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On Two Properties of Enlargements of Maximal Monotone Operators

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Dedicated to Stephen Simons on the occasion of his 70th birthday.

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We give an answer to an open problem regarding the full enlargeability of a maximal monotone operator $S : X \rightrightarrows X^*$ by S^{se}, the smallest enlargement belonging to a certain class of enlargements associated to S. Moreover, we prove the weak* closedness of the set $S_{h_S}(\varepsilon_1, x) + T_{h_T}(\varepsilon_2, x)$ under a weak generalized interior-point regularity condition.

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1. Introduction

Enlargements of monotone operators have been intensively studied, among the works which are dealing with this subject we mention here [8, 9, 10, 11, 12, 13, 16, 25, 30]. A well-known example of an extension of a maximal monotone multifunction is the ε-subdifferential of a proper convex lower semicontinuous function f defined on a Banach space, denoted by $\partial_\varepsilon f$, which has been introduced in [7]. It is known since 1970 that the classical convex subdifferential of f, ∂f, is a maximal monotone operator (see [26]).

In general, given $S : X \rightrightarrows X^*$ an arbitrary monotone operator defined on a Banach space, an enlargement $S^e : \mathbb{R}_+ \times X \rightrightarrows X^*$ of S can be defined as being

$$S^e(\varepsilon, x) := \{x^* \in X^* : \langle y^* - x^*, y - x\rangle \geq -\varepsilon \text{ for all } (y, y^*) \in G(S)\}.$$

Introduced in [10], this enlargement has some remarkable properties similar to those of the ε-subdifferential. Several properties of S^e were studied, like local boundedness,

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demiclosedness of the graph, Lipschitz continuity and the Brøndsted-Rockafellar property (see [11, 12, 30]). Let us notice that the first paper where the Brondsted-Rockafellar type property for the enlargement of a maximal monotone operator has been established is [31, Proposition 6.17] (see also [28, Theorem 29.9]).

The next step in the study of enlargements of monotone operators was to develop this theory from a more abstract point of view, though in a systematic way, by defining a family of enlargements $\mathbb{E}(S)$ associated to S (see [30]). The subfamily of $\mathbb{E}(S)$ containing the enlargements with closed graph is denoted by $\mathbb{E}_c(S)$. The *biggest element* of $\mathbb{E}_c(S)$ is S^e, while the *smallest* one is S^{se} that is defined by $S^{se} = \cap_{E \in \mathbb{E}_c(S)} E$ (see [30]).

In [8] the notion of *full enlargeability* is introduced and studied. However the characterization of the maximal monotone operators S having the property that they are fully enlargeable by S^{se} is left as an open problem. In the first part of this paper we provide an answer to this question by showing that the operators S which are fully enlargeable by S^{se} are precisely those for which for all $x \in D(S)$ the function $\sigma_S(x, \cdot)$ is continuous at any point $x^* \in S(x)$, uniformly in $S(x)$ (see Section 2 for the definition of the function σ_S).

In the second part of the paper we show, by using some techniques from convex analysis, that under a weak regularity condition expressed via the *intrinsic relative algebraic interior* the set $S_{h_S}(\varepsilon_1, x) + T_{h_T}(\varepsilon_2, x)$ is weak* closed. Here h_S is a *representative function* of S and $S_{h_S} : \mathbb{R}_+ \times X \rightrightarrows X^*$, $S_{h_S}(\varepsilon, x) := \{x^* \in X^* : h_S(x, x^*) \le \varepsilon + \langle x^*, x\rangle\}$ is the so-called *enlargement of S with respect to h_S* (cf. [12]). As a particular instance we derive a regularity condition that guarantees the weak* closedness of the set $S^e(\varepsilon_1, x) + T^e(\varepsilon_2, x)$. In case X is *reflexive* or when the maximal monotone operators S and T are *strongly-representable* we obtain an improvement of a recent result published in [16], the proof of which relies on tools from the functional analysis.

