Journal of Convex Analysis
Volume 16 (2009), No. 3&4, 987–992

A Multiplicity Theorem in $\mathbf{R}^n$

Biagio Ricceri
Department of Mathematics, University of Catania,
Viale A. Doria 6, 95125 Catania, Italy
ricceri@dmi.unict.it

Dedicated to Stephen Simons on the occasion of his 70th birthday.

Received: April 30, 2008

The aim of this paper is to establish the following result:

Theorem 1. *Let X be a finite-dimensional real Hilbert space, and let $J : X \to \mathbf{R}$ be a C^1 function such that*

$$\liminf_{\|x\|\to+\infty} \frac{J(x)}{\|x\|^2} \geq 0. \tag{1}$$

Moreover, let $x_0 \in X$ and $r, s \in \mathbf{R}$, with $0 < r < s$, be such that

$$\inf_{x\in X} J(x) < \inf_{\|x-x_0\|\leq s} J(x) \leq J(x_0) \leq \inf_{r\leq\|x-x_0\|\leq s} J(x). \tag{2}$$

Then, there exists $\hat{\lambda} > 0$ such that the equation

$$x + \hat{\lambda} J'(x) = x_0 \tag{3}$$

has at least three solutions.

We will proceed as follows. We first give the proof of Theorem 1. Then, we discuss in detail the finite-dimensionality assumption on X. More precisely, we will show not only that it can not be dropped, but also that it is very hard to imagine some additional condition (different from being x_0 a local minimum of J) under which one could adapt the given proof to the infinite-dimensional case. We finally conclude presenting an application of Theorem 1 to a discrete boundary value problem.

The proof of Theorem 1 is based on Theorem 1 of [3], a particular case of which reads as follows:

Theorem 2. *Let X be a Hausdorff topological space, and let $J, \Phi : X \to \mathbf{R}$ be such that, for each $\lambda > 0$, the function $J + \lambda\Phi$ has sequentially compact sublevel sets. Let M be the set (possibly empty) of all global minima of J, and assume that*

$$\inf_X \Phi < \beta := \min\left\{\inf_M \Phi, \sup_X \Phi\right\},$$

with the convention $\inf_\emptyset \Phi = +\infty$.

Then, at least one of the following assertions holds:

ISSN 0944-6532 / $ 2.50

(a) *For each* $t \in]\inf_X \Phi, \beta[$, *the restriction of* J *to* $\Phi^{-1}(t)$ *has a unique global minimum, say* $\hat{x}_t$, *and the function* $t \to \hat{x}_t$ *is continuous on* $]\inf_X \Phi, \beta[$.

(b) *There exists* $\lambda^* > 0$ *such that the function* $J + \lambda^*\Phi$ *has at least two global minima on* X.

Now, we can prove Theorem 1.

Proof of Theorem 1. We are going to apply Theorem 2. To this end, consider the function $\Phi : X \to \mathbf{R}$ defined by putting

$$\Phi(x) = \begin{cases} \|x - x_0\|^2 & \text{if } \|x - x_0\| < r, \\ r^2 & \text{if } r \leq \|x - x_0\| \leq s, \\ \|x - x_0\|^2 + r^2 - s^2 & \text{if } \|x - x_0\| > s. \end{cases}$$

Clearly, Φ is continuous. From (1), it readily follows that

$$\lim_{\|x\| \to +\infty} (J(x) + \lambda \|x - x_0\|^2) = +\infty$$

for all $\lambda > 0$. So, since J is continuous too, the function $J + \lambda\Phi$ has sequentially compact sublevel sets for all $\lambda > 0$. Let M and β be defined as in Theorem 2. Since M is closed and X is finite-dimensional, in view of the first inequality in (2), we have

$$\inf_{x \in M} \|x - x_0\| > s$$

and this clearly implies that

$$\beta > r^2.$$

Now, let $g :]0, \beta[\to X$ be any function such that

$$\Phi(g(t)) = t$$

for all $t \in]0, \beta[$. In particular, we then have

$$r \leq \|g(r^2) - x_0\| \leq s,$$

$$\|g(t) - x_0\| < r$$

for all $t \in]0, r^2[$, and

$$\|g(t) - x_0\| > s$$

for all $t \in]r^2, \beta[$. From this, it clearly follows that the function g is discontinuous at r^2. This shows that (a) of Theorem 2 does not hold (since, otherwise, $t \to \hat{x}_t$ would be a continuous function satisfying $\Phi(\hat{x}_t) = t$ for all $t \in]0, \beta[$). Consequently, there exists $\lambda^* > 0$ such that the function $J + \lambda^*\Phi$ has at least two global minima on X, say x_1 and x_2. By the last inequality in (2), we have

$$J(x_0) + \lambda^*\Phi(x_0) = J(x_0) < J(x) + \lambda^* r^2 = J(x) + \lambda^*\Phi(x)$$

for all $x \in X$ satisfying $r \leq \|x - x_0\| \leq s$. Consequently, both x_1 and x_2 belong to one of the open sets $\{x \in X : \|x - x_0\| < r\}$ and $\{x \in X : \|x - x_0\| > s\}$. By the definiton of Φ, this implies that x_1 and x_2 are two local minima of the function $x \to J(x) + \lambda^*\|x - x_0\|^2$. But, this function satisfies the Palais-Smale condition ([5], Example 38.25) and so, by Corollary 1 of [2], it possesses at least three critical points which are solutions of equation (3), with $\hat{\lambda} = \frac{1}{2\lambda^*}$. $\square$

As we said at the beginning, the finite-dimensionality assumption on X can not be dropped.

