Journal of Convex Analysis
Volume 16 (2009), No. 2, 501–514

Proximal Smoothness and the Exterior Sphere Condition

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Proximal smoothness and the exterior sphere condition are compared. It is shown via examples that these two properties are not necessarily equivalent, and geometric conditions are provided under which the equivalence holds.

Keywords: Proximal smoothness, interior and exterior sphere conditions, wedged sets, proximal analysis, nonsmooth analysis

1. Introduction

Let S be a nonempty closed subset of $\mathbb{R}^n$. For $x \in S$, a vector $\zeta \in \mathbb{R}^n$ is said to be *proximal normal to S at x* provided that there exists $\sigma = \sigma(x, \zeta) \geq 0$ such that

$$\langle \zeta, s - x \rangle \leq \sigma \|s - x\|^2 \quad \forall s \in S, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the standard inner product and Euclidean norm, respectively. The relation (1) is commonly referred to as the *proximal normal inequality.* No nonzero ζ satisfying (2) exists if $x \in \operatorname{int} S$, but this may also occur for $x \in \operatorname{bdry} S$, as is the case when S is the epigraph of the function $f(z) = -|z|$ and $x = (0, 0)$. For such points, the only proximal normal is $\zeta = 0$. In view of (1), the set of all proximal normals to S at x is a convex cone, and we denote it by $N_S^P(x)$.

Let $x \in \operatorname{bdry} S$, and suppose that $0 \neq \zeta \in \mathbb{R}^n$ and $r > 0$ are such that

$$B\left(x + r\frac{\zeta}{\|\zeta\|}; r\right) \cap S = \emptyset, \tag{2}$$

where $B(z; \rho)$ denotes the open ball of radius ρ centered at z (the closed ball being denoted by $\bar{B}(z; \rho)$). Then ζ is a proximal normal to S at x and we say that ζ is *realized*

*Research conducted in part while visiting Concordia University, Montreal, Canada.
†Research supported by the Natural Sciences Engineering Research Council of Canada.

ISSN 0944-6532 / $ 2.50

by an r-sphere. Note that ζ is then also realized by an r'-sphere for any $0 < r' < r$. One can show that ζ being realized by an r-sphere is equivalent to the proximal normal inequality holding with $\sigma = \frac{1}{2r}$; that is,

$$\left\langle \frac{\zeta}{\|\zeta\|}, s - x \right\rangle \leq \frac{1}{2r}\|s - x\|^2 \quad \forall s \in S. \tag{3}$$

Our general reference regarding proximal normals as well as other constructs of proximal analysis is Clarke, Ledyaev, Stern and Wolenski [7].

For a point $x \in \operatorname{bdry} S$, if there exists $r > 0$ such that *some* $0 \neq \zeta \in N_S^P(x)$ is realized by an r-sphere, then we say that S satisfies an *exterior r-sphere condition at x*. In sphere terminology, it is equivalent to the existence of $y_x \notin S$ such that

$$B(y_x; r) \cap S = \emptyset \quad \text{and} \quad \|x - y_x\| = r.$$

If this holds (for a single $r > 0$) at every boundary point x, then S is said to satisfy an exterior r-sphere condition, and if there exists such an r, we simply say that S satisfies the *uniform exterior sphere condition.*

Of course, when $S = \operatorname{cl}(\operatorname{int} S)$ (the closure of the interior), the uniform exterior sphere condition is equivalent to $(\operatorname{int} S)^c$ (the complement of the interior) satisfying a uniform *interior* sphere condition. The latter condition is a well known one in control theory, and is important in deriving regularity properties of the minimal time function; see e.g. Cannarsa and Frankowska [4], Cannarsa and Sinestrari [4, 5] and Sinestrari [18].

If, for a point $x \in \operatorname{bdry} S$, $r > 0$ is such that *every* $0 \neq \zeta \in N_S^P(x)$ is realized by an r-sphere, then S is said to be *r-proximally smooth* at x. Paralleling the preceding terminology, if this holds at every boundary point x for some positive r, then we say that S is r-proximally smooth, and if there exists such an r, S is simply said to be *uniformly proximally smooth.*

In addition to several other interesting consequences, uniform proximal smoothness of S implies that $N_S^P(x) \neq \{0\}$ for all $x \in \operatorname{bdry} S$. Furthermore, if S is closed and convex, then the proximal normal inequality holds at every $x \in S$ with $\sigma = 0$; hence this class of sets is uniformly proximally smooth, and every $x \in \operatorname{bdry} S$ is realized by an r-sphere of arbitrarily large radius. Apparently, uniform proximal smoothness was first studied by Federer [11], who referred to the property as *positive reach.* In Clarke, Stern and Wolenski [8] (see also Canino [2] and Shapiro [17]), proximal smoothness was studied in detail in a Hilbert space setting, and tie-ins were made with smoothness properties of the euclidean distance function $d_S(\cdot)$ on an open tube around S, but in the present work, we will not rely on those results. We also refer the reader to Poliquin and Rockafellar [13], Poliquin, Rockafellar and Thibault [14], Rockafellar and Wets [16], Colombo and Marigonda [9] and Colombo, Marigonda and Wolenski [10] for investigations and applications of related properties such as *prox-regularity* and *φ-convexity.*[1]

