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External Analysis of Boundary Points of Convex Sets: Supporting Cones and Drops

José Pedro Moreno
Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, Madrid 28049, Spain
josepedro.moreno@uam.es

Alberto Seeger
Department of Mathematics, University of Avignon, 33 rue Louis Pasteur, 84000 Avignon, France
alberto.seeger@univ-avignon.fr

Dedicated to Stephen Simons on the occasion of his 70th birthday.

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Let K be a solid closed convex set in a normed space. We study the geometry of the boundary of K with the help of some multivalued maps defined on the exterior of K. Supporting cones and drops are the main tools of analysis.

Keywords: Convex set, boundary point, supporting cone, drop, multivalued map

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1. Introduction

This work deals with a central issue of classical and modern convex analysis, namely, the description of some geometric properties of boundary points of convex sets. Convexity adds a lot of structure to a set and enables one to use separation arguments and many other useful mathematical tools. Despite this fact, the boundary of a convex set remains a complex mathematical object, specially so in the context of a general normed space. Finite dimensionality is sometimes a useful assumption, but it does not necessarily help in getting a better intuition of the situation, unless, of course, one considers a convex set in the plane or in a three dimensional Euclidean space.

In the sequel, $(X, \|\cdot\|)$ is a real normed space of dimension greater than or equal to 2. Unless explicitly stated otherwise, no special assumption is made concerning the structure of the closed unit ball B_X of the space X. The purpose of this work is studying the geometric nature of the boundary points of a given element K in the class

$$\Xi(X) \equiv \text{ solid closed convex proper subsets of } X.$$

As usual, a proper subset of X is any set different from the whole space X. That K is

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solid simply means that its topological interior is nonempty. On some occasions we ask K to be bounded, but this is done only if the need arises.

Most of the properties of boundary points that we study are classical: smoothness, strict convexity, extremality, etc. However, other properties like, for instance, local uniform rotundity, are less standard. Strict convexity and local uniform rotundity will not be treated here, but in the companion paper [23].

The methodology we use is something that deserves a comment. Our strategy consists in introducing a few specially relevant multivalued maps of the form $F_K : X \rightrightarrows X$ (or $F_K : X \to 2^X$ in the usual set-theoretical notation) and see what happens with the set $F_K(x)$ when the argument $x \in X$ approaches the boundary of K from the exterior. The expression "external analysis" used in the title is in tune with this strategy.

It is also of interest to see what happens with $F_K(x)$ when x remains away from the boundary or, what is perhaps more striking, when x escapes to infinity. The last situation occurs if one wishes to examine the boundary of K by moving farther and farther away from K.

The double arrow notation is well suited for emphasizing the multivalued character of F_K. Which example of multivalued map F_K do we have in mind? Perhaps the most natural way of apprehending the geometric nature of the boundary of K is by using the map $E_K : X \rightrightarrows X$ whose value at a general argument x is given by

$$E_K(x) = \bigcup_{\alpha>0} \alpha(K - x). \tag{1}$$

Alternatively, one may consider the variant

$$T_K(x) = \mathrm{cl}\left[\bigcup_{\alpha>0} \alpha(K - x)\right] \tag{2}$$

obtained from (1) by performing a closure operation. The multivalued map $T_K : X \rightrightarrows X$ will play a protagonic role in all this paper. In the same vein, one may consider also the drop

$$C_K(x) = x + [0,1](K - x) = x + \bigcup_{\alpha\in[0,1]} \alpha(K - x) \tag{3}$$

with vertex at x, and its closed version

$$D_K(x) = \mathrm{cl}\,(x + [0,1](K - x))\,. \tag{4}$$

Drops and closed drops are sets whose introduction in the mathematical literature goes back at least to Daneš [7]. They have been extensively used in connection with the geometric analysis of Banach spaces by Penot [26], Rolewicz [17, 28], Montesinos [21], and other authors. Supporting cones and closed drops are closely related mathematical objects. To avoid unnecessary repetitions, drops will be mentioned only when their use is essential.

The sets listed in (1)–(4) are viewed as functions of the parameter $x \in X$, which supposedly moves in the exterior of K. Examples coming from the theory of illumination of convex sets are treated in [23].

Enough has been said about motivation and methodology. We end this introductory section by fixing the basic notation used in this work. One writes $\mathrm{int}(C)$, $\mathrm{ext}(C)$, and ∂C, for indicating, respectively, the interior, the exterior, and the boundary of a given set C. The symbol S_X refers to the unit sphere of X.

2. Preliminary results on supporting cones

Convex cones have revealed in last decades as an important tool to study many mathematical problems in nonlinear analysis. Recall that a convex cone is a set stable under addition and under multiplication by positive scalars.

The sets introduced in (1) and (2) have a clear geometric interpretation. According to the terminology in vogue in the seventies, $E_K(x)$ is the cone of inner displacements for K relative to x. As Schneider [29] and many other authors, we refer to $T_K(x)$ as the supporting cone to K at x. The notation that we use is consistent with the fact that $T_K(x)$ coincides with the tangent cone to K at x when x belongs to K. The usual practice in convex analysis is declaring $T_K(x)$ to be empty if the point x does not belong to K. We do not follow here this convention, preferring instead to keep the definition (2) in force even if x is in the exterior of K. The same remark applies to the set $E_K(x)$. The basic properties of $E_K(x)$ and $T_K(x)$ are recalled in the next lemma.

Lemma 2.1. *Let $K \in \Xi(X)$ and $x \in X$. Then,*

(a) $E_K(x)$ *and* $T_K(x)$ *are solid convex cones.*
(b) $T_K(x) = \mathrm{cl}\left[E_{\mathrm{int}(K)}(x)\right]$
(c) $E_{\mathrm{int}(K)}(x) = \mathrm{int}[T_K(x)]$.

Proof. The statement (a) is part of the folklore on supporting cones. A proof of (b) for the particular case $x \in K$ is given, e.g., in Laurent's book [18, Chapter 1]. We propose a different proof that takes care of the case $x \in \mathrm{ext}(K)$ as well. One just needs to check that

$$T_K(x) \subset \mathrm{cl}[E_{\mathrm{int}(K)}(x)], \tag{5}$$

the reverse inclusion being trivial. Let $d \in T_K(x)$ and write $d = \lim_{n\to\infty} \alpha_n(z_n - x)$ with $\alpha_n > 0$ and $z_n \in K$. Pick $\bar{z} \in \mathrm{int}(K)$ and define

$$\hat{z}_n = (1-\lambda_n)z_n + \lambda_n \bar{z}$$

with $\{\lambda_n\}_{n\in\mathbb{N}}$ being a sequence a positive reals going to 0 fast enough so that

$$\lim_{n\to\infty} \lambda_n \alpha_n (z_n - \bar{z}) = 0.$$

By construction, each $\hat{z}_n$ belongs to the interior of K and $d = \lim_{n\to\infty} \alpha_n(\hat{z}_n - x)$. This completes the proof of (5) since, by definition,

$$E_{\mathrm{int}(K)}(x) = \bigcup_{\alpha>0} \alpha\,\{\mathrm{int}(K) - x\}\,.$$

Part (c) is a direct consequence of (b). $\square$

3. Pointedness and solidity of supporting cones

Pointedness is a property that usually comes into discussion while dealing with convex cones. By definition, a convex cone Q is pointed if its lineality space $\mathrm{lin}(Q) = Q \cap -Q$ is contained in $\{0\}$. Geometrically speaking, this means that Q contains no line passing through the origin.