2. Preliminary notions and results

In order to make the paper self-contained we begin by introducing some preliminary notions. Consider X a real separated locally convex space and X^* its topological dual space. The notation $\omega(X^*, X)$ stands for the weak* topology induced by X on X^*, while by $\langle x^*, x\rangle$ we denote the value of the linear continuous functional $x^* \in X^*$ at $x \in X$. For a subset C of X we denote by $\mathrm{cl}(C)$, $\mathrm{co}(C)$, $\mathrm{lin}(C)$, $\mathrm{core}(C)$ and ${}^{ic}C$ its *closure*, *convex hull*, *linear hull*, *algebraic interior* and *intrinsic relative algebraic interior*, respectively. Let us note that if C is a convex set, then (cf. [34]):

(i) $x \in \mathrm{core}(C)$ if and only if $\bigcup_{\lambda>0} \lambda(C - x) = X$;
(ii) $x \in {}^{ic}C$ if and only if $\bigcup_{\lambda>0} \lambda(C - x)$ is a closed linear subspace of X.

We also consider the *indicator function* of the set C, denoted by δ_C, which is zero for $x \in C$ and $+\infty$ otherwise.

For a function $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ we denote by $\mathrm{dom}(f) = \{x \in X : f(x) < +\infty\}$ its *domain* and by $\mathrm{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \le r\}$ its *epigraph*. We call f *proper* if $\mathrm{dom}(f) \ne \emptyset$ and $f(x) > -\infty$ for all $x \in X$. By $\mathrm{cl}(f)$ we denote the *lower semicontinuous hull* of f, namely the function whose epigraph is the closure of $\mathrm{epi}(f)$ in $X \times \mathbb{R}$, that is $\mathrm{epi}(\mathrm{cl}(f)) = \mathrm{cl}(\mathrm{epi}(f))$. The function $\mathrm{co}(f)$ is the greatest convex function majorized by f.

Having $f : X \to \overline{\mathbb{R}}$ a proper function, for $x \in \operatorname{dom}(f)$ we define the *ε-subdifferential* of f at x, where $\varepsilon \geq 0$, by

$$\partial_\varepsilon f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle - \varepsilon \text{ for all } y \in X\}.$$

For $x \notin \operatorname{dom}(f)$ we take $\partial_\varepsilon f(x) := \emptyset$. The set $\partial f(x) := \partial_0 f(x)$ is then the classical *subdifferential* of f at x.

The *Fenchel-Moreau conjugate* of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by $f^*(x^*) = \sup_{x\in X}\{\langle x^*, x\rangle - f(x)\}$ for all $x^* \in X^*$. We mention here some important properties of conjugate functions. We have the so-called *Young-Fenchel inequality* $f^*(x^*) + f(x) \geq \langle x^*, x\rangle$ for all $(x, x^*) \in X \times X^*$. If f is proper, then f is convex and lower semicontinuous if and only if $f^{**} = f$ (see [14, 34]). As a consequence we have that in case f is convex and $\operatorname{cl}(f)$ is proper, then $f^{**} = \operatorname{cl}(f)$ (cf. [34, Theorem 2.3.4]).

One can give the following characterizations for the subdifferential and ε-subdifferential of a proper function f by means of conjugate functions (see [14, 34]):

$$x^* \in \partial f(x) \Leftrightarrow f(x) + f^*(x^*) = \langle x^*, x\rangle$$

and, respectively,

$$x^* \in \partial_\varepsilon f(x) \Leftrightarrow f(x) + f^*(x^*) \leq \langle x^*, x\rangle + \varepsilon.$$

Having $f, g : X \to \overline{\mathbb{R}}$ two proper functions we denote by $f\Box g : X \to \overline{\mathbb{R}}$, $f\Box g(x) = \inf_{u\in X}\{f(u) + g(x - u)\}$ their *infimal convolution*. We say that the infimal convolution is *exact at* $x \in X$ if the infimum in the definition is attained, while $f\Box g$ is said to be *exact* if it is exact at every $x \in X$.