[1] A set S is said to be r-prox-regular at a point $x_0 \in \operatorname{bdry} S$ if there exists $\delta > 0$ such that $\left\langle \frac{\zeta}{\|\zeta\|}, s - x \right\rangle \leq \frac{1}{2r}\|s - x\|^2$ for all x and s in $S \cap B(x_0; \delta)$ and for all $0 \neq \zeta \in N_S^L(x)$ (the limiting normal cone to S at x, see [7]). It was proven in [14] that a property termed uniformly r-prox-regularity coincides with uniform r-proximal smoothness. A closed set S is said to be φ-convex if there exists a continuous function $\varphi : S \longrightarrow [0, +\infty[$ such that $\langle \zeta, s - x \rangle \leq \varphi(x)\,\|\zeta\|\,\|s - x\|^2$ for all $x, s \in S$ and for all $\zeta \in N_S^P(x)$. Clearly uniform proximal smoothness coincides with φ-convexity if φ is a constant function. See [9] for results concerning φ-convexity.

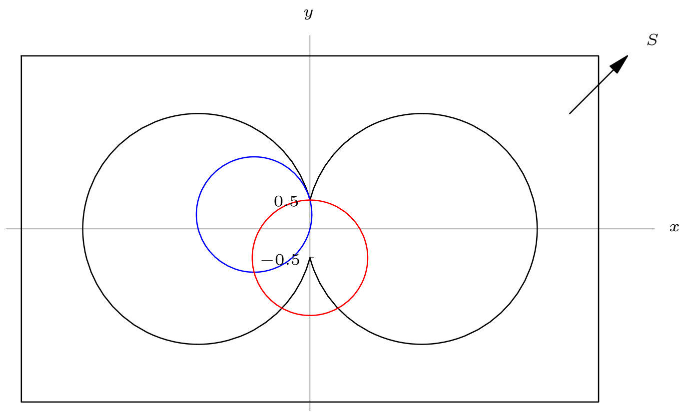

Figure 1.1: Example 2.2

The goal of the present article is to compare the exterior sphere condition with proximal smoothness. Obviously, if S is r-proximally smooth, then it satisfies the exterior r-sphere condition. Our purpose here is to answer the following two questions concerning possible reverse implications:

($*$) *If S satisfies an exterior r-sphere condition and S is known to be uniformly proximally smooth, is it necessarily r-proximally smooth?*

($**$) *If S satisfies a uniform exterior sphere condition, is S necessarily uniformly proximally smooth?* (*Here there is no mention of radius.*)

After settling these issues, we will also study the equivalence between S satisfying the uniform interior sphere condition and S being the union of uniform spheres, and in so doing clarify a semantic ambiguity in the literature concerning these properties.

In the next section, we shall see, by means of counterexamples, that the answer to both questions ($*$) and ($**$) is "no". Then in Section 3 geometric conditions will be provided under which equivalence between the uniform exterior sphere condition and uniform proximal smoothness *does* hold. Section 4 is devoted to the comparison of the uniform interior sphere condition and the union of uniform balls property.

2. Counterexamples

As mentioned above, it is clear from the definitions that if S is r-proximally smooth, then it possesses the exterior r-sphere condition. That the reverse implication is not necessarily true is illustrated by the following simple example.

Example 2.1. Let $S := \{(x, |x|) : x \in \mathbb{R}\} \subset \mathbb{R}^2$. This set possesses the exterior r-sphere condition for any $r > 0$, but fails to be r-proximally smooth for any $r > 0$, and thereby provides a negative answer to Question ($**$). Indeed, for each $x \in]0, +\infty[$ the vector $\zeta = (-1, 1)$ is a proximal normal to S at (x, x), but the radius of the sphere which realizes ζ must approach 0 as $x \downarrow 0$.

In the preceding example, the set S has an empty interior. We now will focus our attention on sets S satisfying $S = \mathrm{cl}\,(\mathrm{int}\, S)$, which are of the type commonly used in control theory as targets in minimal time optimal control problems. We shall refer to

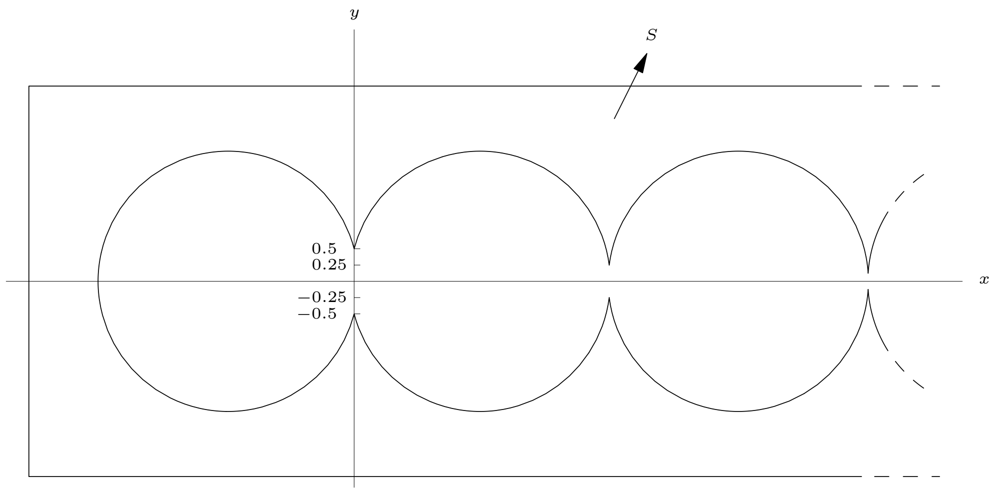

Figure 2.1: Example 2.3

such sets as *standard* sets. Consider the following.