Most probably, the next proposition is partially known. The complete proof is given in order to fix some notation and clarify a couple of things.

Proposition 3.1. *Let $K \in \Xi(X)$.*

(*a*) *$E_K(x)$ is pointed for all $x \in \mathrm{ext}(K)$.*
(*b*) *When $u \in \partial K$, then $E_K(u)$ is pointed if and only if u is an extreme point of K.*
(*c*) *K is line free if and only if $T_K(x)$ is pointed for all $x \in \mathrm{ext}(K)$.*

Proof. The proof of (*a*) is easy and omitted. Let $u \in \partial K$. If $E_K(u)$ is not pointed, then one can find a nonzero vector $d \in X$ such that $d = \alpha(z-u) = -\beta(w-u)$, with $\alpha, \beta > 0$ and $z, w \in K$. One gets $z \neq w$ and

$$u = \frac{\alpha}{\alpha+\beta}\, z + \frac{\beta}{\alpha+\beta}\, w,$$

negating in this way the extremality of u. Conversely, assume that u is not extremal, i.e., it is expressible as midpoint of two different vectors from K, say z and w. Hence, $z-u$ is a nonzero vector belonging to $\mathrm{lin}[E_K(u)]$. This negates the pointedness of $E_K(u)$. We now take care of (*c*). As a preliminary result of its own interest, we show that the equality

$$\mathrm{lin}[T_K(x)] = \mathrm{lin}[\mathrm{rec}(K)] \tag{6}$$

holds for all $x \in \mathrm{ext}(K)$. In the above equality one considers an arbitrary set $K \in \Xi(X)$, and uses the symbol

$$\mathrm{rec}(K) = \left\{ h \in X : h = \lim_{n\to\infty} \alpha_n z_n \text{ with } \{\alpha_n\}_{n\in\mathbb{N}} \to 0^+ \text{ and } z_n \in K,\ \forall n \in \mathbb{N} \right\}$$

for denoting its recession cone. The inclusion $\mathrm{rec}(K) \subset T_K(x)$ is trivial and holds, in fact, for any $x \in X$. Hence, $\mathrm{lin}[\mathrm{rec}(K)] \subset \mathrm{lin}[T_K(x)]$. Conversely, let $x \in \mathrm{ext}(K)$ and $d \in \mathrm{lin}[T_K(x)]$. If $d = 0$, then we are done. Otherwise, one writes

$$d = \lim_{n\to\infty} \alpha_n (z_n - x),$$
$$-d = \lim_{n\to\infty} \beta_n (w_n - x),$$

with $\alpha_n, \beta_n > 0$ and $z_n, w_n \in K$. By adding both equalities and rearranging, one gets

$$\lim_{n\to\infty} (\alpha_n + \beta_n)\,(\xi_n - x) = 0,$$

where the convex combination

$$\xi_n = \frac{\alpha_n}{\alpha_n+\beta_n}\, z_n + \frac{\beta_n}{\alpha_n+\beta_n}\, w_n$$

lies in K. Since

$$(\alpha_n + \beta_n)\,\mathrm{dist}\,[x, K] \leq (\alpha_n + \beta_n)\|\xi_n - x\|$$

and $x \notin K$, it follows that $\alpha_n + \beta_n \to 0$, i.e., $\alpha_n \to 0$ and $\beta_n \to 0$. Hence, $d \in \mathrm{rec}(K)$ and $-d \in \mathrm{rec}(K)$. This completes the proof of (6). Suppose now that K is line free, i.e., it does not contain a line. Given that K is a closed convex set, its recession cone admits the equivalent characterization

$$\mathrm{rec}(K) = \{h \in X : \hat{z} + \mathbb{R}_+ h \subset K\},$$

where the choice of $\hat{z} \in K$ is irrelevant. The assumption that K is line free amounts then to saying that $\mathrm{lin}[\mathrm{rec}(K)] = \{0\}$. Hence, $T_K(x)$ is pointed for each $x \in \mathrm{ext}(K)$. The "if" part of (c) is clear. In fact, if $T_K(x)$ is pointed for some $x \in X$, then K is line free according to (6). □

We now go beyond Proposition 3.1(c) and say a few words on the degree of pointedness of $T_K(x)$. Our purpose is showing that if K is bounded, then $T_K(x)$ becomes "highly" pointed as $\|x\| \to \infty$. The converse statement is also true. In fact, one has the following result.

Theorem 3.2. *A set $K \in \Xi(X)$ is bounded if and only if*

$$\lim_{\|x\|\to\infty} \mathrm{diam}\,[T_K(x) \cap S_X] = 0. \tag{7}$$

Proof. Let us start by deriving a upper bound for the diameter of $T_K(x) \cap S_X$. We claim that, for every $x \in \mathrm{ext}(K)$,

$$\mathrm{diam}\,[T_K(x) \cap S_X] \leq 2\,\frac{\mathrm{diam}(K)}{\mathrm{dist}\,[x, K]}\,, \tag{8}$$

where $\mathrm{dist}\,[x, K] = \sup_{z \in K} \|x - z\|$ stands for the distance from x to the set K. Take any $x \in \mathrm{ext}(K)$. Since

$$T_K(x) \cap S_X = \mathrm{cl}[E_K(x)] \cap S_X = \mathrm{cl}[E_K(x) \cap S_X],$$

it follows that

$$\begin{aligned}
\mathrm{diam}\,[T_K(x) \cap S_X] &= \mathrm{diam}\,[E_K(x) \cap S_X] \\
&= \sup_{u,v \in K} \left\| \frac{u - x}{\|u - x\|} - \frac{v - x}{\|v - x\|} \right\| \\
&\leq 2 \sup_{u,v \in K} \frac{\|u - v\|}{\max\{\|u - x\|, \|v - x\|\}} \\
&\leq \frac{2}{\mathrm{dist}\,[x, K]} \sup_{u,v \in K} \|u - v\|\,.
\end{aligned}$$

Of course, the third line is obtained by applying the Massera-Schaeffer inequality [20]

$$\left\| \frac{a}{\|a\|} - \frac{b}{\|b\|} \right\| \leq 2\,\frac{\|a - b\|}{\max\{\|a\|, \|b\|\}}\,,$$

which holds for any pair a, b of nonzero vectors in X. This confirms our claim. Now, observe that if K is a bounded set, then $\mathrm{diam}(K)$ is finite and $\mathrm{dist}\,[x, K] \to \infty$ as $\|x\| \to \infty$. So, the inequality (8) takes care of the announced behavior of $\mathrm{diam}[T_K(x) \cap S_X]$. If K is unbounded, then it contains a sequence $\{x_n\}_{n\in\mathbb{N}}$ going to infinity in norm. Since K is convex and solid, one may assume that each x_n lies in the interior of K. In such a case, $T_K(x_n) = X$ and $\mathrm{diam}[T_K(x_n) \cap S_X] = 2$ for all $n \in \mathbb{N}$. Hence, $\mathrm{diam}[T_K(x_n) \cap S_X]$ does not go to 0 despite the fact that $\|x_n\| \to \infty$. □

Remark 3.3. The constant 2 appearing in (8) comes from the Massera-Schaeffer inequality. It is the best possible constant in a general normed space X, but it can be sharpened if $\|\cdot\|$ has a bit more structure. For instance, in a Hilbert space setting the constant 2 can be changed by 1.

