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On Existence Theorems for Monotone and Nonmonotone Variational Inequalities

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We make a comparison among various existence theorems for monotone and nonmonotone variational inequalities and give necessary conditions for the solvability of the variational inequality problem.

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1. Introduction

Let us assume that E is a a reflexive Banach space over the reals, $K \subset E$ is a nonempty, closed and convex set, $A : K \to E^*$ a map to the dual space E^* equipped with the *weak** topology. The variational inequality problem (VIP) defined by K and A consists of finding a point $u \in K$ such that

$$\langle Au, v-u\rangle \geq 0, \quad \forall v \in K, \tag{VIP}$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between E^* and E. There are two standard approaches to the existence of solutions of (VIP), namely with or without some monotonicity requirements and in each of these two directions we have a lot of existence results. Therefore, the aim of this paper is to analyze various existence results, disseminated in various papers, and try to do a comparison in order to find the most general and/or the easiest to handle. The paper is organized as follows.

In Sect. 2 we consider the existence results where the conditions imposed on A are related merely to the kind of continuity. The main assumptions regarding the continuity are due to Ky Fan ([13], see also [12], [14]) and to H. Brezis ([1], see also [2], [3]), who introduced the assumption of hemicontinuity in the sense of Fan (F-hemicontinuity in short) and of

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pseudomonotonicity in the sense of Brezis (B-pseudomonotonicity), respectively. Both these assumptions in the finite dimensional case and with K compact generalize the well known theorem by Hartmann and Stampacchia (see Theorem 3.1), as the Example 2.8 shows. The connection between the existence results of Brezis and Ky Fan was already recognized by Brezis which in ([1]) stated:
"Le théorème 24 et ses corollaires généralisent donc des résultats de F. Browder et de P. Hartman and G. Stampacchia. Il sont à rapprocher des résultats de Ky-Fan".

As a matter of fact, if K is open and $\dim E < \infty$, the B-pseudomonotonicity is equivalent to the continuity and the same happens for the F-hemicontinuity as Proposistions 2.9 and 2.10 prove. If $\dim E = \infty$ and the map A is defined on the whole space E, both the Brezis and Fan assumptions ensure that A is continuous on the finite dimensional subspaces of E (see Proposition 2.11).

Propoposition 2.12 states the first comparison result between the Fan and Brezis assumptions, namely if K is convex and closed an F-hemicontinuous map is also B-pseudomonotone. Moreover, if E is a Hilbert space and $A : E \to E^*$ is a linear and continuous operator then the F-hemicontinuity is equivalent to the following property:

$$\text{if } u_n \text{ weakly converges to } 0, \text{ then } \liminf_n \langle Au_n, u_n\rangle \geq 0 \qquad \text{(Ernst-Théra)}$$

(see Proposition 2.13). The last property is proved by E. Ernst and M. Théra ([10]) to be a necessary and sufficient condition for the existence of a solution to (VIP) when A is linear and continuous from E to E^* and K is an arbitrary bounded closed and convex subset of E. Then, the F-hemicontinuity, under the above assumptions, is a necessary condition for the existence of solution of (VIP) and it results equivalent to the B-pseudomonotonicity.

The remaining part of Section 2 is devoted to Variational Inequalities related to merely closed and convex subsets K of E. In this case, an additional assumption on A is needed, namely some kind of coercivity. Proposition 2.17, shows that the coercivity condition related to B-pseudomonotone operators is less general than the one related to F-hemicontinuity. However, Theorem 2.18 shows that a condition firstly introduced by Hartman and Stampacchia is sufficient to ensure the existence of a solution to (VIP) assuming that the operator A is B-pseudomonotone or F-hemicontinuous. In Theorem 2.19 we recall an existence result in which the second condition of B-pseudomonotonicity, namely

2.2) For each $v \in K$ the function $u \to \langle Au, u - v\rangle$ is lower bounded on the bounded subset of K.

is replaced by the condition

2.2′) A is continuous on any finite dimensional subspace.

Finally we discuss a theorem due to B. Ricceri ([28]) which deals with variational inequalities related to operators A only *weakly** continuous. In this case, besides the usual assumptions on K, it is supposed that K has nonempty relative interior (that is the interior of K in its convex hull), and it is shown that this assumption cannot be removed. Then the open problem arises whether assuming that K has the quasi-relative interior nonempty (that is the set of the points of K such that the tangent cone is a subspace) some existence theorem can be provided under a suitable assumption of continuity on

A. It is worth mentioning that in the applications arising from equilibrium problems the constraint set K has quasi-relative interior nonempty, whereas its relative interior is empty.