For a function $f : A \times B \to \overline{\mathbb{R}}$, where A and B are nonempty sets, we denote by $f^\top$ the *transpose* of f, namely the function $f^\top : B \times A \to \overline{\mathbb{R}}, f^\top(b, a) = f(a, b)$ for all $(b, a) \in B \times A$. We consider also the *projection operator* $\operatorname{pr}_A : A \times B \to A$, $\operatorname{pr}_A(a, b) = a$ for all $(a, b) \in A \times B$.

In the following we recall some notions and results concerning monotone operators. For the rest of the section we assume that X is a nonzero real Banach space. A set-valued operator $S : X \rightrightarrows X^*$ is said to be *monotone* if

$$\langle y^* - x^*, y - x\rangle \geq 0, \text{ whenever } x^* \in S(x) \text{ and } y^* \in S(y).$$

The *graph* of S is denoted by

$$G(S) = \{(x, x^*) : x^* \in S(x)\} \subseteq X \times X^*,$$

while its *domain* is the set $D(S) = \{x \in X : S(x) \neq \emptyset\}$. The monotone operator S is called *maximal monotone* if its graph is not properly contained in the graph of any other monotone operator $S' : X \rightrightarrows X^*$. The classical example of a maximal monotone operator is the subdifferential of a proper, convex and lower semicontinuous function (see [26]). However, there exist maximal monotone operators which are not subdifferentials (see [28]).

To an arbitrary monotone operator $S : X \rightrightarrows X^*$ we associate the *Fitzpatrick function* $\varphi_S : X \times X^* \to \overline{\mathbb{R}}$, defined by

$$\varphi_S(x, x^*) = \sup\{\langle y^*, x\rangle + \langle x^*, y\rangle - \langle y^*, y\rangle : y^* \in S(y)\},$$

which is obviously convex and strong-weak* lower semicontinuous. Introduced by Fitzpatrick in [15] in 1988 and rediscovered after some years in [12, 21], it proved to be very important in the theory of maximal monotone operators, revealing important connections between convex analysis and monotone operators (see [1, 2, 3, 4, 5, 12, 19, 22, 24, 27, 28, 29, 35] and the references therein). Considering the function $c : X \times X^* \to \mathbb{R}$, $c(x, x^*) = \langle x^*, x\rangle$ for all $(x, x^*) \in X \times X^*$, we get the equality $\varphi_S(x, x^*) = (c + \delta_{G(S)})^*(x^*, x)$ for all $(x, x^*) \in X \times X^*$, where we are considering the natural injection $X \subseteq X^{**}$. Let S be a maximal monotone operator. Then $\varphi_S(x, x^*) \geq \langle x^*, x\rangle$ for all $(x, x^*) \in X \times X^*$ and $G(S) = \{(x, x^*) \in X \times X^* : \varphi_S(x, x^*) = \langle x^*, x\rangle\}$ (see [15]). Motivated by these properties of the Fitzpatrick function, the notion of *representative function* of a monotone operator was introduced and studied in the literature.

Definition 2.1. For $S : X \rightrightarrows X^*$ a monotone operator, we call **representative function** of S a convex and strong-weak* lower semicontinuous function $h_S : X \times X^* \to \overline{\mathbb{R}}$ fulfilling

$$h_S \geq c \quad \text{and} \quad G(S) \subseteq \{(x, x^*) \in X \times X^* : h_S(x, x^*) = \langle x^*, x\rangle\}. \tag{1}$$

We observe that if $G(S) \neq \emptyset$ (in particular if S is maximal monotone), then every representative function of S is proper. It follows immediately that the Fitzpatrick function associated to a maximal monotone operator is a representative function of the operator.

Proposition 2.2. *Let $S : X \rightrightarrows X^*$ be a maximal monotone operator and h_S be a representative function of S. Then:*

(i) $\varphi_S(x, x^*) \leq h_S(x, x^*) \leq \varphi_S^*(x^*, x)$ *for all* $(x, x^*) \in X \times X^*$;
(ii) *the canonical restriction of* $h_S^{*\top}$ *to* $X \times X^*$ *is also a representative function of* S;
(iii) $\{(x, x^*) \in X \times X^* : h_S(x, x^*) = \langle x^*, x\rangle\} = \{(x, x^*) \in X \times X^* : h_S^{*\top}(x, x^*) = \langle x^*, x\rangle\} = G(S)$.

