathfrak{S}^\bullet)$ as well. Thus we have $\fint U_n d\varphi^\bullet \uparrow \fint U d\varphi^\bullet$.

2) The results obtained in 1) combine to furnish

$$\begin{aligned}\fint u d\xi = \fint u d\vartheta^\bullet = \fint U d\varphi^\bullet &= \fint \left(\fint u(x,y) d\psi^\bullet(y)\right) d\varphi^\bullet(x) \\ &\geqq \fint \left(\fint f(x,y) d\psi^\bullet(y)\right) d\varphi^\bullet(x).\end{aligned}$$

Thus from 0) we obtain the assertion. □

1.15. Specialization of 1.13. Assume that $\varphi : \mathfrak{S} \to [0,\infty]$ and $\psi : \mathfrak{T} \to [0,\infty]$ are measures on σ algebras and $\bullet = \sigma$. First note the obvious fact that in this case the outer envelopes fulfil $\varphi^\sigma = \varphi^\star$ and $\psi^\sigma = \psi^\star$. Thus after the final remark in the subsection on the Choquet integral the second member in the assertion of 1.13 for $f : X \times Y \to [0,\infty]$ is the *iterated upper integral* $\overline{\int}\big(\overline{\int} f(x,y) d\psi(y)\big) d\varphi(x)$. As to the first member, we know from [7], 3.3 that $\vartheta = \varphi \times \psi$ is an outer σ premeasure, and shall see in Section 3 that its maximal extension $\Theta = \vartheta^\sigma|\mathfrak{C}(\vartheta^\sigma)$ is the so-called *primitive product measure* of φ and ψ. It is an immediate verification that $\vartheta^\sigma = \Theta^\sigma = \Theta^\star$. Hence the first member becomes $\fint f d\vartheta^\sigma = \fint f d\Theta^\star = \overline{\int} f d\Theta$. Thus in the present special case the assertion in 1.13 reads

$$\overline{\int} f d\Theta \geqq \overline{\int}\left(\overline{\int} f(x,y) d\psi(y)\right) d\varphi(x) \quad \text{for all } f : X \times Y \to [0,\infty].$$

This version appears for example in Rao [10], Lemma 7, p. 371 and is attributed to M. H. Stone.

2. The Fubini-Tonelli Type Theorem for $\bullet$ Premeasures

The two fundamental inequalities of the previous section are intended to lead to Fubini-Tonelli type theorems. It is obvious that this aim requires substantial relations between the inner and the outer $\bullet$ envelopes of the involved set functions. Our favorite source for these relations is the method of *complemental pairs* of inner and outer $\bullet$ premeasures developed in [6], Part I and summarized in [7], Section 4. We start to recall this procedure.

2.1. Preliminaries on Complemental Pairs

The present subsection assumes a nonvoid set X and $\bullet = \star\sigma\tau$. We define a pair of lattices $\mathfrak{S}$ and $\mathfrak{P}$ with $\varnothing$ in X to be $\bullet$ *complemental* iff $\mathfrak{P} \subset (\mathfrak{S}\top\mathfrak{S}_\bullet)\perp$ and $\mathfrak{S} \subset (\mathfrak{P}\top\mathfrak{P}^\bullet)\perp$. An obvious example is a *ring* $\mathfrak{S} = \mathfrak{P}$ in X. The most important example is $\mathfrak{S} = \mathrm{Comp}(X)$ and $\mathfrak{P} = \mathrm{Op}(X)$ in a Hausdorff topological space X. In the present subsection we fix a $\bullet$ complemental pair $\mathfrak{S}$ and $\mathfrak{P}$ in X. We quote some basic results from [6], Sections 2–4.

Inner Theorem 2.1. *Let $\varphi : \mathfrak{S} \to [0, \infty[$ be an inner $\bullet$ premeasure, and define $\xi := \varphi_\bullet|\mathfrak{P}$. Then*

i) $\xi : \mathfrak{P} \to [0, \infty]$ *is an outer $\bullet$ premeasure with $\mathfrak{P} \subset \mathfrak{C}(\varphi_\bullet) \subset \mathfrak{C}(\xi^\bullet)$.*
ii) $\varphi_\bullet \leqq \xi^\bullet$, *and $\varphi_\bullet|\mathfrak{P}^\bullet = \xi^\bullet|\mathfrak{P}^\bullet$. Hence $\xi^\bullet$ is inner regular $\mathfrak{S}_\bullet$ at $\mathfrak{P}^\bullet$.*
iii) $\varphi_\bullet(A) = \xi^\bullet(A)$ *for all $A \in \mathfrak{C}(\varphi_\bullet)$ with $\xi^\bullet(A) < \infty$. Hence $\varphi_\bullet = \xi^\bullet$ on all members of $\mathfrak{C}(\varphi_\bullet)$ which are upward enclosable $[\xi^\bullet < \infty]^\sigma$.*
iv) *If $\varphi_\bullet$ is outer regular $\mathfrak{P}^\bullet$ at $\mathfrak{S}_\bullet$, then $\xi^\bullet|\mathfrak{S} < \infty$.*

Outer Theorem 2.2. *Let $\xi \to [0, \infty]$ be an outer $\bullet$ premeasure with $\xi^\bullet|\mathfrak{S} < \infty$, and define $\varphi := \xi^\bullet|\mathfrak{S}$. Then*

i) $\varphi : \mathfrak{S} \to [0, \infty[$ *is an inner $\bullet$ premeasure with $\mathfrak{S} \subset \mathfrak{C}(\xi^\bullet) \subset \mathfrak{C}(\varphi_\bullet)$.*
ii) $\varphi_\bullet \leqq \xi^\bullet$, *and $\varphi_\bullet = \xi^\bullet$ on $\mathfrak{S}_\bullet$. Hence $\varphi_\bullet$ is outer regular $\mathfrak{P}^\bullet$ at $\mathfrak{S}_\bullet$.*
iii) $\varphi_\bullet(A) = \xi^\bullet(A)$ *for all $A \in \mathfrak{C}(\xi^\bullet)$ upward enclosable $\mathfrak{S}$.*