Example 2.2. Let S be the standard region inside the rectangle and outside the two large circles in Figure 1.1. This set satisfies an exterior 1-sphere condition. (Observe that the non-vertically oriented circle has 1 as radius.) But, while S is clearly uniformly proximally smooth, it fails to be 1-proximally smooth, since the unit vector $(0, -1)$ normal to S at $(0, \frac{1}{2})$ cannot be realized by 1-sphere. This shows that Question $(*)$ has a negative answer.

While it is true that the set S in the previous example is not 1-proximally smooth, it *is* r-proximally smooth for any $0 < r \leq \frac{1}{2}$, and therefore it does not address Question $(**)$. The following example does so.

Example 2.3. Let S be the standard region inside the infinite rectangle and outside the circles (of radius 2) of Figure 2.1. The intersection of two consecutive circles C_n and C_{n+1} consists of two points of the form $p_n = (a_n, \frac{1}{2n})$ and $q_n = (a_n, -\frac{1}{2n})$, where $a_n \in \mathbb{R}$ and $n \geq 1$. Then S satisfies the exterior 1-sphere condition, but fails to be r-proximally smooth for any $r > 0$, since vertical proximal normals at p_n and q_n are realizable only by spheres of radius at most $\frac{1}{2n}$.

The set S of the previous example is connected, but it fails to be compact. The following example is a two dimensionsal compact counterexample for Question $(**)$ in which the set S is not connected.[2]

Example 2.4. Let S be the infinite union of the "curved" triangles of Figure 2.2. The curved sides of these triangles are arcs of unit circles tangent to the horizontal bases of the triangles. The points a_n and b_n are chosen in such a way that the sequences $|b_n - a_n|$ and $|a_{n+1} - b_n|$ converge to 0 and such that the curved triangles converge to a point (included in the set S); note that S is therefore compact. Clearly S satisfies the exterior 1-sphere condition but fails to be r-proximally smooth for any $r > 0$. Indeed, the radius

[2]It is an open question whether a two dimensions connected and compact counterexample to Question $(**)$ exists. Example 2.4 due to Zvi Artstein (private communication). A similiar example, but in another context, can be also found in Marigonda [12].

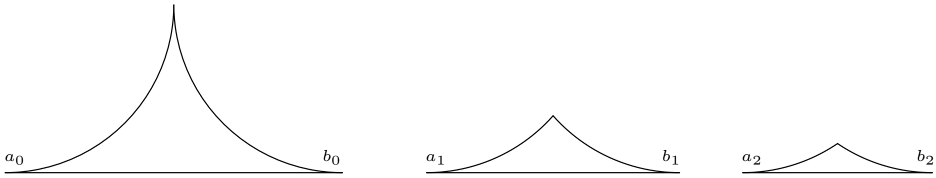

Figure 2.2: Example 2.4

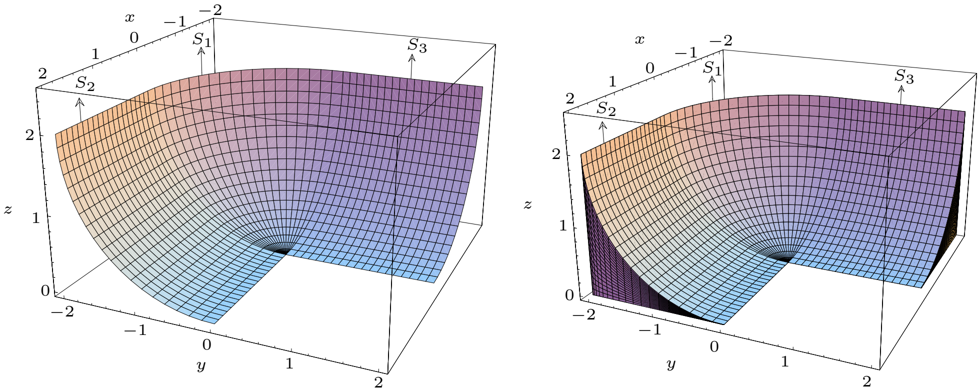

Figure 3.1: Example 2.5

of the spheres which realize horizontal proximal normals at the points b_n must approach 0.

We shall conclude this section with a third negative example for Question $(**)$, but where S is a three dimensions compact and connected set. It will play a role in the next section as well; see Remark 3.10 below.

