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Construction of Weakly Dense, Norm Divergent Sequences*

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Let X be a separable Banach space. We provide an explicit construction of a sequence in X that tends to ∞ in norm but which is weakly dense.

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Our interest in the result stated in the Abstract was motivated by two theorems. First, in their work on hypercyclic operators, K. Chan and R. Sanders proved the following:

Theorem 1 (Chan and Sanders [3]). *For any p, $2 \le p < \infty$, there is a bounded linear operator $T : \ell_p(\mathbb{Z}) \to \ell_p(\mathbb{Z})$ that is weakly hypercyclic but is not hypercyclic.*

That is, there is a vector $x_p \in \ell_p(\mathbb{Z})$ such that $\{x_p, T(x_p), \cdots, T^n(x_p), \cdots\}$ is a weakly dense set and such that for no vector $x \in \ell_p(\mathbb{Z})$ is it true that $\{x, T(x), \cdots, T^n(x), \cdots\}$ is norm dense. In fact, the proof in [3] shows the existence of a vector x_p such that $\|T^n(x_p)\| \to \infty$ while $\{T^n(x_p) \mid n \in \mathbb{N}\}$ is dense in $\ell_p(\mathbb{Z})$ with the weak topology. Moreover, Chan and Sanders remark that, in fact, one can construct a weakly dense sequence $(x_n) \subset \ell_2(\mathbb{Z})$ such that $\|x_n\| \to \infty$ ([3, page 49]).

Second, V. Kadets has proved the following result (see also [6]):

Theorem 2 (Kadets [7]). *For any Banach space X, for any sequence (c_n) of positive*

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real numbers such that $\sum_{n=1}^{\infty} c_n^{-2} = \infty$, *there is a sequence* $(x_n) \subset X$, $\|x_n\| = c_n$ *for all* n, *such that* 0 *is in the weak closure of the sequence* $\{x_n \mid n \in \mathbb{N}\}$.

The proof of this theorem uses Dvoretzky's theorem [4] and consequently is non-constructive. In fact, S. Shkarin [8] recently rediscovered the Kadets result; his proof also makes use of Dvoretzky's theorem.

In this note, we show that Kadets' result yields, as a simple consequence, that for any separable Banach space X, there is a sequence $(x_n) \subset X$ such that $\|x_n\| \to \infty$ while $\{x_n \mid n \in \mathbb{N}\}$ is weakly dense in X.

Moreover, we give a version of Theorem 2 whose proof provides a rather simple, explicit sequence of vectors and which uses the pigeon hole principle instead of Dvoretzky's theorem.

To begin, we show that the conclusion of Theorem 2 can be strengthened.

Theorem 3. *Let* X *be a separable Banach space. Suppose that there is a sequence* $(x_n) \subset X$ *with the following two properties:*

(1) $\|x_n\| \to \infty$ *as* $n \to \infty$,
(2) 0 *is in the weak closure of* $\{x_n \mid n \in \mathbb{N}\}$.

Then there is a sequence $(y_n) \in X$ *such that the following hold:*

(1) $\|y_n\| \to \infty$ *as* $n \to \infty$,
(2) $\{y_n \mid n \in \mathbb{N}\}$ *is weakly dense in* X.

Proof. Consider any sequence (z_r) that is norm dense in X. Let (x_n) be a sequence in $X \setminus \{0\}$ such that $(\|x_n\|)$ diverges to ∞ and 0 belongs to the weak closure of (x_n). Let $x_n^* \in X^*$ of norm 1 such that $x_n^*(x_n) = \|x_n\|$. Let $\gamma_{n,r}$ defined by

$$\gamma_{n,r} = \begin{cases} \dfrac{x_n^*(z_r)}{|x_n^*(z_r)|} & \text{if } x_n^*(z_r) \neq 0, \\ 1 & \text{if } x_n^*(z_r) = 0. \end{cases}$$