After this we turn to a natural combination of the two theorems. We define an inner $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0, \infty[$ to be $\bullet$ *tame for $\mathfrak{S}$ and $\mathfrak{P}$* iff $\varphi_\bullet$ is outer regular $\mathfrak{P}^\bullet$ at $\mathfrak{S}_\bullet$; equivalent is the much simpler condition that each $S \in \mathfrak{S}$ is contained in some $P \in \mathfrak{P}^\bullet$ with $\varphi_\bullet(P) < \infty$. Likewise we define an outer $\bullet$ premeasure $\xi : \mathfrak{P} \to [0, \infty]$ to be $\bullet$ *tame for $\mathfrak{S}$ and $\mathfrak{P}$* iff $\xi^\bullet$ is inner regular $\mathfrak{S}_\bullet$ at $\mathfrak{P}^\bullet$ and fulfils $\xi^\bullet|\mathfrak{S} < \infty$. For these particular $\bullet$ premeasures one extracts from 2.1 and 2.2 the previous main result [6], 4.6 = [7], 4.6 which follows.

Theorem 2.3. *There is a one-to-one correspondence between*

– *the inner $\bullet$ premeasures φ on $\mathfrak{S}$ which are $\bullet$ tame for $\mathfrak{S}$ and $\mathfrak{P}$, and*
– *the outer $\bullet$ premeasures ξ on $\mathfrak{P}$ which are $\bullet$ tame for $\mathfrak{S}$ and $\mathfrak{P}$,*

via $\varphi \mapsto \xi := \varphi_\bullet|\mathfrak{P}$ and $\xi \mapsto \varphi := \xi^\bullet|\mathfrak{S}$. Under this correspondence we have

i) $\mathfrak{C}(\varphi_\bullet) = \mathfrak{C}(\xi^\bullet) =: \mathfrak{C}$.
ii) $\varphi_\bullet \leqq \xi^\bullet$.
iii) $\varphi_\bullet = \xi^\bullet$ *on all members of $\mathfrak{C}$ which are upward enclosable $[\xi^\bullet < \infty]^\sigma$.*

We define a pair of $\bullet$ premeasures $\varphi : \mathfrak{S} \to [0, \infty[$ and $\xi : \mathfrak{P} \to [0, \infty]$ as above to be $\bullet$ *complemental for $\mathfrak{S}$ and $\mathfrak{P}$*.

An important specialization of 2.3 is for *Radon premeasures* on Hausdorff topological spaces. It is due to Schwartz [11], Part I and is the source of the present method of complemental pairs. For Radon premeasures we also refer to [7], 4.3 and to the earlier presentation in Bourbaki [1]. Let as above be $\mathfrak{S} = \mathrm{Comp}(X)$ and $\mathfrak{P} = \mathrm{Op}(X)$ on a Hausdorff topological space X, and let $\varphi : \mathfrak{S} \to [0, \infty[$ be a Radon premeasure on X. We recall that the envelopes $\varphi_\bullet$ are the same for $\bullet = \star\sigma\tau$. One defines φ to be *locally* *finite* iff each $S \in \mathfrak{S}$ is contained in some $P \in \mathfrak{P}$ with $\varphi_\bullet(P) < \infty$: that means iff φ is $\bullet$ tame for $\mathfrak{S}$ and $\mathfrak{P}$. We call the outer $\bullet$ premeasure $\xi := \varphi_\bullet|\mathfrak{P}$ which is $\bullet$ complemental to φ the *hull* of φ. We recall from 2.3 that $\varphi_\bullet \leqq \xi^\bullet$, and that $\varphi_\bullet = \xi^\bullet$ on all members of $\mathfrak{C}(\varphi_\bullet) = \mathfrak{C}(\xi^\bullet)$ which are upward enclosable $[\xi^\bullet < \infty]^\sigma$.

Next we note a simple and almost amusing specialization of 2.3. We make use of [7], 3.7 and 3.3.

Specialization 2.4. *Let $\mathfrak{S}$ be a ring in X and $\varphi : \mathfrak{S} \to [0, \infty[$ be isotone with $\varphi(\varnothing) = 0$ and modular, that is a finite content. Then φ is downward $\bullet$ continuous at $\varnothing$ iff it is upward $\bullet$ continuous, that is an inner $\bullet$ premeasure iff it is an outer $\bullet$ premeasure - all this called $\bullet$* continuous *for short. In this case*

i) $\mathfrak{C}(\varphi_\bullet) = \mathfrak{C}(\varphi^\bullet) =: \mathfrak{C}$.
ii) $\varphi_\bullet \leqq \varphi^\bullet$.
iii) $\varphi_\bullet = \varphi^\bullet$ *on all members of $\mathfrak{C}$ which are upward enclosable $[\varphi^\bullet < \infty]^\sigma$.*

Proof. We know that $\mathfrak{S}$ and $\mathfrak{P} := \mathfrak{S}$ form a $\bullet$ complemental pair. Then the inner $\bullet$ premeasure φ on $\mathfrak{S}$ and the outer $\bullet$ premeasure $\xi := \varphi$ on $\mathfrak{P} = \mathfrak{S}$ are $\bullet$ tame and $\bullet$ complemental for $\mathfrak{S}$ and $\mathfrak{P} = \mathfrak{S}$. $\square$

We want to note that in case $\bullet = \sigma$ the last specialization has a wide but complicated extension in [4], 7.5 = [7], 3.10: The assertions remain true whenever $\mathfrak{S}$ is a lattice with $\varnothing$ in X and $\varphi : \mathfrak{S} \to [0, \infty[$ is both an inner and an outer σ premeasure.