Indeed, we have the following

Example 3. Consider $L^2([0,1])$ with the usual inner product. Let $\varphi : \mathbf{R} \to \mathbf{R}$ be a bounded C^1 function such that $\varphi(t) = 0$ for all $t \in]-\infty, \delta]$ (for some $\delta > 0$) and $\varphi(1) < -\frac{1}{2}$.

For each $u \in L^2([0,1])$, put

$$J(u) = \int_0^1 tu^2(t)dt + \varphi\left(\int_0^1 u^2(t)dt\right).$$

Clearly, J is a C^1 functional on $L^2([0,1])$ satisfying (1). Also, note that $J(1) < 0$. Moreover, if $\int_0^1 u^2(t)dt < \delta$, then $J(u) \geq J(0) = 0$. So, (2) is also satisfied taking $x_0 = 0$ and $0 < r < s < \sqrt{\delta}$. Now, let $\lambda \in \mathbf{R}$ be fixed, and let $u \in L^2([0,1])$ satisfy the equation

$$u + \lambda J'(u) = 0.$$

This means that

$$\int_0^1 \left(1 + 2\lambda\left(t + \varphi'\left(\int_0^1 u^2(\tau)d\tau\right)\right)\right) u(t)v(t)dt = 0$$

for all $v \in L^2([0,1])$. Consequently, we have

$$\left(1 + 2\lambda\left(t + \varphi'\left(\int_0^1 u^2(\tau)d\tau\right)\right)\right) u(t) = 0$$

a.e. in $[0,1]$. From this, it clearly follows that $u = 0$.

Remark 4. In this remark, we want to discuss the possibility of adapting the above proof of Theorem 1 to the infinite-dimensional case, under some appropriate additional assumption. So, besides the hypotheses of Theorem 1, assume that X is infinite-dimensional. Let us begin the discussion considering the case when the last inequality in (2) is satisfied supposing that x_0 is a local minimum for J. We call this the trivial case. Assume also that J is sequentially weakly lower semicontinuous and that, for each $\lambda > 0$, the functional $x \to J(x) + \lambda\|x - x_0\|^2$ satisfies the Palais-Smale condition. By the first inequality in (1), there exists $\tilde{x} \in X$, with $\|\tilde{x} - x_0\| > s$, such that $J(\tilde{x}) < J(x_0)$. Then, for each $\lambda \in \left]0, \frac{J(x_0)-J(\tilde{x})}{\|\tilde{x}-x_0\|^2}\right[$, the functional $x \to J(x) + \lambda\|x - x_0\|^2$, being coercive and sequentially weakly lower semicontinuous, has a global minimum different from x_0 (by the choice of λ). But x_0 turns out to be a local minimum for this functional, and so the conclusion follows from Corollary 1 of [2]. Hence, in the trivial case, everything is immediate, without the need of resorting to Theorem 2. Now, suppose that the trivial case does not occur. Then, in view of the sequential compactness condition required in Theorem 2, we can not consider on X the strong topology. So, it would be reasonable to consider X equipped with the weak topology. Assume that. In this case, when we consider a function $g :]0, \beta[\to X$ such that $\Phi(g(t)) = t$ for all $t \in]0, \beta[$, we can not infer that g is discontinuous at r^2 with respect to the weak topology. This is due to the fact that, since $\dim(X) = \infty$, the norm is not sequentially weakly continuous. So, the information that the map $t \to \hat{x}_t$ would be weakly continuous (given in (a) of Theorem 2) is not enough to conclude that (b) has to hold. Therefore, we need to know that the map $t \to \hat{x}_t$ would be strongly continuous (if (b) does not hold). In this connection, we could resort to Corollary 2.13 of [4] which just ensures the strong continuity of this map on the interval $]t^*, +\infty[$, where $t^* = \inf\{t > 0 : \inf_{\|x-x_0\|=t} J(x) < J(x_0)\}$, provided that J is sequentially weakly continuous. Hence, assume that J is so. In our case, since x_0 is not a local minimum of J and since the function $t \to \inf_{\|x-x_0\|=t} J(x)$ is non-increasing ([4], Lemma 2.1), we have $t^* = 0$. However, just because of the monotonicity of $t \to \inf_{\|x-x_0\|=t} J(x)$, the last inequality in (2) can not be satisfied.

Remark 5. We also remark that Theorem 1 is no longer true without condition (1). In this connection, consider the function $J : \mathbf{R} \to \mathbf{R}$ so defined

$$J(x) = \begin{cases} 0 & \text{if } x \leq 1, \\ -(x-1)^2 & \text{if } x > 1. \end{cases}$$

Clearly, J is a C^1 function satisfying (2), with $x_0 = 0$, and the equation $x + \lambda J'(x) = 0$ has at most two solutions for all $\lambda \in \mathbf{R}$. Likewise, the functions (in $\mathbf{R}$) $J(x) = x^2$ and $J(x) = x$ provide counter-examples to the validity of Theorem 1 when, respectively, either the first or the last inequality in (2) does not hold.