Then

$$\|z_r + \gamma_{n,r} x_n\| \geq |x_n^*(z_r + \gamma_{n,r} x_n)| = |x_n^*(z_r)| + \|x_n\| \geq \|x_n\|,$$

for all $r \leq n \in \mathbb{N}$. We apply the square ordering to

$$\begin{array}{cccccc}
z_1 + \gamma_{1,1}x_1 & \rightarrow & z_1 + \gamma_{2,1}x_2 & & z_1 + \gamma_{3,1}x_3 & \rightarrow \\
& \swarrow & & \nearrow & & \swarrow \\
z_2 + \gamma_{2,2}x_2 & & z_2 + \gamma_{3,2}x_3 & & z_2 + \gamma_{4,2}x_4 & \\
\downarrow & \nearrow & & \swarrow & & \\
z_3 + \gamma_{1,3}x_3 & & z_3 + \gamma_{4,3}x_4 & & z_3 + \gamma_{5,3}x_5 & \cdots \\
\cdots & \cdots & \cdots & \cdots & &
\end{array}$$

obtaining a sequence (y_s). Since $(\|x_n\|)$ diverges to ∞, then $(\|y_s\|)$ diverges to ∞ too. Moreover,

$$z_r \in \overline{\{y_s : s \in \mathbb{N}\}}^{w(X,X^*)},$$

for all $r \in \mathbb{N}$ and hence $\overline{\{y_s : s \in \mathbb{N}\}}^{w(X,X^*)} = X$. $\square$

The rest of this note is devoted to providing an explicit construction of a sequence (x_n) satisfying the conclusion of Theorem 2, for any separable Banach space X. We begin with $X = \ell_1$, considered as a real Banach space. Our argument uses the pigeon hole principle.

Proposition 4. *Consider the sequence (x_n) in ℓ_1 obtained by any ordering of the set $\cup_{k=1}^{\infty}\{\sqrt{k}(e_{m_1} - e_{m_2}) : 1 \leq m_1 < m_2 \leq (2k)^{k+1}\}$. We have that $(\|x_n\|)$ diverges to ∞ and 0 belongs to the weak closure of (x_n).*

Proof. Consider a natural number k and $\varphi_1, \ldots, \varphi_k$ in the unit sphere of ℓ_∞. We denote $K_0 = \{1, \ldots, (2k)^{k+1}\}$. If

$$\varphi_j = (\alpha_m^j)_{m=1}^{\infty}$$

we put

$$J_p^1 = \left\{ m \leq (2k)^{k+1} : \alpha_m^1 \in \left[\frac{p}{k}, \frac{p+1}{k}\right] \right\}$$

for $-k \leq p \leq k-1$, then

$$\{1, \ldots, (2k)^{k+1}\} = \bigcup_{p=-k}^{k-1} J_p^1.$$

Thus there exists a p_1, $-k \leq p_1 \leq k-1$, such that

$$\mathrm{Card}(J_{p_1}^1) \geq (2k)^k.$$

Choose a subset $K_1 \subset J_{p_1}^1$ such that $\mathrm{Card}(K_1) = (2k)^k$. Now we define

$$J_p^2 = \left\{ m \in K_1 : \alpha_m^2 \in \left[\frac{p}{k}, \frac{p+1}{k}\right] \right\}$$

for $-k \leq p \leq k-1$, then

$$K_1 = \bigcup_{p=-k}^{k-1} J_p^2$$

and $\mathrm{Card}(K_1) = (2k)^k$. Thus there exists a p_2, $-k \leq p_2 \leq k-1$, such that $\mathrm{Card}(J_{p_2}^2) \geq (2k)^{k-1}$, and as before we let $K_2 \subset J_{p_2}^2$ such that $\mathrm{Card}(K_2) = (2k)^{k-1}$. We continue by induction. For $l < k$, we assume that $(K_j)_{j=1}^l$ and $(p_j)_{j=1}^l$ have been found so that $K_l \subset K_{l-1} \subset \ldots \subset K_2 \subset K_1$, $-k \leq p_j \leq k-1$,

$$K_j \subset \left\{ m \in K_{j-1} : \alpha_m^j \in \left[\frac{p_j}{k}, \frac{p_j+1}{k}\right] \right\}$$