We add a little reformulation which is a simple consequence of the definition.

Remark 2.5. Let $\varphi : \mathfrak{S} \to [0, \infty]$ be isotone with $\varphi(\varnothing) = 0$. For the subsets of X then *upward enclosable* $[\varphi^\bullet < \infty]^\bullet$ is equivalent to *upward enclosable* $[\varphi < \infty]^\bullet$.

2.2. The Fubini-Tonelli Type Theorem

We turn to the announced Fubini-Tonelli type theorem for $\bullet$ premeasures. The present subsection assumes $\bullet = \sigma\tau$, and

- a pair of lattices $\mathfrak{S}$ and $\mathfrak{P}$ with $\varnothing$ in the nonvoid set X,
- a pair of lattices $\mathfrak{T}$ and $\mathfrak{Q}$ with $\varnothing$ in the nonvoid set Y,

with the product lattices $\mathfrak{R} = (\mathfrak{S} \times \mathfrak{T})^\star$ and $\mathfrak{N} = (\mathfrak{P} \times \mathfrak{Q})^\star$ in $X \times Y$. We first note a basic lemma.

Lemma 2.6. *We have the implications*

- $\mathfrak{P} \subset (\mathfrak{S}\top\mathfrak{S}_\bullet)\perp$ *and* $\mathfrak{Q} \subset (\mathfrak{T}\top\mathfrak{T}_\bullet)\perp \Longrightarrow \mathfrak{N} \subset (\mathfrak{R}\top\mathfrak{R}_\bullet)\perp$,
- $\mathfrak{S} \subset (\mathfrak{P}\top\mathfrak{P}^\bullet)\perp$ *and* $\mathfrak{T} \subset (\mathfrak{Q}\top\mathfrak{Q}^\bullet)\perp \Longrightarrow \mathfrak{R} \subset (\mathfrak{N}\top\mathfrak{N}^\bullet)\perp$.

Thus if the pairs $\mathfrak{S}\&\mathfrak{P}$ and $\mathfrak{T}\&\mathfrak{Q}$ are both $\bullet$ complemental then the pair $\mathfrak{R}\&\mathfrak{N}$ is $\bullet$ complemental as well.

Proof. We prove the first relation. To be shown is that $P \in \mathfrak{P}$ and $Q \in \mathfrak{Q}$ fulfil $P \times Q \in (\mathfrak{R}\top\mathfrak{R}_\bullet)\perp$ or $(P \times Q)' = (P' \times Y) \cup (X \times Q') \in \mathfrak{R}\top\mathfrak{R}_\bullet$. It suffices to see for $S \in \mathfrak{S}$ and $T \in \mathfrak{T}$ that

$$(P' \times Y) \cap (S \times T) = (P' \cap S) \times T \quad \text{and} \quad (X \times Q') \cap (S \times T) = S \times (Q' \cap T)$$

are $\in \mathfrak{R}_\bullet$. But by assumption these formations are in $\mathfrak{S}_\bullet \times \mathfrak{T}$ and $\mathfrak{S} \times \mathfrak{T}_\bullet$ and hence both in $\mathfrak{R}_\bullet$. $\square$

In the sequel we assume that the pairs $\mathfrak{S}\&\mathfrak{P}$ and $\mathfrak{T}\&\mathfrak{Q}$ are both $\bullet$ complemental.

Theorem 2.7. *Assume that $\varphi : \mathfrak{S} \to [0,\infty[$ and $\psi : \mathfrak{T} \to [0,\infty[$ are inner $\bullet$ premeasures with $\vartheta = \varphi \times \psi : \mathfrak{R} \to [0,\infty[$. If as in 2.1 one forms*

$$\xi = \varphi_\bullet|\mathfrak{P} \quad \textit{and} \quad \eta = \psi_\bullet|\mathfrak{Q} \quad \textit{with } \rho = \xi \times \eta : \mathfrak{N} \to [0,\infty],$$

then $\rho = \vartheta_\bullet|\mathfrak{N}$, and hence 2.1.i) implies that ρ is an outer $\bullet$ premeasure with $\mathfrak{N} \subset \mathfrak{C}(\vartheta_\bullet) \subset \mathfrak{C}(\rho^\bullet)$. The functions $f : X \times Y \to [0,\infty]$ which are measurable $\mathfrak{C}(\vartheta_\bullet)$ and have $[f > 0]$ upward enclosable $[\rho^\bullet < \infty]^\sigma$ fulfil

$$\begin{aligned}\fint f d\vartheta_\bullet &= \fint \left(\fint f(x,y) d\psi_\bullet(y) \right) d\varphi_\bullet(x) \\ &= \fint \left(\fint f(x,y) d\eta^\bullet(y) \right) d\xi^\bullet(x) = \fint f d\rho^\bullet.\end{aligned}$$

We note for $f : X \times Y \to [0,\infty]$ from the definition of the Choquet integral that $\fint f d\rho^\bullet < \infty$ implies that $[f > 0]$ is upward enclosable $[\rho^\bullet < \infty]^\sigma$.