What sort of geometric meaning is conveyed by the condition (7)? The first thing that comes to mind is that for a parameter x large in norm, the convex cone $T_K(x)$ is still solid but looks like a half-line. Although this observation is consistent with intuition, it requires a formal justification. In the sequel the notation $\mathbb{R}_+e$ indicates the half-line emanating from the origin and having $e \in S_X$ as direction. The theorem below clarifies the meaning of the expression $T_K(x) \approx \mathbb{R}_+e_x$ for $\|x\|$ large. As we shall see in a moment, the above expression must be understood as asymptotic approximation in the space

$$\mathcal{Q}(X) \equiv \text{ nontrivial closed convex cones in } X$$

equipped with the truncated Pompeiu-Hausdorff metric[1]

$$\varrho(Q_1, Q_2) = \mathrm{haus}(Q_1 \cap B_X, Q_2 \cap B_X).$$

That a convex cone is nontrivial means that it is different from the singleton $\{0\}$ and different from the whole space X.

Theorem 3.4. *$K \in \Xi(X)$ is bounded if and only if there exists a function $x \in X \mapsto e_x \in S_X$ such that*

$$\lim_{\|x\|\to\infty} \varrho(T_K(x), \mathbb{R}_+e_x) = 0. \tag{9}$$

Proof. The proof greatly simplifies if distances between closed convex cones are measured by means of the expression

$$\hat{\varrho}(Q_1, Q_2) = \max\left\{ \sup_{a\in Q_1\cap S_X} \mathrm{dist}[a, Q_2], \sup_{b\in Q_2\cap S_X} \mathrm{dist}[b, Q_1] \right\}.$$

In a Hilbert space setting, ϱ and $\hat{\varrho}$ are exactly the same metric. In a general normed space, $\hat{\varrho}$ is not truely a metric because it does not satisfy the triangular inequality. This fact has no incidence in the proof of the theorem. The only thing one needs to know is that[2]

$$\hat{\varrho}(Q_1, Q_2) \;\le\; \varrho(Q_1, Q_2) \;\le\; 2\,\hat{\varrho}(Q_1, Q_2). \tag{10}$$

[1]It is not clear who was the first author that considered ϱ as tool for measuring distances between closed convex cones. In any case, for closed linear subspaces of a Banach space, the use of ϱ goes back at least to Gurariĭ [13]. The survey paper [16] contains a wealth of information on general properties of ϱ and several other metrics on $\mathcal{Q}(X)$.

[2]This chain of inequalities can be found, for instance, in [16, Theorem 3.8]. As a matter of fact, the factor 2 in the last part of (10) can be changed by something better, namely, the sphericity coefficient of the normed space X. The details can be consulted in [16, Section 3].

A natural candidate as vector e_x is anyone satisfying

$$e_x \in T_K(x) \cap S_X, \tag{11}$$

that is, $x \mapsto e_x$ is taken as a selection of the multivalued map $T_K(\cdot) \cap S_X$. The choice (11) implies that $\mathbb{R}_+ e_x \subset T_X(x)$, and therefore

$$\hat{\varrho}(T_K(x), \mathbb{R}_+ e_x) = \sup_{a \in T_X(x) \cap S_X} \operatorname{dist}[a, \mathbb{R}_+ e_x].$$

Another consequence of (11) is that

$$\operatorname{dist}[a, \mathbb{R}_+ e_x] \leq \|a - e_x\| \leq \operatorname{diam}[T_K(x) \cap S_X]$$

for all $a \in T_X(x) \cap S_X$. By putting all the pieces together, one gets

$$\varrho(T_K(x), \mathbb{R}_+ e_x) \leq 2 \operatorname{diam}[T_K(x) \cap S_X]. \tag{12}$$

If K is bounded, then (9) follows as a consequence of (12) and Theorem 3.2. If K is not bounded, then one can take a sequence $\{x_n\}_{n \in \mathbb{N}}$ in the interior of K and such that $\|x_n\| \to \infty$. In such a case, $T_K(x_n) = X$ for all $n \in \mathbb{N}$, and the condition (9) is violated. □

If $\|\cdot\|$ derives from an inner product $\langle \cdot, \cdot \rangle$, then it is possible to define the maximal angle $\theta_{\max}(Q)$ of any closed convex cone Q in X. One simply writes

$$\theta_{\max}(Q) = \sup_{a, b \in Q \cap S_X} \arccos\langle a, b \rangle.$$

The term $\alpha_K(x) = \theta_{\max}(T_K(x))$ is sometimes called the aperture angle of x relative to K. Aperture angles have been considered in a number of papers, specially when K is a convex polytope of dimension two or three. See Figure 3.1 for an illustration of this concept and references [1, 5, 24] for various applications. Following the terminology of [14], one refers to the coefficient

$$\sigma(Q) = \cos\left(\frac{\theta_{\max}(Q)}{2}\right)$$

as the angular index of pointedness of the closed convex cone Q. The last conclusion of Corollary 3.5 says that $\sigma(T_K(x)) \to 1$ as $\|x\| \to \infty$. This is a more formal way of expressing that $T_K(x)$ becomes highly pointed when $\|x\|$ becomes large.

Corollary 3.5. *Let X be a Hilbert space and $K \in \Xi(X)$ be bounded. Then,*

$$\cos[\alpha_K(x)] \;\geq\; 1 - \frac{1}{2}\left(\frac{\operatorname{diam}(K)}{\operatorname{dist}[x, K]}\right)^2 \tag{13}$$

for all $x \in \operatorname{ext}(K)$. In particular, $\alpha_K(x) \to 0$ as $\|x\| \to \infty$.

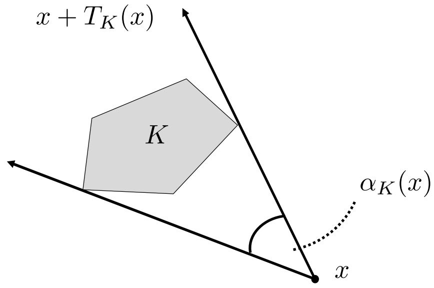

Figure 3.1: Aperture angle of x relative to K.

Proof. Since X is a Hilbert space, we leave aside the Massera-Schaeffer inequality and use instead the Dunkl-Williams inequality [8], which asserts that

$$\left\| \frac{a}{\|a\|} - \frac{b}{\|b\|} \right\| \leq \left(\frac{2}{\|a\| + \|b\|} \right) \|a - b\|$$

for any pair a, b of nonzero vectors in X. By proceeding as in the proof of Theorem 3.2, one gets this time

$$\operatorname{diam}[T_K(x) \cap S_X] \leq \frac{\operatorname{diam}(K)}{\operatorname{dist}[x, K]} \tag{14}$$

for every $x \in \operatorname{ext}(K)$. Passing from diameters to angles is a matter of exploiting the general identity

$$\|a - b\|^2 = 2(1 - \langle a, b \rangle) \quad \forall a, b \in S_X. \tag{15}$$

The inequality (13) is obtained by combining (14) and (15). The details are omitted. □

In fact, one can write the above corollary in a stronger form: $K \in \Xi(X)$ is bounded if and only if $\alpha_K(x) \to 0$ as $\|x\| \to \infty$. However, we prefer the actual formulation of Corollary 3.5 because it underlines the role of (13) as bound for the aperture angle $\alpha_K(x)$. The next example shows that this bound is optimal.