Section 3 is devoted to study existence theorems for monotone Variational Inequalities. We remark at first that the monotone approach is due to G. Stampacchia and recall the famous Hartmann-Stampacchia theorem (Theorem 3.1) in which the operator $A : E \to E^*$ is supposed to be monotone, continuous on finite dimensional subspaces of K, or alternatively, defined on the whole space, monotone and hemicontinuous on line segments. The Hartmann-Stampacchia theorem has been generalized by [23] and [24] assuming the map $A : K \to E^*$ pseudomonotone in the sense of Karamardian and continuous on finite dimensional subspaces of E (see Theorem 3.4). However, the result of Theorem 3.4 is in turn generalized by Theorem 3.6 in which the map A is assumed to be Karamardian-pseudomonotone and lower hemicontinuous on line segments. This results generalizes Theorems 3.1 and 3.4 because a monotone and lower hemicontinuous map on K is not necessarily continuous on finite dimensional subspaces of K (see Example 3.8). Moreover, we have also taken the opportunity to quote some other kind of continuity assumptions used by [15] in order to obtain surjectivity results and an interesting existence result in the framework of nonreflexive Banach space (see Theorem 3.9).

2. The approach without monotonicity

Various kinds of continuity are requested in the statement of existence theorems present in the literature. In 1968 H. Brezis introduced ([1], see also [2], [3]) a kind of lower semicontinuity which he called "pseudomonotonicity". The definition (2.1 below specificates the general pseudomonotonicity with respect to the weak topology of Banach reflexive space.

Definition 2.1. A map A from K to X^* is called pseudomonotone in the sense of Brezis (B-pseudomonotone) iff

2.1) For each sequence u_n weakly converging to u (in short $u_n \rightharpoonup u$) in K and such that $\limsup_n \langle Au_n, u_n - u\rangle \leq 0$ it results that:

$$\liminf_n \langle Au_n, u_n - v\rangle \geq \langle Au, u - v\rangle, \quad \forall v \in K.$$

2.2) For each $v \in K$ the function $u \to \langle Au, u - v\rangle$ is lower bounded on bounded subsets of K.

Another kind of lower semicontinuity is the hemicontinuity in the sense of Ky Fan (see [12], [13], [5]), which we call F-hemicontinuity to avoid confusion with another notion to be introduced in the sequel and which we report in the particular case of a Banach reflexive space.

Definition 2.2. A mapping $A : K \to E^*$ is F-hemicontinuous iff for all $v \in K$ the function $u \mapsto \langle Au, u - v\rangle$ is weakly lower semicontinuous on K.

Moreover we recall the following other kinds of continuity, which will be used together with some kind of monotonicity assumptions:

Definition 2.3 (Hemicontinuity along line segments (see [31], [19]). $A : K \to E^*$ is hemicontinuous along line segments, iff the function:

$$t \mapsto \langle A(tu + (1-t)v), w \rangle, \quad t \in [0,1]$$

is continuous for all $u, v, w \in K$.

Definition 2.4 (Lower hemicontinuity along line segments (see [5])). $A : K \to E^*$ is lower hemicontinuous along line segments, iff the function:

$$\xi \mapsto \langle A\xi, u - v \rangle$$

is lower semicontinuous for all $u, v \in K$ on the line segments $[u, v]$.

Now we point out the well known reason why (see ([1]), ([28])) a continuity assumption of this type is needed in order to get general existence results. It is well known that, in the finite dimensional case, the following result due to P. Hartmann and G. Stampacchia ([19], see also [25]) holds:

Theorem 2.5. *Assume that* $\dim E < +\infty$ *and let* K *be convex and compact. Let* $A : K \to E^*$ *be a continuous mapping. Then,* (VIP) *admits solutions.*