Proof. 1) From the fundamental inequalities 1.11 and 1.13 combined with 2.1 applied to φ and ψ we see that all functions $f : X \times Y \to [0,\infty]$ fulfil

$$\begin{aligned}\fint f d\vartheta_\bullet &\leqq \fint \left(\fint f(x,y) d\psi_\bullet(y) \right) d\varphi_\bullet(x) \\ &\leqq \fint \left(\fint f(x,y) d\eta^\bullet(y) \right) d\xi^\bullet(x) \leqq \fint f d\rho^\bullet.\end{aligned}$$

Therefore $\vartheta_\bullet \leqq \rho^\bullet$.

2) We claim that $\rho = \vartheta_\bullet|\mathfrak{N}$. To see this note from 1.8 for $P \in \mathfrak{P}$ and $Q \in \mathfrak{Q}$ that

$$\rho(P \times Q) = \xi(P)\eta(Q) = \varphi_\bullet(P)\psi_\bullet(Q) = \vartheta_\bullet(P \times Q),$$

and from 1.9 that $\mathfrak{N} \subset (\mathfrak{C}(\varphi_\bullet) \times \mathfrak{C}(\psi_\bullet))^\star \subset \mathfrak{C}(\vartheta_\bullet)$. For $N \in \mathfrak{N}$ we have to prove that $\rho(N) \leqq \vartheta_\bullet(N)$ and thus can assume that $\vartheta_\bullet(N) < \infty$. It follows that $\vartheta_\bullet < \infty$ and hence $\rho < \infty$ on $\{A \in \mathfrak{N} : A \subset N\}$, and then [4], 2.5 implies that $\rho(N) = \vartheta_\bullet(N)$. Thus in fact 2.1 can be applied to ϑ and ρ, and in particular ρ is an outer $\bullet$ premeasure.

3) Now let $f : X \times Y :\to [0,\infty]$ be measurable $\mathfrak{C}(\vartheta_\bullet)$, that is $[f \geqq t] \in \mathfrak{C}(\vartheta_\bullet)$ for $0 < t < \infty$, and $[f > 0]$ be upward enclosable $[\rho^\bullet < \infty]^\sigma$. Then 2.1.*iii*) implies that $\vartheta_\bullet([f \geqq t]) = \rho^\bullet([f \geqq t])$ for $0 < t < \infty$ and hence $\fint f d\vartheta_\bullet = \fint f d\rho^\bullet$. Thus we obtain the final assertion. $\square$

The simplest special case is the one which results from the specialization 2.4 of 2.3.

Specialization 2.8. *Assume that*

– $\varphi : \mathfrak{S} \to [0,\infty[$ *is a content on the ring $\mathfrak{S}$ in X and $\bullet$ continuous,*

– $\psi : \mathfrak{T} \to [0, \infty[$ *is a content on the ring* $\mathfrak{T}$ *in* Y *and* $\bullet$ *continuous,*

so that $\vartheta = \varphi \times \psi : \mathfrak{R} \to [0, \infty[$ *is a content on the ring* $\mathfrak{R} = (\mathfrak{S} \times \mathfrak{T})^\star$ *in* $X \times Y$ *and* $\bullet$ *continuous as well. Then the functions* $f : X \times Y \to [0, \infty]$ *which are measurable* $\mathfrak{C}(\vartheta_\bullet) = \mathfrak{C}(\vartheta^\bullet)$ *and have* $[f > 0]$ *upward enclosable* $[\vartheta^\bullet < \infty]^\sigma$ *fulfil*

$$\begin{aligned} ⨍ f d\vartheta_\bullet &= ⨍ \left(⨍ f(x,y) d\psi_\bullet(y) \right) d\varphi_\bullet(x) \\ &= ⨍ \left(⨍ f(x,y) d\psi^\bullet(y) \right) d\varphi^\bullet(x) = ⨍ f d\vartheta^\bullet. \end{aligned}$$

As above we note for $f : X \times Y \to [0, \infty]$ that $⨍ f d\vartheta^\bullet < \infty$ implies that $[f > 0]$ is upward enclosable $[\vartheta^\bullet < \infty]^\sigma$.

3. The Connection with the Traditional Fubini-Tonelli Theorems

The notion of product for traditional measures has been a delicate one from the start. It appears that the development culminated in the two notions of the *primitive product* and the *c.l.d.* (:=complete locally determined) *product* of measures due to Fremlin [3], Section 251. We start to describe the connection with the present formations.

3.1. Preliminaries on the Product of Measures

The present section assumes a pair of measures $\alpha : \mathfrak{A} \to [0, \infty]$ and $\beta : \mathfrak{B} \to [0, \infty]$ on σ algebras $\mathfrak{A}$ in X and $\mathfrak{B}$ in Y, and their product $\pi := \alpha \times \beta$ on the algebra $\mathfrak{G} := (\mathfrak{A} \times \mathfrak{B})^\star$, which is upward σ continuous and hence an outer σ premeasure. We form the restrictions

$$\begin{aligned} \varphi &:= \alpha|[\alpha < \infty] = \alpha|\mathfrak{S} \quad \text{on } \mathfrak{S} := [\alpha < \infty], \\ \psi &:= \beta|[\beta < \infty] = \beta|\mathfrak{T} \quad \text{on } \mathfrak{T} := [\beta < \infty], \end{aligned}$$

and their product $\vartheta := \varphi \times \psi$ on the ring $\mathfrak{R} := (\mathfrak{S} \times \mathfrak{T})^\star$, which are finite contents and σ continuous in the sense of 2.4 and hence inner and outer σ premeasures. We also consider the restriction $\delta := \pi|[\pi < \infty]$ on the ring $[\pi < \infty]$. Of course $\mathfrak{R} \subset [\pi < \infty] \subset \mathfrak{G}$ and $\vartheta = \delta|\mathfrak{R}$. We proceed to the relations between the inner and outer σ envelopes of these formations.