Example 3.6. Consider the set $K = \{z \in \mathbb{R}^2 : z_1^2 + z_2^2 \leq 1, z_1 \geq 0\}$ and the sequence $\{x_n\}_{n \in \mathbb{N}}$ given by $x_n = (-n, 0)$. A matter of computation shows that $\operatorname{dist}[x_n, K] = n$, $\operatorname{diam}(K) = 2$, and

$$\operatorname{diam}[T_K(x_n) \cap S_X] = \frac{2}{\sqrt{1 + n^2}}.$$

Hence, the product $\operatorname{dist}[x_n, K] \operatorname{diam}[T_K(x_n) \cap S_X]$ converges to $\operatorname{diam}(K)$. This shows that (14) cannot be sharpened. This is what we mean by saying that (13) is optimal.

And what about the degree of solidity of $T_K(x)$ when x gets away from K? There are manifold ways of measuring the degree of solidity of a closed convex cone Q in a normed space X. Perhaps the best known tool is the Frobenius solidity coefficient

$$\Phi_{\text{frob}}[Q] = \sup\{r : \|z\| = 1, \ r \in [0, 1], \ z + rB_X \subset Q\}. \tag{16}$$

As a mnemotechnic rule for this concept one can use

$$\Phi_{\text{frob}}[Q] = \left\{ \begin{array}{l} \text{radius of the largest ball centered} \\ \text{at a unit vector and contained in } Q, \end{array} \right. \tag{17}$$

but the latter equality is slightly abusive in the context of a general normed space X. Although the supremum (16) is always well defined, the largest ball mentioned in (17) is not necessarily unique and may even fail to exist.

The Frobenius solidity coefficient has been extensively studied in [14, 15] and used in concrete applications by Freund and collaborators [9, 10, 11, 12].

The next result is similar in spirit to Theorem 3.2. This time the emphasis is not in pointedness, but in solidity. That X is a reflexive Banach space helps in deriving a short and elegant proof, but most likely one could dispense with such assumption.

Theorem 3.7. *Let X be a reflexive Banach space. Then, the set $K \in \Xi(X)$ is bounded if and only if*

$$\lim_{\|x\|\to\infty} \Phi_{\text{frob}}[T_K(x)] = 0. \tag{18}$$

Proof. If the set K is not bounded, then one proceeds as in the proof of Theorem 3.2. The key observation is that $\Phi_{\text{frob}}[T_K(x)] = 1$ for all $x \in \text{int}(K)$. Suppose now that K is bounded. As shown in [15, Corollary 9], when X is Banach and reflexive, the function Φ_{frob} is Lipschitz continuous on the metric space $(\mathcal{Q}(X), \varrho)$. More precisely,

$$|\Phi_{\text{frob}}(Q_1) - \Phi_{\text{frob}}(Q_2)| \le 2\varrho(Q_1, Q_2) \quad \forall Q_1, Q_2 \in \mathcal{Q}(X). \tag{19}$$

Since $\Phi_{\text{frob}}(\mathbb{R}_+ e) = 0$ for all $e \in S_X$, the Lipschitz condition (19) yields

$$\Phi_{\text{frob}}[T_K(x)] \le 2\varrho(T_K(x), \mathbb{R}_+ e_x).$$

Of course, one chooses e_x as in Theorem 3.4 and let $\|x\| \to \infty$. □

Geometrically speaking, the limit in (18) indicates that $T_K(x)$ looses its solidity as x escapes to infinity. This is a natural thing to happen if K is bounded.

4. Semicontinuity properties of supporting cones

Recall that a multivalued map $\Gamma : X \rightrightarrows X$ on a normed space is said to be lower-semicontinuous at a reference point $\bar{x} \in X$ if

$$\Gamma(\bar{x}) \subset \liminf_{x\to\bar{x}} \Gamma(x).$$

This amounts to saying that for each open set O with $\Gamma(\bar{x}) \cap O \neq \emptyset$, there exists a neighborhood N of $\bar{x}$ such that $\Gamma(x) \cap O \neq \emptyset$ for all $x \in N$. The natural counterpart of lower-semicontinuity is not upper-semicontinuity in the sense of Berge [4], but a concept called outer-semicontinuity. One says that Γ is outer-semicontinuous at $\bar{x} \in X$ if

$$\limsup_{x\to\bar{x}} \Gamma(x) \subset \Gamma(\bar{x}).$$

Outer-semicontinuity of Γ at each point in X is equivalent to saying that the graph of Γ is closed. The upper and lower limits mentioned above are understood in the Painlevé-Kuratowski sense, i.e.,

$$\limsup_{x\to\bar{x}} \Gamma(x) = \{y \in X : \liminf_{x\to\bar{x}} \text{dist}[y, \Gamma(x)] = 0\},$$
$$\liminf_{x\to\bar{x}} \Gamma(x) = \{y \in X : \lim_{x\to\bar{x}} \text{dist}[y, \Gamma(x)] = 0\}.$$

Calculus rules for evaluating these sets can be found in the standard books [3, 27].

Proposition 4.1. *If $K \in \Xi(X)$, then $E_K : X \rightrightarrows X$ and $T_K : X \rightrightarrows X$ are lower-semicontinuous multivalued maps.*

Proof. Since lower Painlevé-Kuratowski limits are blind with respect to topological closure, it is enough to prove the lower-semicontinuity of E_K. Take any reference point $\bar{x} \in X$ and consider an open set O in X such that $E_K(\bar{x}) \cap O \neq \emptyset$. Hence, there exist $\alpha > 0$ and $z \in K$ such that $\alpha(z - \bar{x}) \in O$. Since O is open, one can find a neighborhood N of $\bar{x}$ such that $\alpha(z - x) \in O$ for all $x \in N$. This yields $E_C(x) \cap O \neq \emptyset$ for all $x \in N$, and confirms the lower-semicontinuity of E_K at $\bar{x}$. □

Remark 4.2. There exists a rich literature devoted to the lower limit of the Bouligand tangent cone map $\mathcal{T}_K : X \rightrightarrows X$ associated to a closed set K that is not necessarily convex. One can not avoid thinking of the works by Cornet [6], Penot [25], and Treiman [30]. These authors analyze the behavior of $\mathcal{T}_K(x)$ as x moves in K and approaches the reference point $\bar{x} \in \partial K$. By contrast, we are interested in the case in which x moves in the exterior of K.

Remark 4.3. Drops and closed drops also behave in a lower-semicontinuous manner with respect to changes in the parameter x. More formally, if $K \in \Xi(X)$, then $C_K : X \rightrightarrows X$ and $D_K : X \rightrightarrows X$ are lower-semicontinuous multivalued maps.