Example 2.5. Consider the following three surfaces in $\mathbb{R}^3$, shown in the left picture of Figure 3.1:

- S_1 is the part of the sphere $x^2 + y^2 + (z-2)^2 = 4$ with $x \leq 0$, $y \leq 0$ and $z \leq 2$.
- S_2 is the part of the cylinder $y^2 + (z-2)^2 = 4$ with $0 \leq x \leq 2$, $-2 \leq y \leq 0$ and $z \leq 2$.
- S_3 is the part of the cylinder $x^2 + (z-2)^2 = 4$ with $-2 \leq x \leq 0$, $0 \leq y \leq 2$ and $z \leq 2$.

Now define S to be the region between the surface $S_1 \cup S_2 \cup S_3$ and the plane $z = 0$, as is shown in the right picture of Figure 3.1. Clearly S is a standard set. Moreover, S satisfies the exterior 1-sphere condition, but it fails to be uniformly proximally smooth. Indeed, similarly to Example 2.1, for each $x \in]0,2[$ the vector $\zeta = (0,1,0)$ is a proximal normal to S at $(x,0,0)$, but the radius of the sphere which realizes ζ necessarily approaches 0 as $x \downarrow 0$.

3. Conditions for equivalence

Let $S \subset \mathbb{R}^n$ be a closed set, not necessarily standard. We introduce a geometric hypothesis, for a given point $x \in \operatorname{bdry} S$ and scalar $\delta > 0$:

($H_{x,\delta}$): For every $y \in (S \cap B(x;\delta)) \backslash \{x\}$ and all $t \in [0,1[$ one has

$$S \cap I(x,y,t) \neq \emptyset,$$

where $I(x,y,t)$ denotes the open interval from $(1-t)y + tx$ to x.

Remark 3.1. Clearly a convex set S satisfies hypothesis ($H_{x,\delta}$) for all boundary points x and all $\delta > 0$. The hypothesis is also satisfied by locally convex or, more generally, locally star-shaped sets. An example of set which satisfies ($H_{x,\delta}$) but fails to be locally star-shaped is the epigraph of the function

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Proposition 3.2. *Let S be a closed set with $x \in \operatorname{bdry} S$, and assume that hypothesis ($H_{x,\delta}$) holds. Then S is $\frac{\delta}{2}$-proximally smooth at x.*

Proof. We proceed via contradiction. Assume that S is not $\frac{\delta}{2}$-proximally smooth at x. Then there exist a unit vector $\zeta_0 \in N_S^P(x)$ and $s_0 \in S \setminus \{x\}$ such that

$$\langle \zeta_0, s_0 - x \rangle > \frac{1}{\delta} \|s_0 - x\|^2. \tag{4}$$

The last inequality yields $s_0 \in B(x;\delta)$. Then by ($H_{x,\delta}$) we obtain the existence of a sequence $0 < t_i < 1$ such that $t_i \longrightarrow 1$ and $(1-t_i)s_0 + t_i x \in S$. Since $\zeta_0 \in N_S^P(x)$, there exists $\sigma > 0$ such that

$$\langle \zeta_0, s - x \rangle \leq \sigma \|s - x\|^2 \quad \forall s \in S.$$

Hence

$$(1-t_i)\langle \zeta_0, s_0 - x \rangle \leq \sigma (1-t_i)^2 \|s_0 - x\|^2,$$

which yields

$$\langle \zeta_0, s_0 - x \rangle \leq \sigma (1-t_i) \|s_0 - x\|^2. \tag{5}$$

Now using the two inequalities (4) and (5) we obtain that $\frac{1}{\delta} < \sigma(1-t_i)$. Letting $i \longrightarrow +\infty$ in the preceding inequality provides the desired contradiction. □

A point $x \in \operatorname{bdry} S$ is said to be *normal* if the proximal normal cone at x is a half line; that is, there exists a vector $0 \neq \zeta$ such that

$$N_S^P(x) := \{t\zeta : t \geq 0\}.$$

Remark 3.3.

- Clearly proximal smoothness holds at normal points.
- If S is standard and has C^2-smooth boundary near x, then x is normal.
- If S satisfies an interior sphere condition at x, then either x is normal or $N_S^P(x) = \{0\}$.
- If x is normal, then S does not necessarily satisfy an interior sphere condition at x. This is illustrated by the cylinder with heart-shaped cross-section shown in Figure 3.2, where x is the cusp at the top. Note that ($H_{x,\delta}$) is satisfied at this cusp.

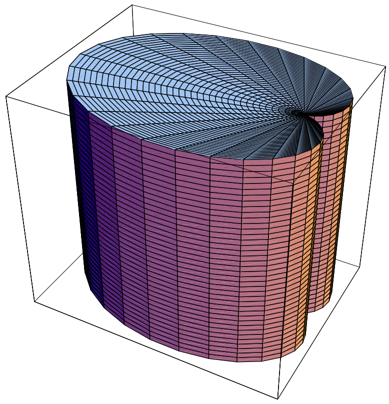

Figure 3.2: Heart cylinder

- On the other hand, hypothesis $(H_{x,\delta})$ can fail at a normal point even if S is uniformly proximally smooth. An example is the epigraph of $f(x) = -x^2$ at the origin.