Now, let us pass from a finite dimensional space E to a reflexive Banach space with $\dim E = +\infty$, remaining K compact. In order to show the existence of a solution we can proceed in the following way. Denote by $\mathcal{U}$ the family of all the finite dimensional linear subspaces of E meeting K and consider $\mathcal{U}$ as a direct set, with the set-theoretic inclusion, indexed by $s \in S$. Let us assume that $A : K \to E^*$ is *weakly** continuous. Due to Theorem 2.5, there exists a solution $u_s \in K \bigcap U_s$, i.e.

$$\langle Au_s, v - u_s \rangle \geq 0, \quad \forall v \in K \cap U_s \, . \tag{1}$$

Since K is compact, the net $\{u_s\}$ admits some cluster point u_0 in K and there exists a sub-net of $\{u_s\}$, say $\{u_\alpha\}$, converging to u_0. Thus, considering for each $v \in K$, the equality

$$\langle Au_\alpha, v - u_\alpha \rangle = \langle Au_\alpha, v - u_0 \rangle + \langle Au_\alpha, u_0 - u_\alpha \rangle \tag{2}$$

and observing that in virtue of (1)

$$\liminf_\alpha \langle Au_\alpha, v - u_\alpha \rangle \geq 0$$

and

$$\lim_\alpha \langle Au_\alpha, v - u_0 \rangle = \langle Au_0, v - u_0 \rangle,$$

in order to obtain our goal it is enough that

$$\limsup_\alpha \langle Au_\alpha, u_0 - u_\alpha \rangle \leq 0 \tag{3}$$

If we look at the B-pseudomonotonicity and at the F-hemicontinuity assumptions, we can see that these definitions involve $\liminf$ and $\limsup$ of the above term $\langle Au_\alpha, u - u_\alpha \rangle$.

The existence theorems in the infinite dimensional case with K weakly compact are the following.

Theorem 2.6 (see [1]). *Let K be a nonempty convex and weakly compact subset of E and A a B-pseudomonotone mapping from K to E^*. Then* (VIP) *admits solutions.*

Theorem 2.7 (see [13], [14], [5]). *Let K be a nonempty, convex and weakly compact subset of E, and $A : K \to E^*$ an F-hemicontinuous mapping. Then* (VIP) *admits solutions.*

Let us observe that Theorem 2.6 generalizes Theorem 2.7 when $\dim E < +\infty$. In fact, a continuous mapping $A : K \to E$ is also B-pseudomonotone and F-hemicontinuous but the reverse is not ensured as the following example shows:

Example 2.8. Fix $a > 1$ and consider the function defined on $[0, 1]$ by

$$f(x) = \begin{cases} -1/x + a, & x \in]1/2a, 1] \\ -a, & x \in]0, 1/2a] \\ a, & x = 0 \end{cases}$$

f(x) is F-hemicontinuos on $[0, 1]$, but obviously not continuous. The point $x_0 = 1/a$ solves the variational inequality corresponding to the function f and to the interval $[0, 1]$.

In general a B-pseudomonotone mapping $A : K \to E$ is continuous in K if K is an open subset of E (see Proposition 21 of [1]). Moreover we have the following results for an F-hemicontinuous map, which easily follow from the subsequent Proposition 2.12 together with Proposition 21 of [1].

Proposition 2.9. *Let $A : K \to E^*$ be F-hemicontinuous, with K an open subset of E. Then, if E is finite dimensional, A maps bounded sets into bounded sets.*

Proposition 2.10. *Let $A : K \to E^*$ be F-hemicontinuous, with E finite dimensional and K a nonempty, open subset of E. Then A is continuous on K.*

If the mapping A acts on the whole space E the following result holds.

Proposition 2.11. *Let $A : E \to E^*$ F-hemicontinuous. Then A is continuous on the finite dimensional subspaces of E , that is, for each finite dimensional subspace $E_0 \subset E$ and for each fixed $v \in E$, the mapping:*

$$E_0 \ni u \to \langle Au, v\rangle \in \mathbb{R}$$

is continuous.

Proof. Fix $v \in E$ and let $E_1 = \{\lambda u + \mu v,\ \lambda, \mu \in \mathbb{R}, u \in E_0\}$ the linear space generated by E_0 and v. Let $j : E_1 \to E$ the injection map, together with its dual map $j^* : E^* \to E_1^*$. Because of (2.10) the map $j^*Aj : E_1 \to E_1^*$ is continuous. Since in E_1 the strong and the weak topologies coincide, this means that for each fixed $w \in E_1$ the map $z \mapsto \langle j^*Ajz, w\rangle$ is continuous. In particular, we can choose $w = v$ and $z \in E_0$ and the thesis follows immediately for the fact that $\langle j^*Aju, v\rangle = \langle Au, v\rangle$, $\forall u, v \in E_1$. □

A first comparison between B-pseudomonotonicity and F-hemicontinuity is given by the next proposition.