Proposition 4.4. *Let $K \in \Xi(X)$. Then, the map $T_K : X \rightrightarrows X$ is outer-semicontinuous at each point in* $\mathrm{ext}(K)$.

Proof. Let $\bar{x} \in \mathrm{ext}(K)$. Since upper Painlevé-Kuratowski limits are blind with respect to topological closure, it is enough to prove the inclusion $\limsup_{x \to \bar{x}} E_K(x) \subset T_K(\bar{x})$. Take convergent sequences $\{x_n\}_{n\in\mathbb{N}} \to \bar{x}$ and $\{y_n\}_{n\in\mathbb{N}} \to \bar{y}$ such that $y_n \in E_K(x_n)$ for all $n \in \mathbb{N}$. We must prove that $\bar{y} \in T_K(\bar{x})$. Suppose that $\bar{y} \neq 0$, otherwise we are done. By construction, $y_n = \alpha_n(z_n - x_n)$, with $\alpha_n > 0$ and $z_n \in K$. If $\limsup_{n\to\infty} \alpha_n = \infty$, then $\lim_{k\to\infty} \alpha_{n_k} = \infty$ for some subsequence $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$, and a passage to the limit in $z_{n_k} = x_{n_k} + (1/\alpha_{n_k})\, y_{n_k}$ contradicts the fact that $\bar{x} \in \mathrm{ext}(K)$. Therefore, $\{\alpha_n\}_{n\in\mathbb{N}}$ is a bounded sequence. Taking a subsequence if necessary, one may suppose that $\{\alpha_n\}_{n\in\mathbb{N}}$ converges to some nonnegative real α. The case $\alpha = 0$ leads to $\bar{y} \in \mathrm{rec}(K)$, which in turn implies that $\bar{y} \in T_K(\bar{x})$. The case $\alpha > 0$ leads to $\bar{x} + (1/\alpha)\, \bar{y} \in K$. Again, one gets $\bar{y} \in T_K(\bar{x})$. □

The proof we gave of Proposition 4.4 is short and self-contained. In a finite dimensional setting we could have exploited a more abstract and general result by Luc and Wets [19, Theorem 4.1] on the outer-semicontinuity of the convex conic hull operation.

5. Equipping the exterior of K with an order relation

5.1. A good way of approaching the boundary

The multivalued map $T_K : X \rightrightarrows X$ has no chance of being outer-semicontinuous at boundary points of K. To see this, just let the parameter x approach the boundary of K from the interior. Nonentheless, the situation is somewhat different if x moves just in the exterior of K.

If a sequence $\{x_n\}_{n\in\mathbb{N}}$ lies in the exterior of K and approaches a boundary point u, it does not follow that $\limsup_{n\to\infty} T_K(x_n) \subset T_K(u)$. However, this inclusion holds true if one constructs $\{x_n\}_{n\in\mathbb{N}}$ in such a way that each x_n is "well positioned" with respect to u. This idea is elaborated next. A key ingredient in the discussion is the relation

$$y \gg_K x \quad \Longleftrightarrow \quad y \in x - T_K(x)$$

between vectors of X. Figure 5.1 displays the positions of x and y relative to K.

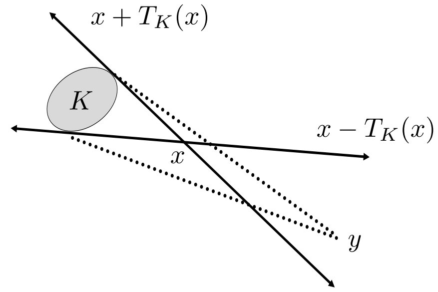

Figure 5.1: y is "greater or equal" to x in the sense of the relation $\gg_K$. The drop $D_K(y)$ captures x.

As mentioned before, supporting cones and closed drops are closely related mathematical objects. It is not surprising altogether to see that

$$\begin{cases} \text{when } x \text{ and } y \text{ are in the exterior of } K \in \Xi(X), \\ \text{one has } y \gg_K x \text{ if and only if } x \in D_K(y). \end{cases} \tag{20}$$

Proving (20) offers no difficulty at all. However, it is important to underline that x and y must be taken in the exterior of the set K. Figure 5.1 shows that the drop with vertex at y captures the point x.

As a first use of the relation $\gg_K$, we state a fundamental monotonicity lemma for the multivalued map T_K.

Lemma 5.1. *Let $K \in \Xi(X)$. If $y, x \in X$ satisfies $y \gg_K x$, then $T_K(y) \subset T_K(x)$.*

Proof. We start by proving that

$$y \in x - E_K(x) \quad \Longrightarrow \quad E_K(y) \subset E_K(x). \tag{21}$$

By homogeneity, it is enough to check that $K - y \subset E_K(x)$. Let $\xi = z - y$ with $z \in K$. The assumption $y \in x - E_K(x)$ means that $x - y = \alpha(w - x)$ with $\alpha > 0$ and $w \in K$. The vector

$$\begin{aligned} x + \frac{1}{1+\alpha}(z-y) &= x + \frac{1}{1+\alpha}\left[z - x + \alpha(w-x)\right] \\ &= \frac{1}{1+\alpha}\, z + \frac{\alpha}{1+\alpha}\, w \end{aligned}$$

belongs to K because it is a convex combination of z and w. Hence,

$$\frac{1}{1+\alpha}\,\xi = \frac{1}{1+\alpha}\,(z-y) \in K - x.$$

This shows that $\xi \in E_K(x)$ and completes the proof of (21). Consider now the case $y \gg_K x$, i.e.,

$$x - y = \lim_{n\to\infty} \alpha_n(w_n - x)$$

with $\alpha_n > 0$ and $w_n \in K$. Note that $y = \lim_{n\to\infty} y_n$ with

$$y_n = x - \alpha_n(w_n - x) \in x - E_K(x).$$

The implication (21) yields $E_K(y_n) \subset E_K(x)$. By taking lower Painlevé-Kuratowski limits on both sides, one gets

$$T_K(y) \subset \liminf_{n\to\infty} T_K(y_n) = \liminf_{n\to\infty} E_K(y_n) \subset \mathrm{cl}[E_K(x)] = T_K(x),$$

the leftmost inclusion being due to Proposition 4.1. □

The next proposition justifies our subsequent use of a terminology that is proper to the theory of ordered spaces.