Upon imposing uniformity on δ in hypothesis $(H_{x,\delta})$, we obtain the following as a direct consequence of Proposition 3.2.

Corollary 3.4. *Let S be a nonempty closed set and assume that there exists $\delta > 0$ such that every $x \in \operatorname{bdry} S$ is either normal, or $(H_{x,\delta})$ holds. Then if S satisfies an exterior r-sphere condition, S is uniformly proximally smooth. (In particuar, it is $\min\{r, \frac{\delta}{2}\}$-proximally smooth.)*

Another corollary ensues from the previous one, and the fact that the uniform exterior sphere condition is weaker than uniform proximal smoothness:

Corollary 3.5. *Under the assumptions of the preceding corollary, S satisfies the uniform exterior sphere condition if and only if it is uniformly proximally smooth.*

Remark 3.6.

- The set S of Example 2.1 does not satisfy the hypothesis of the preceding two corollaries. Indeed, no points are normal and no uniform $\delta > 0$ can be found.
- The set S of Example 2.2 does satisfy the hypotheses, with a maximum δ of $\frac{1}{n}$.
- Clearly the sets of Example 2.3, Example 2.4 and Example 2.5 do not satisfy the hypotheses. Indeed, consider points p_n for Example 2.3, b_n for Example 2.4 and $(x, 0, 0)$ for Example 2.5.

We will now introduce a more important geometric condition than $(H_{x,\delta})$. Recall that a closed set S is said to be *wedged* (or *epi-Lipschitz*) at a boundary point x, if near x the set S can be viewed, after application of an orthogonal matrix, as the epigraph of a Lipschitz continuous function. Specifically, there exists an open neighborhood V of x, a unit vector e, and for the hyperplane

$$H := \{x' : \langle e, x' - x \rangle = 0\}$$

through x, a Lipschitz continuous function $f : H \cap V \longrightarrow \mathbb{R}$ such that for some open neighborhood W of x one has

$$W \cap S = W \cap \{x' + te : x' \in H \cap V \text{ and } f(x') \leq t < \infty\}.$$

This geometric definition was introduced by Rockafellar in [15]. The property is also characterizable in terms of the nonemptiness of the topological interior of the Clarke tangent cone which is also equivalent to the pointedness of the Clarke normal cone; see [7] and [16]. If S is wedged at x for all $x \in \operatorname{bdry} S$, then we simply say that S is wedged, and in this case it is clear that S is standard. Note that normality of a point does not imply wedgedness at that point; again, consider the heart cylinder in Figure 3.2.

The following theorem asserts that wedgedness of S at a boundary point x guarantees equivalence of the exterior sphere condition and proximal smoothness in a local sense. Here uniform proximal smoothness (resp. the uniform exterior sphere condition) of S *near* x connotes the existence of $r > 0$ and $\delta > 0$ such that S is r-proximally smooth (resp. satisfies the exterior r-sphere condition) at y for all $y \in \operatorname{bdry} S \cap B(x;\delta)$.

Theorem 3.7. *Let S be a nonempty closed set and let $x \in \operatorname{bdry} S$. Assume that S is wedged at x. Then S satisfies a uniform exterior sphere condition for all boundary points near x iff S is uniformly proximally smooth near x.*

Before giving the proof of this result, we shall require the following lemma.

Lemma 3.8. *Let $f : U \longrightarrow \mathbb{R}$ be a K-Lipschitz function defined on an open, convex and bounded set $U \subset \mathbb{R}^n$. Let $\operatorname{epi}(f) := \{(x,\alpha) : x \in U,\ \alpha \geq f(x)\}$ (the epigraph of f) satisfies the exterior r-sphere condition. Then $\operatorname{epi}(f)$ is $\frac{r}{(1+K^2)^{\frac{3}{2}}}$-proximally smooth.*

In the lemma's statement, the definitions of proximal smoothness and the exterior sphere condition are extended to the locally closed set $\operatorname{epi}(f)$ in the obvious way. Prior to giving the proof, we need to recall some definitions and facts from nonsmooth analysis.

The following definition has relevance for a wider class of functions, but let us assume that $f : W \to \mathbb{R}$ where $W \subset \mathbb{R}^n$ is open, and that f is K-Lipschitz near a point x. A vector $\zeta \in \mathbb{R}^n$ is a *proximal subgradient of f at x* provided that

$$(\zeta, -1) \in N^P_{\operatorname{epi}(f)}(x, f(x)),$$

and this is equivalent to the existence of $\sigma = \sigma(x,\zeta) \geq 0$ and $\delta > 0$ such that the following *proximal subgradient inequality* holds:

$$f(y) - f(x) + \sigma \|y - x\|^2 \geq \langle \zeta, y - x \rangle \quad \forall y \in B(x;\delta) \cap W.$$

The *proximal subdifferential of f at x*, denoted $\partial_P f(x)$ is the set of all the proximal subgradients at x. Under the present Lipschitz assumption, one can employ the previous inequality to show that $\|\zeta\| \leq K$ for every $\zeta \in \partial_P f(x)$. In addition, it is an exercise using the proximal normal inequality to show that every nonzero proximal normal (ζ, θ) to $\operatorname{epi}(f)$ at $(x, f(x))$ has $\theta < 0$, and therefore the cone $N^P_{\operatorname{epi}(f)}(x, f(x))$ is generated by those vectors of the form $(\zeta, -1)$. We shall also require the following result, proven in Theorem 5.1 of Clarke, Stern and Wolenski [8]:

Proposition 3.9. *Let $f : U \longrightarrow \mathbb{R}$ be a K-Lipschitz function defined on an open, convex and bounded set $U \subset \mathbb{R}^n$, and suppose that $\sigma > 0$ is such that for each $x \in U$, there exists $\zeta \in \mathbb{R}^n$ such that*

$$f(y) - f(x) + \sigma\|y - x\|^2 \geq \langle \zeta, y - x \rangle \quad \forall y \in U. \tag{6}$$

Then for each $x \in U$, (6) holds for all $\zeta \in \partial_P f(x)$.

We are now in position to prove the lemma.

Proof of Lemma 3.8. Assume that epi(f) satisfies the exterior r-sphere condition and let $x \in U$. Then there exists $(0,0) \neq (\zeta, \theta) \in N^P_{\text{epi}(f)}(x, f(x))$ such that

$$\left\langle \frac{(\zeta,\theta)}{\|(\zeta,\theta)\|}, (y,\alpha) - (x, f(x)) \right\rangle \leq \frac{1}{2r}\|(y,\alpha) - (x, f(x))\|^2$$

for all $y \in U$ and for all $\alpha \geq f(y)$. Since f is Lipschitz we have that $\theta < 0$ and then

$$\left\langle \frac{(\frac{\zeta}{-\theta}, -1)}{\|(\frac{\zeta}{-\theta}, -1)\|}, (y,\alpha) - (x, f(x)) \right\rangle \leq \frac{1}{2r}\|(y,\alpha) - (x, f(x))\|^2$$

for all $y \in U$ and for all $\alpha \geq f(y)$. Hence $\xi := \frac{\zeta}{-\theta} \in \partial_P f(x)$ is such that

$$\left\langle \frac{(\xi, -1)}{\|(\xi, -1)\|}, (y,\alpha) - (x, f(x)) \right\rangle \leq \frac{1}{2r}\|(y,\alpha) - (x, f(x))\|^2 \tag{7}$$

for all $y \in U$ and for all $\alpha \geq f(y)$.

The K-Lipschitz assumption on f implies $\|\xi\| \leq K$, so we have $\|(\xi, -1)\| \leq \sqrt{1 + K^2}$. Then a straightforward calculation using $(f(y) - f(x))^2 \leq K^2\|y - x\|^2$ readily yields

$$f(y) \geq -\frac{(1 + K^2)^{\frac{3}{2}}}{2r}\|y - x\|^2 + < \xi, y - x > + f(x) \tag{8}$$

for all $y \in U$. Then by Proposition 3.9, inequality (8) holds for all $\xi \in \partial_P f(x)$. That is,

$$f(y) \geq -\frac{(1 + K^2)^{\frac{3}{2}}}{2r}\|y - x\|^2 + < \xi, y - x > + f(x)$$

for all $y \in U$ and for all $\xi \in \partial_P f(x)$. This implies that

$$\alpha - f(x) + \frac{(1 + K^2)^{\frac{3}{2}}}{2r}\left[\|y - x\|^2 + (\alpha - f(x))^2\right] \geq \langle \xi, y - x \rangle$$

for all $y \in U$, for all $\alpha \geq f(y)$ and for all $\xi \in \partial_P f(x)$. It follows that

$$\left\langle \frac{(\xi, -1)}{\|(\xi, -1)\|}, (y,\alpha) - (x, f(x)) \right\rangle \leq \frac{(1 + K^2)^{\frac{3}{2}}}{2r}\|(y,\alpha) - (x, f(x))\|^2$$

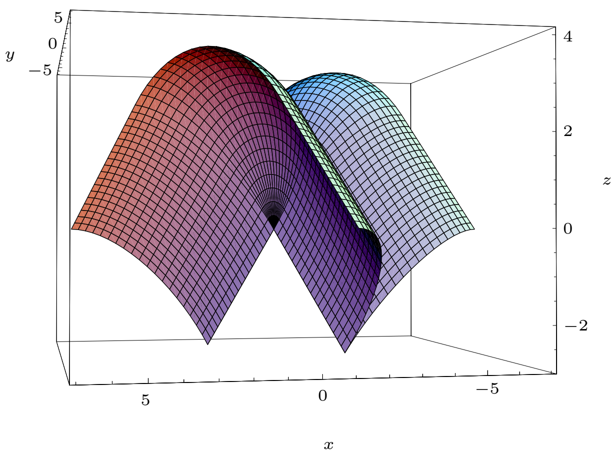

Figure 3.3: Epigraph example

for all $y \in U$, for all $\alpha \geq f(y)$ and for all $\xi \in \partial_P f(x)$. Since $N^P_{\mathrm{epi}(f)}(x, f(x))$ is generated by those vectors of the form $(\zeta, -1)$, we obtain

$$\left\langle \frac{(\zeta,\theta)}{\|(\zeta,\theta)\|}, (y,\alpha) - (x, f(x)) \right\rangle \leq \frac{(1+K^2)^{\frac{3}{2}}}{2r} \|(y,\alpha) - (x, f(x))\|^2$$

for all $y \in U$, for all $\alpha \geq f(y)$ and for all $(0,0) \neq (\zeta,\theta) \in N^P_{\mathrm{epi}(f)}(x, f(x))$. Therefore $\mathrm{epi}(f)$ is $\frac{r}{(1+K^2)^{\frac{3}{2}}}$-proximally smooth. □