Remark 3.1. i) From 1.3 and from the definition we have $\alpha^\sigma = \varphi^\sigma$ and $\beta^\sigma = \psi^\sigma$, and $\pi^\sigma = \delta^\sigma \leqq \vartheta^\sigma$, where in the last case $<$ can happen.

ii) For $E \subset X \times Y$ we have

$$\vartheta^\sigma(E) < \infty \implies E \text{ is upward enclosable } \mathfrak{R}^\sigma \implies \vartheta^\sigma(E) = \pi^\sigma(E).$$

Proof. i) We present an example for $<$: Let X be uncountable and α the cardinality on $\mathfrak{A} = \mathfrak{P}(X)$, and $\beta = 0$ on $\mathfrak{B} = \mathfrak{P}(Y)$ in nonvoid Y. Then on the one hand $\pi = 0$, and $X \times Y \in \mathfrak{G}$ implies that $\pi^\sigma = 0$. On the other hand $X \times Y$ is not upward enclosable $\mathfrak{R}^\sigma$, so that $\vartheta^\sigma(X \times Y) = \infty$.

ii) The first $\implies$ is clear. For the second $\implies$ it is to be shown that $\vartheta^\sigma(E) \leqq \pi^\sigma(E)$. Assume that $\pi^\sigma(E) < c < \infty$, and take a sequence $(A_n)_n$ in $\mathfrak{G}$ with $A_n \uparrow A \supset E$ and

$\lim_{n\to\infty} \pi(A_n) < c$. Also take a sequence $(R_n)_n$ in $\mathfrak{R}$ with $R_n \uparrow R \supset E$, which exists by assumption. Then $A_n \cap R_n \in \mathfrak{R}$ with $A_n \cap R_n \uparrow A \cap R \supset E$ and

$$\vartheta^\sigma(E) \leqq \lim_{n\to\infty} \vartheta(A_n \cap R_n) = \lim_{n\to\infty} \pi(A_n \cap R_n) \leqq \lim_{n\to\infty} \pi(A_n) < c.$$

The assertion follows. □

Next define $\mathfrak{H} \subset \mathfrak{P}(X \times Y)$ to consist of the subsets $H \subset X \times Y$ such that $H \subset (P \times Y) \cup (X \times Q)$ for some $P \in \mathfrak{S}$ with $\varphi(P) = \alpha(P) = 0$ and some $Q \in \mathfrak{T}$ with $\psi(Q) = \beta(Q) = 0$. Thus $\mathfrak{H}$ is hereditary and fulfils $\mathfrak{H}^\sigma = \mathfrak{H}$, and the $H \in \mathfrak{H}$ have $\pi^\sigma(H) = \pi^\star(H) = 0$.

Lemma 3.2.

i) *Each $E \in \mathfrak{G}$ with $\pi(E) < \infty$ is of the form $E = R \cup H$ with $R \in \mathfrak{R}$ and $H \in \mathfrak{H} \cap \mathfrak{G}$.*
ii) *Each $E \in \mathfrak{G}^\sigma$ with $\pi^\sigma(E) < \infty$ is of the form $E = R \cup H$ with $R \in \mathfrak{R}^\sigma$ and $H \in \mathfrak{H}$.*

Proof. *i)* From [4], 20.2.2) we have

$$E = \bigcup_{l=1}^{r} A_l \times B_l \quad \text{with } A_1, \cdots, A_r \in \mathfrak{A} \text{ pairwise disjoint and } B_1, \cdots, B_r \in \mathfrak{B},$$

so that the $A_l \times B_l$ are pairwise disjoint and $\pi(E) = \sum_{l=1}^{r} \alpha(A_l)\beta(B_l) < \infty$. It follows that $E = R \cup H$, where R is the union of the $A_l \times B_l$ with $\alpha(A_l) < \infty$ and $\beta(B_l) < \infty$, while H is the union of the $A_l \times B_l$ with either $\alpha(A_l) = \infty$ and hence $\beta(B_l) = 0$ or $\beta(B_l) = \infty$ and hence $\alpha(A_l) = 0$. Thus the representation $E = R \cup H$ is as required.

ii) Take a sequence $(E_n)_n$ in $\mathfrak{G}$ with $E_n \uparrow E$ and hence $\pi(E_n) < \infty$. From *i)* we have $E_n = R_n \cup H_n$ with $R_n \in \mathfrak{R}$ and $H_n \in \mathfrak{H} \cap \mathfrak{G}$. We can assume that $R_n \uparrow$ to some $R \in \mathfrak{R}^\sigma$ and $H_n \uparrow$ to some $H \in \mathfrak{H}^\sigma = \mathfrak{H}$. Then $E = R \cup H$. □

Lemma 3.3. *Let $\lambda : \mathfrak{L} \to [0, \infty[$ be a finite content on a ring. If $(E_n)_n$ is a decreasing sequence in $\mathfrak{L}$ and $(S_n)_n$ is a sequence in $\mathfrak{L}$ with $S_n \subset E_n$ and $\lambda(S_n) = \lambda(E_n)$ for $n \in \mathbb{N}$, then $\lambda(S_1 \cap \cdots \cap S_n) = \lambda(E_n)$ for $n \in \mathbb{N}$.*

Proof. The induction step $1 \leqq n \Rightarrow n + 1$: For $D := S_1 \cap \cdots \cap S_n$ we have

$$\lambda(D \cap S_{n+1}) + \lambda(D \cup S_{n+1}) = \lambda(D) + \lambda(S_{n+1}) = \lambda(E_n) + \lambda(E_{n+1}).$$

Now $D \subset D \cup S_{n+1} \subset E_n \cup E_{n+1} = E_n$ and hence $\lambda(D \cup S_{n+1}) = \lambda(E_n)$. It follows that $\lambda(D \cap S_{n+1}) = \lambda(E_{n+1})$. □

Remark 3.4. $\vartheta_\sigma = \delta_\sigma$.