Proposition 5.2. *Let $K \in \Xi(X)$ be line free. Then,*

(*a*) *The relation $\gg_K$ is reflexive and transitive on X.*
(*b*) *Restricted to* ext(K)*, the relation $\gg_K$ is also antisymmetric.*

Proof. Reflexivity is trivial because $0 \in T_K(x)$ for all $x \in X$. Take $x, y, z \in X$ such that $z \gg_K y$ and $y \gg_K x$. By adding $y - z \in T_K(y)$ and $x - y \in T_K(x)$, one gets

$$x - z \in T_K(y) + T_K(x).$$

But, in view of Lemma 5.1, one has $T_K(y) \subset T_K(x)$. The conclusion is that

$$x - z \in T_K(x) + T_K(x) = T_K(x),$$

i.e., $z \gg_K x$. Finally, let $x, y \in \mathrm{ext}(K)$ be such that $x \gg_K y$ and $y \gg_K x$. One can write

$$y - x \in T_K(y) \subset T_K(x),$$
$$x - y \in T_K(x)\,,$$

where the inclusion is obtained thanks to Lemma 5.1. Hence, $x-y$ belongs to the lineality space of $T_K(x)$. Since $T_K(x)$ is pointed by Proposition 3.1, it follows that $x = y$. This proves antisymmetry on ext(K). □

If K is line free, then $\gg_K$ is an order relation on ext(K). Otherwise, $\gg_K$ is just a pre-order. In either case, one can always speak about $\gg_K$-increasing sequences, $\gg_K$-minorized sequences, and so on. Without further ado, we state:

Theorem 5.3. *Let $K \in \Xi(X)$ and $u \in \partial K$. Then,*

$$\lim_{\substack{x\to u\\ x\gg_K u}} T_K(x) = T_K(u), \tag{22}$$

where the above notation indicates that $\{T_K(x_n)\}_{n\in\mathbb{N}}$ converges to $T_K(u)$ in the Painlevé-Kuratowski sense for any sequence $\{x_n\}_{n\in\mathbb{N}} \to u$ that is $\gg_K$-minorized by u.

Proof. Needless to say, the $\gg_K$-minorization assumption means that $x_n \gg_K u$ for all $n \in \mathbb{N}$. By Lemma 5.1, one gets $T_K(x_n) \subset T_K(u)$ for all $n \in \mathbb{N}$. Hence, $\limsup_{n\to\infty} T_K(x_n) \subset T_K(u)$. The lower counterpart $T_K(u) \subset \liminf_{n\to\infty} T_K(x_n)$ is taken care by Proposition 4.1. □

The next result is a short complement to Theorem 5.3. Proposition 5.4 is an easy result, but it deserves to be properly recorded.

Proposition 5.4. *Let $K \in \Xi(X)$. If a sequence in X is $\gg_K$-minorized by a point in the boundary of K, then the sequence never meets the interior of K.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be $\gg_K$-minorized by $u \in \partial K$. In such a case, $x_n \in u - T_K(u)$ for all $n \in \mathbb{N}$. If some x_{n_0} were in the interior of K, then

$$x_{n_0} - u \in \mathrm{int}(K) - u \subset \mathrm{int}[T_K(u)].$$

This is a clear contradiction because the sets $-T_K(u)$ and $\mathrm{int}[T_K(u)]$ do not intersect. □

The minorization condition $x \gg_K u$ cannot be dropped from the limit appearing on the left-hand side of (22). Figure 5.2 is worth dozens of words. If $\{x_n\}_{n\in\mathbb{N}}$ is constructed as in Figure 5.2, then the set $\limsup_{n\to\infty} T_K(x_n)$ is not even convex!

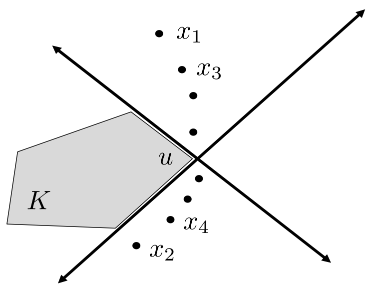

Figure 5.2: A bad way of approaching the boundary point u.

Corollary 5.5. *Let X be an Euclidean space and let $K \in \Xi(X)$. For a point u in the boundary of K, the following conditions are equivalent:*

(a) *u is smooth in the sense that $T_K(u)$ is a half-space.*

(b) *For all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|x - u\| < \delta \text{ and } x \gg_K u \quad \Longrightarrow \quad \Phi_{\mathrm{frob}}[T_K(x)] > 1 - \varepsilon.$$

Proof. For nontrivial closed convex cones in a finite dimensional normed space, convergence in the Painlevé-Kuratowski sense is equivalent to convergence with respect to the truncated Pompeiu-Hausdorff metric ϱ (cf. [27, Section 4]). In view of Theorem 5.3 and the Lipschitz continuity of Φ_{frob} on $(\mathcal{Q}(X), \varrho)$, one has

$$\lim_{\substack{x\to u\\ x\gg_K u}} \Phi_{\mathrm{frob}}[T_K(x)] = \Phi_{\mathrm{frob}}[T_K(u)]. \tag{23}$$

On the other hand, in a Hilbert space setting, the function $\Phi_{\rm frob}$ attains its maximum only[3] at half-spaces. More precisely, for $Q \in \mathcal{Q}(X)$, one has the equivalence

$$\Phi_{\rm frob}[Q] = 1 \quad \Longleftrightarrow \quad Q \text{ is a half-space.} \tag{24}$$

The combination of (23) and (24) yields the equivalence between (a) and (b). $\square$

A convex cone is "highly" solid when its Frobenius solidity coefficient is near 1. So, roughly speaking, the geometric message conveyed by Corollary 5.5 is as follows:

$$u \in \partial K \text{ is smooth} \Longleftrightarrow \begin{cases} \text{for all } x \text{ near and } \gg_K \text{-minorized by } u, \\ \text{the convex cone } T_K(x) \text{ is highly solid.} \end{cases}$$

5.2. The role of $\gg_K$-monotonicity

As a second use of the relation $\gg_K$, we state a monotonicity lemma for the distance function associated to K. The strict version of this lemma is restricted to a class of convex sets that we call externally qualified. A set $K \in \Xi(X)$ is declared externally qualified if $E_K(x) \cup \{0\}$ is closed for all $x \in \text{ext}(K)$. External qualification occurs, for instance, when K is bounded or when K is a polyhedron in a finite dimensional space.

Lemma 5.6. *Let $K \in \Xi(X)$ and $y, x \in X$ be such that $y \gg_K x$. Then,*

$$\text{dist}[y, K] \geq \text{dist}[x, K]. \tag{25}$$

The inequality (25) becomes strict if one assumes, in addition, that K is externally qualified and that y and x are different points in the exterior of K.

Proof. According to the definition of $y \gg_K x$, we must show that

$$\text{dist}[x - d, K] \geq \text{dist}[x, K]$$

for all $d \in T_K(x)$. Since $\text{dist}[\,\cdot\,, K]$ is continuous, it is enough to check the above inequality for $d \in E_K(x)$, that is,

$$\text{dist}[x - \alpha(z - x), K] \geq \text{dist}[x, K]$$

for every $\alpha > 0$ and every $z \in K$. Let $\xi = x - \alpha(z - x)$ with α and z as above. Note that x is expressible in the form

$$x = (1 - \lambda)z + \lambda\xi$$

with $\lambda = (1 + \alpha)^{-1}$ in $]0, 1[$. Hence, the convexity of $\text{dist}[\,\cdot\,, K]$ yields

$$\begin{aligned} \text{dist}[x, K] &\leq (1 - \lambda)\,\text{dist}[z, K] + \lambda\,\text{dist}[\xi, K] \\ &\leq \lambda\,\text{dist}[\xi, K] \\ &\leq \text{dist}[\xi, K], \end{aligned} \tag{26}$$

and completes the first part of the lemma. When $x \in \text{ext}(K)$, also $\xi \in \text{ext}(K)$ and the inequality in (26) is strict. If, in addition, K is externally qualified, then $T_K(x) = E_K(x) \cup \{0\}$, and one ends up with the strict version of (25). $\square$

[3]This is still true if the space X is both gentle and polite, see [15] for definitions and details. In a general normed space X, one may get $\Phi_{\rm frob}[Q] = 1$ even if $Q \in \mathcal{Q}(X)$ is not a half-space. This pathological situation occurs, for instance, in the space of continuous functions $f : [0, 1] \to \mathbb{R}$ equipped with the uniform norm.