Proposition 2.12. *Let $A : K \to E^*$ an F-hemicontinuous mapping, where K is a closed and convex subset of E. Then A is B-pseudomonotone.*

Proof. Condition 2.1 of Definition 2.1 is obviously verified. By contradiction, assume that condition 2.2 is not true. Then, there exists $v^* \in K$ and a bounded subset K^* of K such that the function $u \mapsto \langle Au, u - v^* \rangle$ is not lower bounded on K^*. As a consequence there exists a sequence $\{u_n\}$, $u_n \in K^*, \forall n \in \mathbb{N}$, such that

$$\lim_n \langle Au_n, u_n - v^* \rangle = -\infty \tag{4}$$

Since K^* is bounded, there exists a subsequence $\{u_{k_n}\}$ such that u_{k_n} converges weakly to $u^* \in K$, because K is convex and closed, hence weakly closed. Then, in virtue of F-hemicontinuity we have

$$\liminf_n \langle Au_{k_n}, u_{k_n} - v^* \rangle \geq \langle Au^*, u^* - v^* \rangle \tag{5}$$

which condradicts (4). □

Another characterization of F-hemicontinuity when A is defined on a Hilbert space E is given by the following result.

Proposition 2.13. *Let E be a Hilbert space and $A : E \to E^*$ a linear and continuous operator. Then the following statements are equivalent:*

i) *A is F-hemicontinuous on E.*

ii) *A is such that if $u_n \rightharpoonup 0$, then $\liminf_n \langle Au_n, u_n \rangle \geq 0$.*

Proof. If A is F-hemicontinuous, it is clear that A verifies *ii*). Now suppose that A verifies *ii*). Being A linear and continuous it follows that if $u_n \rightharpoonup u$ then $\lim_n \langle Au_n, w \rangle = \langle Au, w \rangle, \forall w \in E$. Considering $w_n = u_n - u \rightharpoonup 0$, in virtue of assumption *ii*) we have

$$\liminf_n \{ \langle Au_n, u_n - u \rangle - \langle Au, u_n - u \rangle \} \geq 0$$

and also, since $\lim_n \langle Au, u_n - u \rangle = 0$,

$$\liminf_n \langle Au_n, u_n - u \rangle \geq 0$$

Then, for all $v \in K$ we reach

$$\liminf_n (\langle Au_n, u_n - v \rangle + \langle Au_n, v - u \rangle) \geq 0$$

and, taking into account that

$$\lim_n \langle Au_n, v - u \rangle = \langle Au, v - u \rangle$$

we have

$$\liminf_n \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle$$

□

Now let us recall the very interesting result by E. Ernst and M. Théra [10].

Theorem 2.14. *Let E a real Hilbert space and A a linear and continuous operator. Then, the following statements are equivalent:*

i) *A is such that if $u_n \rightharpoonup 0$, then $\liminf_n \langle Au_n, u_n\rangle \geq 0$*

ii) (VIP) *admits solutions for every bounded convex and closed set K.*

This result together with Proposition 2.13 yields to the fact that F-hemicontinuity on E is a necessary condition for the solvability of (VIP) on an arbitrary closed, convex and bounded subset of E when the operator A is linear and continuous and E is a Hilbert space. Moreover, in this case the concepts of B-pseudomonotonicity and F-hemicontinuity are equivalent.

Now let us consider the Variational Inequality related to merely convex and closed subset K of E. The existence theorems related to B-pseudomonotonicity and F-hemicontinuity are the following.