To conclude, we present an application of Theorem 1 to a discrete boundary value problem.

Let $n \in \mathbf{N}$ $(n \geq 2)$ be fixed, and let $f_k : \mathbf{R} \to \mathbf{R}$ $(k = 1, ..., n)$ be n given functions. For $\lambda > 0$, we then consider the classical second-order problem

$$(P_\lambda) \qquad \begin{cases} -(x_{k+1} - 2x_k + x_{k-1}) = \lambda f_k(x_k) & k = 1, ..., n \\ x_0 = x_{n+1} = 0. \end{cases}$$

Set

$$X = \{(x_0, x_1, ..., x_n, x_{n+1}) \in \mathbf{R}^{n+2} : x_0 = x_{n+1} = 0\}.$$

Endow X with the inner product

$$\langle x, y \rangle = \sum_{k=1}^{n+1} (x_k - x_{k-1})(y_k - y_{k-1}).$$

One readily has

$$\langle x, y \rangle = -\sum_{k=1}^{n} (x_{k+1} - 2x_k + x_{k-1}) y_k \,. \tag{4}$$

We also put

$$\gamma = \inf_{x \in X, \|x\|=1} \max_{1 \leq k \leq n} |x_k|$$

and

$$\delta = \sup_{x \in X, \|x\|=1} \max_{1 \leq k \leq n} |x_k| \,.$$

The application of Theorem 1 that we want to present is as follows:

Theorem 6. *Let $f_k : \mathbf{R} \to \mathbf{R}$ $(k = 1, ..., n)$ be n continuous functions such that*

$$\limsup_{|t|\to+\infty} \frac{\int_0^t f_k(\tau)d\tau}{t^2} \leq 0 \tag{5}$$

for all $k = 1, ..., n$. Assume that there exist two numbers r, s, with $0 < r < s$, such that

$$\sum_{k=1}^{n} \sup_{t\in\mathbf{R}} \int_0^t f_k(\tau)d\tau > \sum_{k=1}^{n} \sup_{|t|\leq\delta s} \int_0^t f_k(\tau)d\tau \tag{6}$$

and

$$\sup_{\gamma r\leq|t|\leq\delta s} \int_0^t f_k(\tau)d\tau \leq -\sum_{h=1,h\neq k}^{n} \sup_{|t|\leq\delta s} \int_0^t f_h(\tau)d\tau \tag{7}$$

for all $k = 1, ..., n$.

Then, there exists $\hat{\lambda} > 0$ such that problem $(P_{\hat{\lambda}})$ has at least three solutions.

Proof. For each $x \in X$, put

$$J(x) = -\sum_{k=1}^{n} \int_0^{x_k} f_k(t)dt \ .$$

Clearly, J is a C^1 function on X and, in view of (4), the solutions of problem (P_λ) are exactly the solutions in X of the equation

$$x + \lambda J'(x) = 0 \ .$$

We are going to apply Theorem 1 (with $x_0 = 0$). Note that condition (1) follows from condition (5) (see the proof of Theorem 3 of [1]). Clearly, we have

$$\inf_{x\in X} J(x) = -\sum_{k=1}^{n} \sup_{t\in\mathbf{R}} \int_0^t f_k(\tau)d\tau$$

and

$$\inf_{\|x\|\leq s} J(x) \geq -\sum_{k=1}^{n} \sup_{|t|\leq\delta s} \int_0^t f_k(\tau)d\tau \ .$$

Consequently, the first inequality in (1) is satisfied in view of (6). Finally, let $x \in X$ satisfy $r \leq \|x\| \leq s$. Then, for some $k \in \mathbf{N}$, with $1 \leq k \leq n$, one has $|x_k| \geq \gamma r$, and so, by (7), it clearly follows that

$$J(x) \geq 0$$

which gives the last inequality in (2). Therefore, all the assumptions of Theorem 1 are satisfied, and the conclusion follows. □

Remark 7. It would be interesting to know if, in certain cases, the conclusion of Theorem 1 (in particular, that of Theorem 2) holds for exactly one $\hat{\lambda} > 0$.

References

[1] F. Faraci, A. Iannizzotto: Multiplicity theorems for discrete boundary value problems, Aequationes Math. 74 (2007) 111–118.

[2] P. Pucci, J. Serrin: A mountain pass theorem, J. Differ. Equations 60 (1985) 142–149.

[3] B. Ricceri: Well-posedness of constrained minimization problems via saddle-points, J. Glob. Optim. 40 (2008) 389–397.

[4] M. Schechter, K. Tintarev: Spherical maxima in Hilbert space and semilinear elliptic eigenvalue problems, Differ. Integral Equ. 3 (1990) 889–899.

[5] E. Zeidler: Nonlinear Functional Analysis and its Applications. III, Springer, New York (1985).