Although the next result looks somewhat similar to Theorem 5.3, it is completely different in spirit. Now the sequence $\{x_n\}_{n\in\mathbb{N}}$ does not need to converge to a prescribed reference point, but it must behave in a certain monotonic way.

Theorem 5.7. *Let* $K \in \Xi(X)$.

(*a*) *If* $\{x_n\}_{n\in\mathbb{N}}$ *is* $\gg_K$*-increasing, then* $\lim_{n\to\infty} T_K(x_n) = \bigcap_{n\in\mathbb{N}} T_K(x_n)$.
(*b*) *If* $\{x_n\}_{n\in\mathbb{N}}$ *is* $\gg_K$*-decreasing, then* $\lim_{n\to\infty} T_K(x_n) = \mathrm{cl}\left[\bigcup_{n\in\mathbb{N}} T_K(x_n)\right]$.

Proof. This is a matter of combining Lemma 5.1 and standard results on Painlevé-Kuratowski limits of monotone sequences of sets (cf. [2]). The details are omitted. □

Saying that $\{x_n\}_{n\in\mathbb{N}}$ is $\gg_K$-increasing has its usual meaning, i.e., $x_{n+1} \gg_K x_n$ for all $n \in \mathbb{N}$. Keeping in mind Lemma 5.6, one sees that

$$\mathrm{dist}[x_0, K] \le \mathrm{dist}[x_1, K] \le \mathrm{dist}[x_2, K] \le \ldots,$$

i.e., any $\gg_K$-increasing sequence emanating from $x_0 \in \mathrm{ext}(K)$ remains away from the set K. Of course, this does not mean that $\mathrm{dist}[x_n, K]$ goes to ∞. Incidentally, note that the inequality $x_{n+1} \gg_K x_n$ can be written in the form $-(x_{n+1} - x_n) \in T_K(x_n)$. This difference inclusion corresponds to the Euler discretization scheme for the continuous time differential inclusion $-\dot{\varphi}(t) \in T_K(\varphi(t))$. For a $\gg_K$-decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$, the monotonic behavior of $\mathrm{dist}[x_n, K]$ is in the reverse sense. One gets

$$\mathrm{dist}[x_0, K] \ge \mathrm{dist}[x_1, K] \ge \mathrm{dist}[x_2, K] \ge \ldots,$$

a chain of inequalities implying that $\{x_n\}_{n\in\mathbb{N}}$ remains within a given distance from K.

Remark 5.8. By a K-stream one understands a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathrm{ext}(K)$ satisfying the recurrence relation $x_{n+1} \in D_K(x_n)$ for all $n \in \mathbb{N}$. The concept of K-stream has been used by Montesinos [21, 22] in connection with the analysis of the drop property for convex sets in Banach spaces. In view of (20), a K-stream is nothing else than a $\gg_K$-decreasing sequence in the exterior of K.

6. Miscellaneous results

In order to ensure a smooth presentation of the main ideas, we have left unattended a certain number of less central results. We state them now without lengthy explanations.

The reverse of the implication stated in Lemma 5.1 is not true, even if the attention is restricted to the exterior of K. The next proposition concerns the possibility of writing such a reverse implication. It provides also a sufficient condition for ensuring the injectivity of the map T_K on the exterior of K.

Proposition 6.1. *Suppose that* $K \in \Xi(X)$ *is bounded and that* $x, y \in \mathrm{ext}(K)$. *Then,*

(*a*) $T_K(y) \subset T_K(x)$ *implies* $y \gg_K x$.
(*b*) $T_K(y) = T_K(x)$ *implies* $y = x$.

Furthermore, without the boundedness assumption, the statements (*a*) *and* (*b*) *are false.*

Proof. Let $x, y \in \mathrm{ext}(K)$. In view of (20), one must prove that

$$T_K(y) \subset T_K(x) \quad \Longrightarrow \quad x \in D_K(y). \tag{27}$$

Consider first the case $X = \mathbb{R}^2$. In this particular setting, $T_K(y)$ is a polyhedral convex cone expressible in the form

$$T_K(y) = \mathbb{R}_+(u_1 - y) + \mathbb{R}_+(u_2 - y).$$

The points $u_1, u_2 \in \partial K$ are obtained by writing

$$[y + \mathbb{R}_+(u_i - y)] \cap \operatorname{int}(K) = \emptyset$$

for $i \in \{1, 2\}$. Suppose that $x \notin D_K(x)$. We shall prove that $u - y \notin T_K(x)$ for some $u \in K$, contradicting in this way the inclusion $T_K(y) \subset T_K(x)$. Let us write

$$x = y + \alpha_1(u_1 - y) + \alpha_2(u_2 - y)$$

and examine the orientation of the cone $T_K(x)$ as function of the signs of the "coordinates" α_1 and α_2. Since we are working in a two dimensional setting, the simplest thing to do is drawing a picture (cf. Figure 6.1). One sees that

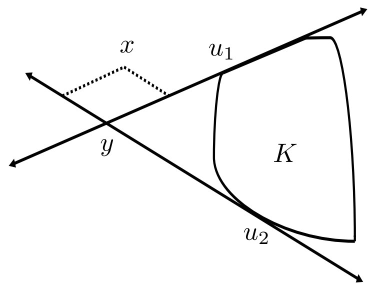

Figure 6.1: The position of x is determined by the coordinates α_1 and α_2. In this figure, $\alpha_1 > 0$ and $\alpha_2 < 0$. Hence, the supporting cone $T_K(x)$ does not contain $u_1 - y$.

$$\begin{aligned} \alpha_2 < 0 &\implies u_1 - y \notin T_K(x), \\ \alpha_1 < 0 &\implies u_2 - y \notin T_K(x), \\ \alpha_1 \geq 0, \alpha_2 \geq 0 &\implies u_1 - y \notin T_K(x) \text{ or } u_2 - y \notin T_K(x). \end{aligned}$$

So, in all circumstances, either $u_1 - y \notin T_K(x)$ or $u_2 - y \notin T_K(x)$. The two dimensional case being settled, we come back to a general normed space X. We handle the implication (27) as follows. Suppose that the inclusion $T_K(y) \subset T_K(x)$ is true. Without loss of generality, one can assume that $0 \in \operatorname{int}(K)$ and that $\{x, y\}$ are linearly independent. Consider the convex set $K' = K \cap H$, where H is the two dimensional space spanned by x and y. We claim that

$$T_{K'}(y) \subset T_{K'}(x). \tag{28}$$

Let d be a nonzero vector in $T_{K'}(y)$. Since $T_{K'}(y) \subset T_K(y) \subset T_K(x)$ and K is bounded, it follows that $d = \beta(z - x)$ with $\beta > 0$ and $z \in K$. Since x and d are in H, the vector $z = x + (d/\beta)$ is also in H. This shows that $d \in T_{K'}(x)$ and confirms the claim (28). By applying (27) to the set K', one gets $x \in D_{K'}(y)$. Hence, $x \in D_K(y)$. The proof of (a) is complete. In order to prove (b), suppose that $T_K(x) = T_K(y)$. In view of (a), one gets $y \gg_K x$ and $x \gg_K y$. Proposition 5.2 yields $x = y$. Finally, to see that the boundedness assumption cannot be omitted, consider the unbounded set $K = \mathbb{R}_+ \times \mathbb{R}$ and the points $x = (-2, 0)$ and $y = (-1, 0)$. $\square$

Remark 6.2. However, T_K could be injective on $\mathrm{ext}(K)$, even if K is unbounded. Consider, for instance, the set $K = \{(z_1, z_2) \in \mathbb{R}^2 : z_2 \geq (z_1)^2\}$.