Remark 3.10. The Lipschitz assumption on the function f is crucial in the preceding lemma, as can be seen as follows. Let S_1, S_2,and S_3 be as in Example 2.5, and then "glue" the surface $S_1 \cup S_2 \cup S_3$ to itself so as to obtain the surface of Figure 3.3, which is the graph of a function $z = f(x,y)$. Take U to be an open, convex and bounded neighborhood of the origin in the plane $z = 0$. Clearly f is not Lipschitz on U. On the other hand, $\mathrm{epi}(f)$ possesses the exterior 1-sphere condition but fails to be proximally smooth.

We are now ready to prove Theorem 3.7.

Proof of Theorem 3.7. We only need to prove the "only if" part of the statement, since the "if" is immediate. To this end, let S be a nonempty closed set and let $x \in \mathrm{bdry}\, S$. Assume that S is wedged at x and that S satisfies a uniform exterior sphere condition at all boundary points near x. Then there exists $\delta_x > 0$ and $r > 0$ such that S satisfies the exterior r-sphere at any point in $\mathrm{bdry}\, S \cap B(x;\delta_x)$, and in view of wedgedness, $S \cap B(x;\delta_x)$ can be viewed as the epigraph of K-Lipschitz function, as described above. Then by the preceding lemma we deduce that the set $S \cap B(x;\delta_x)$ is $\frac{r}{(1+K^2)^{\frac{3}{2}}}$-proximally smooth in the following sense: For all $y \in \mathrm{bdry}\, S \cap B(x;\delta_x)$ and for all $0 \neq \zeta \in N^P_S(y)$ we have

$$\left\langle \frac{\zeta}{\|\zeta\|}, s - y \right\rangle \leq \frac{(1+K^2)^{\frac{3}{2}}}{2r} \|s-y\|^2 \tag{9}$$

for all $s \in S \cap B(x; \delta_x)$. We claim that there exists $r' > 0$ for which S is r'-proximally smooth at any point in $\operatorname{bdry} S \cap B(x; \frac{\delta_x}{2})$. Indeed, if this were not the case, then there would exist three sequences $y_i \in \operatorname{bdry} S \cap B(x; \frac{\delta_x}{2})$, $0 \neq \zeta_i \in N_S^P(y_i)$ and $s_i \in S$ such that

$$\left\langle \frac{\zeta_i}{\|\zeta_i\|}, s_i - y_i \right\rangle > i\|s_i - y_i\|^2.$$

The last inequality yields $\|s_i - y_i\| \leq \frac{1}{i}$, and therefore for i sufficiently large we can assume that $\|s_i - x\| < \delta_x$. Then by (9), for i sufficiently large one has

$$\left\langle \frac{\zeta_i}{\|\zeta_i\|}, s_i - y_i \right\rangle \leq \frac{(1+K^2)^{\frac{3}{2}}}{2r}\|s_i - y_i\|^2,$$

and so $\frac{(1+K^2)^{\frac{3}{2}}}{2r} > i$. This provides the desired contradiction, and therefore S is uniformly proximally smooth on $S \cap B(x; \frac{\delta_x}{2})$. □

Remark 3.11. Example 2.5 shows that wedgedness is crucial in Theorem 3.7. Indeed, the set S of that example is not wedged at the origin, satisfies the exterior sphere condition near the origin, but fails to be proximally smooth near the origin.

If we assume that the boundary of S is compact then we have the following corollary, in which the assertion of Theorem 3.7 is strengthened.

Corollary 3.12. *Let $S \subset \mathbb{R}^n$ be a wedged set with compact boundary. Then S satisfies a uniform exterior sphere condition iff S is uniformly proximally smooth.*

Proof. By Theorem 3.7 we have that for all $x \in \operatorname{bdry} S$ there exists $\delta_x > 0$ and $r_x > 0$ such that S is r_x-proximally smooth at any point in $\operatorname{bdry} S \cap B(x; \delta_x)$. Since $\operatorname{bdry} S$ is compact we deduce the existence of finite sequence $\{x_i\}_{1 \leq i \leq n}$ in $\operatorname{bdry} S$ such that

$$\operatorname{bdry} S \subset \bigcup_{i=1}^{n} B(x_i; \delta_{x_i}).$$

Clearly this implies that S is r-proximally smooth, where $r := \min\{r_{x_1}, r_{x_2}, ...r_{x_n}\}$. □

Remark 3.13. The set of Example 2.3 is wedged but does not have a compact boundary. This shows that the preceding corollary fails if we drop that compactness assumption. On the other hand, the sets of Example 2.4 and Example 2.5 are not wedged but possess compact boundary. This shows that the corollary fails if we drop the wedgedness assumption.