Theorem 2.15 ([1]). *Let $A : K \to E^*$ be B-pseudomonotone and let K be nonempty closed and convex. Moreover, assume that there exists $u_0 \in K$ such that:*

$$\lim_{\|u\|\to\infty, u\in K} \frac{\langle Au, u - u_0\rangle}{\|u\|} = +\infty \tag{6}$$

Then (VIP) *admits solutions.*

Theorem 2.16 ([12], [14]). *Let $A : K \to E^*$ be F-hemicontinuous and let K be a nonempty closed and convex subset of E. Moreover, let us suppose that A satisfies the following condition*

H1) *There exist $K_1 \subset K$ nonempty weakly compact and $K_2 \subset K$ compact such that for every $v \in K \setminus K_1$ there exists $w \in K_2$ such that*

$$\langle Av, v - w\rangle > 0. \tag{7}$$

Then (VIP) *admits solutions.*

It is possible to compare assumption (6) of Theorem 2.15 with assumption *H1)* of Theorem 2.16. In fact we can prove the next result.

Proposition 2.17. *Condition* (6) *implies condition H1).*

Proof. From (6) we derive that there exist two positive constants C and R, such that:

$$\langle Au, u - u_0\rangle > C > 0, \quad \forall u \in K \setminus \overline{B(0,R)}$$

Being $\overline{B(0,R)} = \{v \in E : \|v\| \leq R\}$ weakly compact, condition *H1)* is verified choosing $K_1 = \overline{B(0,R)}$ and $K_2 = \{u_0\}$. □

However, we can provide a version of Theorems 2.15 and 2.16 in which condition (6) and *H1)* are replaced by the following one firstly considered in [19]:

H2) There exist $u_0 \in K$ and $R > \|u_0\|$ such that

$$\langle Av, v - u_0 \rangle > 0, \quad \forall v \in K \cap \{v \in E : \|v\| = R\}.$$

In fact we have the following result

Theorem 2.18. *Let $A : K \to E^*$ be B-pseudomonotone or F-hemicontinuous, where K is nonempty, closed and convex. Moreover, assume that H2) holds. Then,* (VIP) *admits solutions.*

Proof. Let us consider the set

$$K_R = \{v \in K : \|v\| \leq R\}$$

Since K_R is closed, convex and bounded (and nonempty for R big enough), and A B-pseudomonotone or F-hemicontinuous, there exists $u_R \in K_R$ such that:

$$\langle Au_R, v - u_R \rangle \geq 0, \quad \forall v \in K_R \tag{8}$$

Now let us remark that $\|u_R\| < R$. In fact, if we had $\|u_R\| = R$, then assumption *H2)* written for $v = u_R$ would yield $\langle Au_R, u_R - u_0 \rangle > 0$, which contradicts (8). To prove that u_R solves (VIP) fix arbitrarily $w \in K$ and, for $t \in [0, 1]$, consider the point:

$$v_t = (1 - t)u_R + tw$$

which, for t small enough belongs to K_R, hence $\langle Au_R, (1 - t)u_R + tw - u_R \rangle \geq 0$, namely

$$\langle Au_R, w - u_R \rangle \geq 0, \quad \forall w \in K$$

□

It is remarkable that condition 2.2) in the definition of B-pseudomonotonicity can be replaced by the following:

2.2′) A is continuous on any finite dimensional subspace.

In fact, in paper [4] Brezis, Nirenberg and Stampacchia showed the following theorem

Theorem 2.19. *Let K be a convex and closed subset of E and assume that $A : K \to E^*$ verifies conditions* 2.1) *and* 2.2′)*. Moreover, suppose that there exist a compact subset L of K and $u_0 \in L$ such that*

$$\langle Av, v - u_0 \rangle \geq 0, \quad \forall v \in K \setminus L \tag{9}$$

Then (VIP) *admits solutions.*

The following result due to B. Ricceri (see [28]) deserves some comments.

Theorem 2.20. *Assume that K is nonempty, convex and closed, that its relative interior (that is the interior of K in its convex hull) is nonempty, and that A is weakly* continuous. Moreover, let K_1, K_2 be two nonempty compact subsets of K, with $K_2 \subset K_1$ and and K_2 finite dimensional such that for each $v \in K \setminus K_1$ there exists $w \in K_2$ such that*

$$\langle Av, v - w \rangle > 0$$

Then (VIP) *admits solutions.*

In this theorem A is supposed *weakly** continuous but it is also supposed that the relative interior of K is nonempty. This assumption cannot be removed as a counterexample in [16] (see also [30]) shows. However, in many infinite dimensional variational inequalities which model equilibrium problems (see e.g. [6], [7], [8], [9], [17], [20], [21], [22], [27], [18]) the relative interior of the constraints set K is empty, while the quasi-relative interior is nonempty. Then the open problem arises wether assuming that the quasi-relative interior of K is non empty (let us recall that the quasi relative interior of K is the set of the points of K such that the tangent cone is a subspace) and replacing the *weakly** continuity assumption on A with a less general assumption, an appropriate existence theorem can be proved.