The next theorem completes Proposition 5.2 and adds some new material.

Theorem 6.3. *Let $K \in \Xi(X)$. Consider the following conditions:*

(a) ∂K *is line free.*
(b) K *is line free.*
(c) $\gg_K$ *is antisymmetric on* $\mathrm{ext}(K)$.
(d) D_K *is injective on* $\mathrm{ext}(K)$.
(e) T_K *is injective on* $\mathrm{ext}(K)$.

Then, (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d). *One also has* (e) $\Rightarrow$ (a), *but the reverse implication is false.*

Proof. For the sake of readibility, we divide the proof in short and independent portions:

(a) $\Rightarrow$ (b). This is likely to be known. Suppose that ∂K is line free, but $z + \mathbb{R}d \subset K$ for some $z \in K$ and some nonzero vector $d \in X$. We must arrive to a contradiction. Note that the line $L = z + \mathbb{R}d$ must be contained in the interior of K. Take $x \in \mathrm{ext}(K)$ and define $H = z + \mathrm{span}\{x - z, d\}$. By construction, H is the two dimensional affine space that contains L and the point x. Consider the partition $H = P_1 \cup L \cup P_2$, where P_1 and P_2 are open half-spaces. Suppose, for instance, that $x \in P_1$. If one views $K' = K \cap H$ as a subset of H, then everything boils down to working in a two dimensional context. Observe that K' is a closed convex set in H, the line L lies in the interior of K', and x belongs to the exterior of K'. Hence, by a simple convexity argument, there is a line $\hat{L} \subset P_1$ parallel to L and contained in the boundary of K'. It follows that $\hat{L} \subset \partial K$, a contradiction with the fact that ∂K is line free.

(b) $\Rightarrow$ (c). Already established in Proposition 5.2.

(c) $\Rightarrow$ (d). Let $\gg_K$ be antisymmetric on $\mathrm{ext}(K)$. Let $x, y \in \mathrm{ext}(K)$ be such that $D_K(x) = D_K(y)$. Since $x \in D_K(x)$, it follows that $x \in D_K(y)$. In view of (20), one has $y \gg_K x$. A mutatis mutandis argument yields $x \gg_K y$. By antisymmetry, one ends up with $x = y$.

(d) $\Rightarrow$ (a). This implication closes a loop. Suppose that D_K is injective on $\mathrm{ext}(K)$, but that

$$z + \mathbb{R}d \subset \partial K \tag{29}$$

for some $z \in \partial K$ and some nonzero vector $d \in X$. We must arrive to a contradiction. Take $x, y \in \mathrm{ext}(K)$ such that $y - x = d$. From (29) and the very definition of a closed drop, one gets $x + \mathbb{R}d \subset D_K(x)$ and $y + \mathbb{R}d \subset D_K(y)$. In particular, $y \in D_K(x)$ and $x \in D_K(y)$. Now, from the combination of $y \in D_K(x)$ and $K \subset D_K(x)$, one obtains $D_K(y) \subset D_K(x)$. Similarly, $x \in D_K(y)$ and $K \subset D_K(y)$ yield $D_K(x) \subset D_K(y)$. In short, $D_K(x) = D_K(y)$, but $x \neq y$, contradicting the injectivity of D_K.

(e) $\Rightarrow$ (c). Let T_K is injective on $\mathrm{ext}(K)$. Let $x, y \in \mathrm{ext}(K)$ be such that $y \gg_K x$ and $x \gg_K y$. Lemma 5.1 yields $T_K(x) = T_K(y)$. The injectivity of T_K shows that $x = y$.

(b) $\not\Rightarrow$ (e). The set $K = [-1, 0] \times \mathbb{R}_+$ is line free. However, T_K is not injective on $\mathrm{ext}(K)$. To see this, just consider $x = (1, 1)$ and $y = (2, 2)$. $\square$

Figure 6.2 collects the information provided by Proposition 3.1, Theorem 6.3 and Proposition 6.1. It gives a nice overview of the situation, but is not meant to be an exhaustive compte-rendu. An interesting question that remains open is this: which is the geometric property of K, or the boundary of K for that matter, that corresponds to the injectivity of the map T_K on $\mathrm{ext}(K)$? It must be a property lying between boundedness and line freeness.

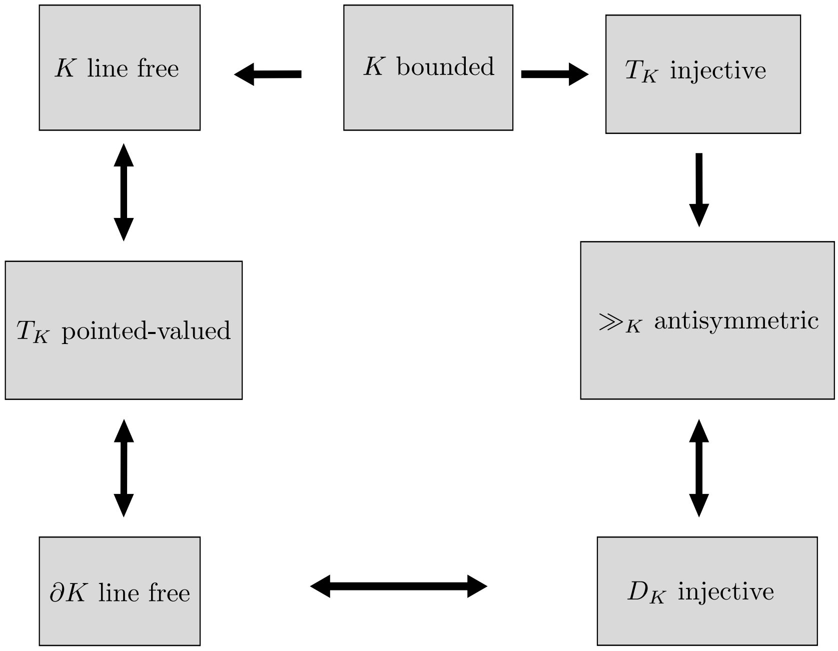

Figure 6.2: Behavior of D_K and T_K on $\mathrm{ext}(K)$. Link to the relation $\gg_K$ and to properties of K. Two-headed arrows indicate equivalence, one-headed arrows indicate nonreversibility of the implication.

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