4. Interior sphere condition

We continue to assume that our set S is standard; that is, $S = \operatorname{cl}(\operatorname{int} S)$. In the control theoretic literature on can find two definitions of the interior r-sphere condition. The first one (see [1, 3, 4]) is complementary to the notion of exterior r-sphere condition which we have been using; that is, for each $x \in \operatorname{bdry} S$ there exists $y_x \in S$ such that

$$x \in \bar{B}(y_x; r) \subset S.$$

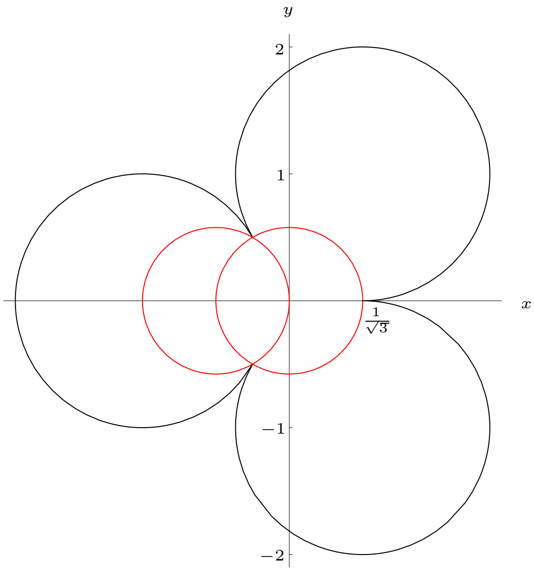

Figure 4.1: Example 4.1

The second one (see [5, 6, 18]) says that for all $x \in S$ there exists $y_x \in S$ such that

$$x \in \bar{B}(y_x; r) \subset S.$$

This means that S is the union of closed r-balls. Equivalently, there exists $S_0 \subset S$ such that $S_0 + r\bar{B} = S$. Clearly, if S is the union of closed r-balls then it satisfies the interior r-sphere condition. The following example shows that the reverse implication is not necessarily true and then the two definitions are *not equivalent*.

Example 4.1. Let S be the closed region inside the three circles of Figure 4.1. Clearly this set satisfies the interior 1-sphere condition (in the first sense) since the three circles are of radius 1. But the origin cannot be covered by a 1-ball contained in S; in fact, the maximal radius for a family of covering balls is $\frac{1}{\sqrt{3}}$. Therefore the interior sphere condition does not hold for S in the second sense.

If a closed set $C \subset \mathbb{R}^n$ is r-proximally smooth, then the complement of its interior, $(\operatorname{int} C)^c$, is the union of closed r-balls. To see why, consider any $x \in (\operatorname{int} C)^c$. If $d_C(x) > r$, then clearly there is an r-ball centered at x which is contained in $(\operatorname{int} C)^c$. If $d_C(x) \leq r$, consider any closest point $s \in C$ to x. Then $\zeta := x - s$ is a proximal normal to C at s. Since ζ is realizable by an r-sphere, there is a closed r-ball centered at $s + r\frac{\zeta}{\|\zeta\|}$ which is contained in $(\operatorname{int} C)^c$, and x is in this ball.

Therefore we have

$$\begin{aligned} & (\operatorname{int} S)^c \text{ is uniformly proximally smooth} \\ \Rightarrow\ & S \text{ is the union of uniform closed balls} \\ \Rightarrow\ & S \text{ has the uniform interior sphere condition.} \end{aligned}$$

The reverse implications are not necessarily true, as shown by Examples 2.3, 2.4 and 2.5. Indeed, in those examples the set $(\operatorname{int} S)^c$ has the union of uniform balls property, but S is not uniformly proximally smooth.

From Corollary 3.12 we obtain the following corollary, which asserts that the wedgedness of S together with boundary compactness guarantee the equivalence between the three properties under consideration.

Corollary 4.2. *Assume that S is wedged and that* $\operatorname{bdry} S$ *is compact. Then the following assertions are equivalent:*

(i) $(\operatorname{int} S)^c$ *is proximally smooth.*
(ii) S *is the union of uniform closed balls.*
(iii) S *possesses the uniform interior sphere condition.*

Remark 4.3. Let us reconsider Example 4.1. We noted that while S has the interior 1-sphere property, it is not the union of closed 1-balls. But it certainly *is* the union of closed r-balls for $r \leq \frac{1}{\sqrt{3}}$. It remains an open question as to whether the uniform interior sphere condition for S implies that S is a union of uniform closed balls. We conclude by expressing this question as a formal conjecture, and in a way that is free of terminology.

Conjecture 4.4. *Suppose that S is a closed set and that there exists $r > 0$ as follows: For each $x \in \operatorname{bdry} S$ there exists $y_x \in S$ for which $x \in \bar{B}(y_x; r) \subset S$. Then there exists $r' > 0$ such that S is the union of balls of radius r'.*

Acknowledgements. The authors wish to thank Professors Z. Artstein and P. Cannarsa for fruitful discussions. We are also grateful to the referees who read the paper with great care and made useful remarks.

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