Before concluding this section we would like to mention that the notion of B-pseudo-monotonicity or F-hemicontinuity can be generalized to vector variational inequalities. In this respect we point out the paper [11] and the references therein contained for some results in the vectorial case. Moreover, it is worth mentioning the paper [15] in which surjectivity results for nonlinear mappings T from a Banach space X to its dual X^* are considered. Many kinds of assumptions on the mappings T are considered, for example B-pseudomonotonicity, hemicontinuity, semimonotonicity, the so called condition (P) (and others could be considered) and their mutual relations are focused in order to state surjectivity results.

3. The monotone approach

The monotone approach is due to Hartmann and Stampacchia [19] (see also [29]) who proved the following

Theorem 3.1. *Let E be a reflexive Banach space and let K be a closed convex set in E. Let $A : K \to E^*$ be monotone, continuous on finite dimensional subspaces of K. [Alternatively, let $A : E \to E^*$ be monotone and hemicontinuous on line segments]. Then, a necessary and sufficient condition in order that a solution of* (VIP) *exist is that there exists a constant R such that at least a solution of the variational inequality*

$$u_R \in K_R,\ \langle Au_R, v - u_R\rangle \geq 0, \quad \forall v \in K_R$$

satisfies the inequality

$$\|u_R\| < R.$$

Let us recall the monotonicity assumption.

Definition 3.2. A map $A : K \to E^*$ is called monotone if

$$\langle Au - Av, u - v\rangle \geq 0, \quad \forall u, v \in K$$

The reason why the assumptions of Theorem 3.1 are that $A : K \to E^*$ is monotone, continuous on finite dimensional subspaces of K, or alternatively, that $A : E \to E^*$ is monotone and hemicontinuous on line segments, reflects also here, the fact that when A is defined on the whole space E, if it is monotone and hemicontinuous along line segments, it is also continuous on finite dimensional subspaces of K. The role of the monotonicity assumption is that, being

$$\langle Au, v - u\rangle \leq \langle Av, v - u\rangle, \quad \forall u, v \in K$$

it is possible to obtain the Minty Lemma if A is hemicontinuous along line segments on K (see the next lemma (3.5) and if $v_n \rightharpoonup u$ in K, to get

$$0 \leq \liminf_n \langle Av_n, v_n - u\rangle, \quad \forall u \in K$$

It is surprising that Brezis, Nirenberg and Stampacchia were the first who noted that in Theorem 3.1, instead of monotonicity of A, it is sufficient to assume that (see [4], p. 297):

$$\langle Au, u - v\rangle \leq 0 \text{ implies } \langle Av, v - u\rangle \geq 0 \text{ for any } u, v \in K.$$

Then Karamardian considered this more general concept of monotonicity, which he called pseudomonotonicity, giving the following definition.

Definition 3.3. The map $A : K \to E^*$ is said to be pseudomonotone in the sense of Karamardian (K-pseudomonotone) iff for all $u, v \in K$

$$\langle Av, u - v\rangle \geq 0 \to \langle Au, u - v\rangle \geq 0$$

Then the authors in [24], using the K-pseudomonotonicity, showed the following Theorem, close to the generalized version of Theorem 3.1.

Theorem 3.4 (see [24]). *Let K be a closed convex set and $A : K \to E^*$ a K-pseudomonotone map which is continuous on finite dimensional subspaces of E. Then the following statements are equivalent:*

a) (VIP) *admits solutions.*

b) *Condition H2) holds, i.e. there exists $u_0 \in K$ and $R > \|u_0\|$ such that*

$$\langle Av, v - u_0\rangle \geq 0, \quad \forall v \in K \cap \{v \in E : \|v\| = R\}$$

c) *There exists a point $u_0 \in K$ such that the set*

$$\{v \in K : \langle F(v), v - u_0\rangle < 0\}.$$

is bounded.