Proof. $\vartheta_\sigma \leqq \delta_\sigma$ is clear since ϑ is a restriction of δ. To prove $\vartheta_\sigma(E) \geqq \delta_\sigma(E)$ for $E \subset X \times Y$ we can assume that $\delta_\sigma(E) > 0$. Fix $\delta_\sigma(E) > c > 0$, and take a sequence $(E_n)_n$ in $[\pi < \infty]$ with $E_n \downarrow\subset E$ and $\pi(E_n) = \delta(E_n) > c$ for $n \in \mathbb{N}$. From the above 3.2.*i)* we have $E_n = R_n \cup H_n$ with $R_n \in \mathfrak{R}$ and $H_n \in \mathfrak{H} \cap \mathfrak{G}$. Thus $\pi(H_n) = 0$ and hence $\delta(R_n) = \delta(E_n)$. From 3.3 it follows that $\delta(R_1 \cap \cdots \cap R_n) = \delta(E_n)$. Thus the $D_n := R_1 \cap \cdots \cap R_n$ form a sequence in $\mathfrak{R}$ with $D_n \downarrow\subset E$ and $\vartheta(D_n) > c$ for $n \in \mathbb{N}$, so that $\vartheta_\sigma(E) \geqq c$. The assertion follows. □

We combine the above facts with the previous specialization 2.4 and with 2.5 to obtain the consequence which follows.

Consequence 3.5.

0) *We have $\vartheta_\sigma = \delta_\sigma \leqq \pi^\sigma = \delta^\sigma \leqq \vartheta^\sigma$, and all these set functions have the same* Carathéodory class $\mathfrak{C}(\cdot) =: \mathfrak{C}$.

i) $\vartheta_\sigma(E) = \vartheta^\sigma(E)$ *for all* $E \in \mathfrak{C}$ *upward enclosable* $\mathfrak{R}^\sigma$.

ii) $\delta_\sigma(E) = \delta^\sigma(E)$ *for all* $E \in \mathfrak{C}$ *upward enclosable* $[\pi < \infty]^\sigma$.

For the sequel we need one more fact.

Remark 3.5. For $E \in \mathfrak{C}$ we have

$$\vartheta_\sigma(E) = \sup\{\pi^\sigma\big(E \cap (P \times Q)\big) : P \in \mathfrak{S} \text{ and } Q \in \mathfrak{T}\}.$$

Proof. To see $\geqq$ we note from 3.5.*0)i)* that

$$\vartheta_\sigma(E) \geqq \vartheta_\sigma\big(E \cap (P \times Q)\big) = \vartheta^\sigma\big(E \cap (P \times Q)\big) \geqq \pi^\sigma\big(E \cap (P \times Q)\big).$$

To see $\leqq$ fix $c < \vartheta_\sigma(E)$. Since ϑ_σ is inner regular $\mathfrak{R}_\sigma$ there is an $M \in \mathfrak{R}_\sigma$ with $M \subset E$ and $c < \vartheta_\sigma(M)$. Now $M \subset P \times Q$ for some $P \in \mathfrak{S}$ and $Q \in \mathfrak{T}$. Thus $M \subset E \cap (P \times Q)$ and hence $c < \vartheta_\sigma\big(E \cap (P \times Q)\big) \leqq \pi^\sigma\big(E \cap (P \times Q)\big)$ from 3.5.*0)*. □

We turn to the comparison with the two product formations due to Fremlin [3], Section 251. First one defines $\gamma : \mathfrak{P}(X \times Y) \to [0, \infty]$ to be

$$\gamma(M) = \inf\left\{\sum_{l=1}^{\infty}\alpha(A_l)\beta(B_l) : (A_l)_l \text{ in } \mathfrak{A} \text{ and } (B_l)_l \text{ in } \mathfrak{B} \text{ with } M \subset \bigcup_{l=1}^{\infty} A_l \times B_l\right\}.$$

It has been proved in [3], 251E that

1) γ is an outer measure in the Carathéodory sense. Thus $\Gamma := \gamma|\mathfrak{C}(\gamma)$ is a measure on the σ algebra $\mathfrak{C}(\gamma)$.
2) $\mathfrak{A} \times \mathfrak{B} \subset \mathfrak{C}(\gamma)$ and hence $\mathfrak{G}^\sigma \subset \mathrm{A}\sigma(\mathfrak{A} \times \mathfrak{B}) \subset \mathfrak{C}(\gamma)$.
3) $\gamma(A \times B) = \Gamma(A \times B) = \alpha(A)\beta(B)$ for $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$.
4) We add the obvious fact that γ is outer regular $\mathfrak{G}^\sigma$.

From these facts γ can be identified as follows.

Remark 3.7. $\gamma = \pi^\sigma$.