and $\mathrm{Card}(K_j) = (2k)^{k-j+1}$, for $j = 1, \ldots, l$. Since

$$k - l + 1 \geq 2$$

if we define again

$$J_p^{l+1} = \left\{ m \in K_l : \alpha_m^{l+1} \in \left[\frac{p}{k}, \frac{p+1}{k}\right] \right\}$$

for $-k \leq p \leq k-1$, then $K_l = \bigcup_{p=-k}^{k-1} J_p^{l+1}$ and $\operatorname{Card}(K_l) = (2k)^{k-l+1} \geq (2k)^2$. Thus there exists a p_{l+1}, $-k \leq p_{l+1} \leq k-1$, such that $\operatorname{Card}(J_{p_{l+1}}^{l+1}) \geq (2k)^{k-l}$. Again we consider $K_{l+1} \subset J_{p_{l+1}}^{l+1}$ such that $\operatorname{Card}(K_{l+1}) = (2k)^{k-l}$.

By taking $l = k-1$, we obtain finally the existence of a K_k with

$$K_k \subset K_{k-1} \subset \ldots \subset K_2 \subset K_1 \subset \{1, \ldots, (2k)^{k+1}\},$$

$$\operatorname{Card}(K_k) = 2k,$$

and

$$|\alpha_m^j - \alpha_r^j| \leq \frac{1}{k}$$

for all $j = 1, \ldots, k$ and all $m, r \in K_k$. Since $\operatorname{Card}(K_k) = 2k \geq 2$, we can take $m_1 < m_2$, such that

$$\{m_1, m_2\} \subset K_k.$$

Consider

$$x = \sqrt{k}(e_{m_1} - e_{m_2}).$$

We have that

$$|\varphi_j(x)| = |\sqrt{k}(\alpha_{m_1}^j - \alpha_{m_2}^j)| \leq \frac{\sqrt{k}}{k} = \frac{1}{\sqrt{k}}, \tag{1}$$

for all $j = 1, \ldots, k$ and

$$\|x\|_1 = 2\sqrt{k}.$$

Let

$$I_k = \left\{ \sqrt{k}(e_{m_1} - e_{m_2}) \ : \ 1 \leq m_1 < m_2 \leq (2k)^{k+1} \right\}$$

for $k = 1, 2, \ldots$. We take

$$(x_n)_{n=1}^{\infty} = \bigcup_{k=1}^{\infty} I_k$$

where (e.g.) we first order the elements of I_1, then of I_2 and so on. Clearly $(\|x_n\|)$ diverges to ∞. We claim that 0 belongs to the weak closure of (x_n). Indeed, given $\phi_1, \ldots, \phi_h$ in $\ell_\infty \setminus \{0\}$ and $\varepsilon > 0$, we consider $k \geq h$ such that

$$\frac{\sup\{\|\phi_1\|, \ldots, \|\phi_h\|\}}{\sqrt{k}} < \varepsilon,$$

and we define

$$\varphi_j = \begin{cases} \dfrac{\phi_j}{\|\phi_j\|} & \text{if } 1 \leq j \leq h, \\[2ex] \dfrac{\phi_h}{\|\phi_h\|} & \text{if } h \leq j \leq k. \end{cases} \tag{2}$$

By (1) we can find an $x_n = \sqrt{k}(e_{m_1} - e_{m_2})$ for a certain pair $\{m_1, m_2\}$, $1 \leq m_1 < m_2 \leq (2k)^{k+1}$ such that

$$\left| \frac{\phi_j}{\|\phi_j\|}(x_n) \right| \leq \frac{1}{\sqrt{k}}.$$

for all $j = 1, \ldots, h$. Hence

$$|\phi_j(x_n)| \leq \frac{\|\phi_j\|}{\sqrt{k}} < \varepsilon,$$

for all $j = 1, \ldots, h$, and we have obtained

$$0 \in \overline{\{x_n : n \in \mathbb{N}\}}^{w(\ell_1, \ell_\infty)}.$$

□

Note that the sequence (x_n) that was constructed in the above proof is particularly simple; namely each x_n is of the form $C(e_i - e_j)$. We now show how the previous proposition yields the general result. Note that given a dense sequence in a separable Banach space, the sequence (z_n) in the following Corollary can be explicitly described.