In Theorem 3.4 it is requested that A is continuous on finite dimensional subspaces of E instead of K and it is not considered the case in which A is, alternatively, hemicontinuous along line segments. It seems that this fact is due to the lack of the corresponding result that a K-pseudomonotone and hemicontinuous along line segments map is also continuous on finite dimensional subspaces. On the other hand, the Minty lemma remains true as the next result shows.

Lemma 3.5 (see [26]). *Let $A : K \to E^*$ a K-pseudomonotone and lower hemicontinuous along line segments map. Then $u \in K$ is a solution of* (VIP) *if and only if u is solution of the Minty variational inequality problem* (MVIP)

$$u \in K : \langle Av, v - u\rangle \geq 0, \quad \forall v \in K \tag{MVIP}$$

However, the above theorem can be generalized assuming that A is lower hemicontinuous along line segments of K as the following result shows.

Theorem 3.6. *Let K be a closed convex set and $A : K \to E^*$ a K-pseudomonotone map which is lower hemicontinuous along line segments. Let us assume that condition H2) holds, namely, there exists $u_0 \in K$ and $R > \|u_0\|$ such that*

$$\langle Av, v - u_0 \rangle \geq 0, \quad \forall v \in K \cap \{v \in E : \|v\| = R\}$$

Then (VIP) *admits solutions.*

Proof. Let us consider the set

$$K_R = \{v \in K : \|v\| \leq R\}$$

Being A a K-pseudomonotone map which is lower hemicontinuous along line segments, in virtue of Corollary 5.1 iii) of [5], there exists a solution $u_R \in K_R$ such that

$$\langle Au_r, v - u_R \rangle \geq 0, \quad \forall v \in K_R$$

Then, proceeding as in the last part of Theorem 2.18 we achieve that u_R is a solution of (VIP) on K. □

Some interesting consequences follow from Theorem 3.6. If K is bounded we obtain the following generalization of Theorem 3.1:

Corollary 3.7. *If K is convex, closed and bounded and A is K-pseudomonotone and lower hemicontinuous along line segments, then* (VIP) *admits solutions.*

Moreover, Theorem 3.6 generalizes Theorem 3.4 because A is lower hemicontinuous along line segments on K, whereas in Theorem 3.4 A is requested to be continuous on finite dimensional subspaces of E. On the other hand a K-pseudomonotone and lower hemicontinuous map in general is not continuous on finite dimensional subspaces as the next example shows:

Example 3.8. Fix $a > 1$ and consider the function defined on $[0, 1]$ by

$$f(x) = \begin{cases} -1/x + a, & x \in]1/2a, 1[\\ -a, & x \in [0, 1/2a] \\ a, & x = 1 \end{cases}$$

It is easy to verify that $f(x)$ is monotone, the function $\xi \to \langle f(\xi), u - v \rangle$ is lower semicontinuous for all $u, v \in [0, 1]$ on the line segment $[u, v]$, but $f(x)$ is not continuous.

Before concluding this section it is worth mentioning some results which generalize our setting of monotone variational inequalities along various directions.

A. Domokos and J. Kolumbán (see e.g. [11]) consider an approach for the theory of variational inequalities which includes variational inequalities defined on nonreflexive Banach spaces as well as generalized vector variational inequalities defined on topological vector spaces. An example of existence results on nonreflexive Banach spaces is given by the following theorem (see [11]).

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $F : \Omega \times \mathbb{R} \to \mathbb{R}$ a function such that $F(\cdot, r)$ is measurable for all $r \in \mathbb{R}^n$, $F(\omega, \cdot)$ is continuous for almost every $\omega \in \Omega$, $F(\cdot, r)$ is monotone nodecreasing for all $r \in \mathbb{R}^n$, $F(\cdot, r) \in L^1(\Omega)$ for all $r \in \mathbb{R}^n$. Let us introduce the Nemitski operator defined by F, namely:*

$$T(f)(\omega) = F(\omega, f(\omega)),$$

which maps L^∞ into L^1 and is continuous, bounded and monotone. Let $K \subset L^\infty = (L^1(\Omega))^*$ be a closed, convex, bounded set. Now let us consider the following variational inequality.

Find $f_0 \in K$ such that:

$$\langle f - f_0, T(f_0) \rangle = \int_\Omega F(\omega, f_0(\omega))(f(\omega) - f_0(\omega))d\omega \geq 0, \quad \forall f \in K. \tag{10}$$

Then, Variational Inequality (10) *admits solutions.*

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