Proof. i) It suffices to prove $\gamma = \pi^\sigma$ on $\mathfrak{G}^\sigma$, because both sides are outer regular $\mathfrak{G}^\sigma$; for γ this is in 4). ii) It suffices to prove $\gamma = \pi^\sigma$ on $\mathfrak{G}$, because both sides are upward σ continuous on $\mathfrak{G}^\sigma$; for γ this is in 1)2). iii) Now let $E \in \mathfrak{G}$ and hence

$$E = \bigcup_{l=1}^{r} A_l \times B_l \quad \text{with } A_1, \cdots, A_r \in \mathfrak{A} \text{ pairwise disjoint and } B_1, \cdots, B_r \in \mathfrak{B}.$$

From 1)2)3) then

$$\gamma(E) = \Gamma(E) = \sum_{l=1}^{r}\Gamma(A_l \times B_l) = \sum_{l=2}^{r}\alpha(A_l)\beta(B_l) = \pi(E) = \pi^\sigma(E).$$

□

The measure $\Gamma : \mathfrak{C}(\gamma) = \mathfrak{C}(\pi^\sigma) = \mathfrak{C} \to [0,\infty]$ is defined to be the *primitive product* of α and β. The *c.l.d. product* $\nu : \mathfrak{C} \to [0,\infty]$ of α and β is defined to be

$$\nu(E) = \sup\{\Gamma\big(E \cap (P \times Q)\big) : P \in \mathfrak{S} \text{ and } Q \in \mathfrak{T}\} \quad \text{for } E \in \mathfrak{C}.$$

Thus we obtain the representations which follow.

Theorem 3.8. *We have $\Gamma = \pi^\sigma|\mathfrak{C} = \delta^\sigma|\mathfrak{C}$ and $\nu = \vartheta_\sigma|\mathfrak{C} = \delta_\sigma|\mathfrak{C}$. Thus $\nu \leqq \Gamma$. For $f : X \times Y \to [0,\infty]$ measurable $\mathfrak{C}$ it follows that*

$$⨍ f d\pi^\sigma = ⨍ f d\delta^\sigma = \int f d\Gamma \quad \text{and} \quad ⨍ f d\vartheta_\sigma = ⨍ f d\delta_\sigma = \int f d\nu.$$

We note at once the connection with the inner σ premeasures.

Proposition 3.9. *Assume that $\alpha = \xi_\sigma|\mathfrak{C}(\xi_\sigma)$ and $\beta = \eta_\sigma|\mathfrak{C}(\eta_\sigma)$ are the maximal inner σ extensions of inner σ premeasures ξ on X and η on Y. Then the c.l.d. product $\nu : \mathfrak{C} \to [0,\infty]$ of α and β is the maximal inner σ extension $\Lambda = \lambda_\sigma|\mathfrak{C}(\lambda_\sigma)$ of the product inner σ premeasure $\lambda = \xi \times \eta$ on $X \times Y$.*

Proof. We know from 1.9 that Λ is an extension of $\alpha \times \beta = \pi$ and hence an extension of $\delta = \pi|[\pi < \infty]$. Thus $\delta_\sigma = \Lambda = \lambda_\sigma$ on $[\pi < \infty] \subset \mathfrak{C}(\lambda_\sigma)$ and hence on $[\pi < \infty]_\sigma$. Now δ_σ is inner regular $[\pi < \infty]_\sigma$; and also λ_σ, because the domain of $\lambda = \xi \times \eta$ is $\subset (\mathfrak{S} \times \mathfrak{T})^\star = \mathfrak{R} \subset [\pi < \infty]$. It follows that $\delta_\sigma = \lambda_\sigma$ on $\mathfrak{P}(X \times Y)$, and hence $\nu = \delta_\sigma|\mathfrak{C} = \delta_\sigma|\mathfrak{C}(\delta_\sigma) = \lambda_\sigma|\mathfrak{C}(\lambda_\sigma) = \Lambda$. □

3.2. The Fubini-Tonelli Theorem

We combine the fundamental inequalities 1.11 and 1.13 with 3.1.i) and 3.5.*0*).

Theorem 3.10. *For all $f : X \times Y \to [0,\infty]$ we have*

$$\begin{aligned} ⨍ f d\delta_\sigma = ⨍ f d\vartheta_\sigma &\leqq ⨍ \Big(⨍ f(x,y) d\psi_\sigma(y) \Big) d\varphi_\sigma(x) \\ &\leqq ⨍ \Big(⨍ f(x,y) d\psi^\sigma(y) \Big) d\varphi^\sigma(x) = ⨍ \Big(⨍ f(x,y) d\beta^\sigma(y) \Big) d\alpha^\sigma(x) \\ &\leqq ⨍ f d\pi^\sigma = ⨍ f d\delta^\sigma \leqq ⨍ f d\vartheta^\sigma. \end{aligned}$$

Next we recall two basic facts from Section 2, this time combined with 2.5.

i) For all $f : X \times Y \to [0,\infty]$ the definition of the Choquet integral implies that

$$⨍ f d\vartheta^\sigma < \infty \Longrightarrow [f > 0] \text{ is upward enclosable } \mathfrak{R}^\sigma,$$

$$⨍ f d\pi^\sigma = ⨍ f d\delta^\sigma < \infty \Longrightarrow [f > 0] \text{ is upward enclosable } [\pi < \infty]^\sigma.$$

ii) For all $f : X \times Y \to [0,\infty]$ measurable $\mathfrak{C}$ we conclude from 2.4.*iii*) that

$$[f > 0] \text{ upward enclosable } \mathfrak{R}^\sigma \Longrightarrow ⨍ f d\vartheta_\sigma = ⨍ f d\vartheta^\sigma,$$

$$[f > 0] \text{ upward enclosable } [\pi < \infty]^\sigma \Longrightarrow ⨍ f d\delta_\sigma = ⨍ f d\delta^\sigma.$$

We combine ii) with 3.10 to obtain the subsequent Fubini-Tonelli theorem. We invoke 3.8 in order to formulate the result in terms of Γ and ν, and leave aside the less important ϑ^σ. For the connection with Section 2 see the final Remark 3.12 below.