Corollary 5. *Given a separable Banach space X, there is a sequence $(z_n) \subset X$ such that $\|z_n\| \to \infty$ and $X = \overline{\{z_n \mid n \in \mathbb{N}\}}^{w(X,X^*)}$.*

Proof. By Theorem 3, it is enough to prove that there exists a sequence (z_n) in $X \setminus \{0\}$ such that $(\|z_n\|)$ diverges to ∞ and 0 belongs to the weak closure of (z_n). Let B_X be the open unit ball of X. Since X is infinite dimensional, the Riesz lemma allows us to construct a sequence (y_n) in B_X such that $\|y_n - y_m\| \geq \frac{1}{2}$ for all $n \neq m$.

Now we define $T : \ell_1 \longrightarrow X$ by

$$T(\alpha_n) = \sum_{n=1}^{\infty} \alpha_n y_n.$$

Since $T(B_{\ell_1}) \subset B_X$, T is continuous. Let $(x_n) \subset \ell_1$ be the sequence obtained in Proposition 4. We are going to check that $z_n = T(x_n)$ defines the sequence that we are looking for. If $A = \{x_n \ : \ n \in \mathbb{N}\}$, we know that $0 \in \overline{A}^{w(\ell_1,\ell_\infty)}$. Since T is weak-weak continuous, we get

$$0 = T(0) \in T\left(\overline{A}^{w(\ell_1,\ell_\infty)}\right) \subset \overline{T(A)}^{w(X,X^*)}.$$

Moreover, we know that for each n there exist unique $k(n),\ m_1(n),\ m_2(n) \in \mathbb{N}$ with $m_1(n) \neq m_2(n)$ such that $x_n = \sqrt{k(n)}(e_{m_1(n)} - e_{m_2(n)})$ and $k(n) \to \infty$ whenever n grows to ∞. Hence $\|x_n\|_1 = \sqrt{2k(n)}$ diverges to ∞. But

$$\begin{aligned}\|T(x_n)\| &= \sqrt{k(n)}\|T(e_{m_1(n)}) - T(e_{m_2(n)})\| \\ &= \sqrt{k(n)}\|y_{m_1(n)} - y_{m_2(n)}\| \geq \frac{1}{2}\sqrt{k(n)}\ ,\end{aligned}$$

for all $n \in \mathbb{N}$ and we have obtained that $\|(z_n)\|$ diverges to ∞ too. □

Comments. 1. Proposition 4 can be adapted to the complex ℓ_1 case. As a consequence, Corollary 5 holds for $\mathbb{R}$ and $\mathbb{C}$.

2. The above argument shows that for every infinite dimensional Banach space (separable or not) one can construct a sequence (x_n) in $X \setminus \{0\}$ such that $(\|x_n\|)$ diverges to ∞ and such that 0 belongs to the weak closure of (x_n).

3. Corollary 5 is much weaker than Theorem 1. Indeed, the result of [3] produces an operator $T : \ell_p(\mathbb{Z}) \to \ell_p(\mathbb{Z}),\ p \geq 2$ which, in turn, produces the sequence (x_n). Shkarin notes there is no bilateral shift on $\ell_p(\mathbb{Z})$ for $1 \leq p < 2$ that is weakly, but not norm, hypercyclic [8]. Although this does not rule out the possibility of finding a bounded operator T on such spaces and a weakly dense sequence (x_n) such that $T(x_n) = x_{n+1}$ for all n, it at least gives an indication that, if they exist, such operators are difficult to come by. (See also related work of S. Grivaux [5]).

4. Using a result of Ball [1, Theorem 7], Shkarin ([8, Proposition 5.4]) shows that if the norms of the sequence (x_n) tend "too rapidly" to infinity (e.g. if $\sum_n \|x_n\|^{-1} < \infty$), then $0 \notin \overline{\{x_n \mid n \in \mathbb{N}\}}^{w(X,X^*)}$.

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