Theorem 3.11. *The functions $f : X \times Y \to [0,\infty]$ which are measurable $\mathfrak{C}$ and have $[f>0]$ upward enclosable $[\pi<\infty]^\sigma$ fulfil*

$$\int f d\nu = \int\!\!\!\!- \left(\int\!\!\!\!- f(x,y) d\psi_\sigma(y) \right) d\varphi_\sigma(x) = \int\!\!\!\!- \left(\int\!\!\!\!- f(x,y) d\beta^\sigma(y) \right) d\alpha^\sigma(x) = \int f d\Gamma.$$

As far as the author can see the above Theorem 3.11 comprises the present versions of the Fubini-Tonelli theorems - with two exceptions that will be discussed thereafter. This statement requires two amendments: On the one hand our theorem contains, as before in Section 2, in contrast to the traditional ones no assertion relative to the legitimacy and to the kind of existence of the respective iterated integrals: for the simple reason that the use of our envelopes and of the Choquet integral makes this problem disappear, and the earlier assertions themselves do not seem to be of particular interest. On the other hand both the previous formations Γ and ν and the present π and ϑ and δ, as well as ϑ and ρ in Section 2, are symmetric in the two factors. Therefore our theorem implies the usual assertions on the equality under the two possible orders in the respective iterated integrals.

We quote a few examples of Fubini-Tonelli theorems in the recent textbook literature, which can be considered to be representative. First of all one often assumes the measures α and β to be σ *finite*: then $X \times Y \in \mathfrak{R}^\sigma$, and hence all functions $f : X \times Y \to [0,\infty]$ have $[f>0]$ upward enclosable $\mathfrak{R}^\sigma$. Examples are Elstrodt [2], Theorem 2.1, p. 175–176, Pap [9], Corollary 162, and Rao [10], Theorem 2, p. 385. Examples of Fubini-Tonelli theorems without this assumption are Fremlin [3], Exercises 252Ycd, Pap [9], Theorem 160 and Rao [10], Theorem 1, p. 381. These theorems assume $\int f d\Gamma < \infty$ and thus result from our 3.11 combined with the above i). Moreover Fremlin [3], Theorem 252G and Pap [9], Theorem 161 are examples of partial assertions which result from the first $\leqq$ in our previous 3.10.

After this we pass to the two particular former Fubini-Tonelli theorems emphasized above. These are Fremlin [3], 252B and [4], 22.9 due to the present author. The basic deviations from the present treatment are that in these theorems the σ finiteness requirement is directed to the *second factor alone*, and that in compensation from the two fundamental inequalities in 3.10 the *left inner one alone* is claimed to turn into an equation. Of course this requires a different technique. It is not clear and rather doubtful whether the idea can be carried through in the present context, and the point does not seem to be in special demand. In [4], 22.9 one assumes, in accord with the spirit of [4], the measures α and β to be as in 3.9, and the assertion is for their product $\nu = \Lambda$. In [3], 252B the situation is somewhat more comprehensive, in that one admits certain cases where instead of α and β their *completions* are required to be as in 3.9. However, the essential difference between the two presentations is that [4], 22.9 is for both $\bullet = \sigma\tau$ in uniform manner, like in the present Section 2, while in [3] the entire treatment of $\bullet = \tau$ is separated from the abstract measure situation $\bullet = \sigma$ and restricted to the *topological* context.

Remark 3.12. In the deduction of Theorem 3.11 the substance of Section 2 has been used in two applications of 2.4, but *not* in form of the previous Fubini-Tonelli Theorem 2.7. In fact, it is quite clear that 3.11 is not an immediate consequence of 2.7. In the sequel we want to show that in the present context the natural application of 2.7 leads to a Fubini-Tonelli theorem which is similar to but different from 3.11: If 2.7 starts from $\varphi : \mathfrak{S} \to [0, \infty[$ and $\psi : \mathfrak{T} \to [0, \infty[$ with $\vartheta = \varphi \times \psi : \mathfrak{R} \to [0, \infty[$, then it is most natural to take $\mathfrak{P} = \mathfrak{A}$ and $\mathfrak{Q} = \mathfrak{B}$, so that $\mathfrak{N} = (\mathfrak{P} \times \mathfrak{Q})^\star = (\mathfrak{A} \times \mathfrak{B})^\star = \mathfrak{G}$. Then one obtains, via $\xi = \varphi_\sigma|\mathfrak{A}$ and $\eta = \psi_\sigma|\mathfrak{B}$ and 3.5.*0*), the outer σ premeasure $\rho = \vartheta_\sigma|\mathfrak{N} = \vartheta_\sigma|\mathfrak{G} \leqq \pi^\sigma|\mathfrak{G} = \pi$. Thus the final assertion for the functions $f : X \times Y \to [0, \infty]$ measurable $\mathfrak{C}(\vartheta_\sigma) = \mathfrak{C}$ reads

$$\begin{aligned} \int\!\!\!\!\!- f d\nu &= \int\!\!\!\!\!- \left(\int\!\!\!\!\!- f(x,y) d\psi_\sigma(y) \right) d\varphi_\sigma(x) \\ &= \int\!\!\!\!\!- \left(\int\!\!\!\!\!- f(x,y) d\eta^\sigma(y) \right) d\xi^\sigma(x) = \int\!\!\!\!\!- f d\rho^\sigma, \end{aligned}$$

whenever $[f > 0]$ is upward enclosable $[\rho < \infty]^\sigma$. Thus compared with 3.11 the assertion is for a larger class of functions f but furnishes a weaker result.

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