Remark 2.3. These properties of representative functions are well-known in the framework of reflexive Banach spaces (see [24]). It is shown in [2] that these characterizations hold also in a general Banach space. For more on the properties of representative functions we refer to [2, 3, 12, 19, 24] and the references therein.

The following particular class of maximal monotone operators has been recently introduced in [17] and also studied in [33].

Definition 2.4. An operator $S : X \rightrightarrows X^*$ is said to be **strongly-representable** whenever there exists a proper convex and strong lower semicontinuous function $h : X \times X^* \to \overline{\mathbb{R}}$ such that

$$h \geq c, h^*(x^*, x^{**}) \geq \langle x^{**}, x^*\rangle \ \ \forall (x^*, x^{**}) \in X^* \times X^{**}$$

and

$$G(S) = \{(x, x^*) \in X \times X^* : h(x, x^*) = \langle x^*, x\rangle\}.$$

In this case h is called a **strong-representative** of S.

If $S : X \rightrightarrows X^*$ is strongly-representable, then S is maximal monotone (see [17, Theorem 4.2], [33, Theorem 8]) and φ_S is a strong-representative of S.

Remark 2.5. Recently was proved that the class of strongly-representable monotone operators coincides with the class of maximal monotone operators of type (NI) (see [18, Theorem 1.2])

Now let us recall the definition of a family of enlargements introduced in [30].

Definition 2.6. ([30]) Let $S : X \rightrightarrows X^*$ be a monotone operator. Define $\mathbb{E}(S)$ as the family of multifunctions $E : \mathbb{R}_+ \times X \rightrightarrows X^*$ satisfying the following properties:

(i) E is an enlargement of S, i.e.:

$$S(x) \subseteq E(\varepsilon, x) \text{ for all } \varepsilon \geq 0 \text{ and } x \in X;$$

(ii) E is non-decreasing, that is for all $x \in X$, $E(\varepsilon_1, x) \subseteq E(\varepsilon_2, x)$ provided that $\varepsilon_1 \leq \varepsilon_2$;

(iii) E satisfies the transportation formula, which means that for every pair $(\varepsilon_1, x^1, v^1)$, $(\varepsilon_2, x^2, v^2) \in G(E)$ and for every $\lambda \in [0, 1]$ it implies that $(\varepsilon, x, v) \in G(E)$, where $\varepsilon := \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2 + \lambda(1-\lambda)\langle v^1 - v^2, x^1 - x^2\rangle$, $x := \lambda x^1 + (1-\lambda)x^2$ and $v := \lambda v^1 + (1-\lambda)v^2$.

A particular choice of E was considered in [10] and it has for $\varepsilon \geq 0$ and $x \in X$ the following definition :

$$S^e(\varepsilon, x) = \{x^* \in X^* : \langle y^* - x^*, y - x\rangle \geq -\varepsilon \text{ for all } (y, y^*) \in G(S)\}.$$

The properties of this enlargement have been intensively studied (see [10, 11, 12, 30]). In case S is maximal monotone the operator S^e belongs to $\mathbb{E}_c(S)$ (the set of enlargements $E \in \mathbb{E}(S)$ such that $G(E)$ is closed with respect to the strong topology on $X \times X^*$) and in fact is the *biggest element* of $\mathbb{E}_c(S)$ (cf. [30]). The enlargement S^e can be characterized via the Fitzpatrick function associated to S: for $\varepsilon \geq 0$ and $x \in X$ we have $S^e(\varepsilon, x) := \{x^* \in X^* : \varphi_S(x, x^*) \leq \varepsilon + \langle x^*, x\rangle\}$. The family $\mathbb{E}_c(S)$ has also a *smallest element*, namely the enlargement S^{se} defined as $S^{se}(\varepsilon, x) = \cap_{E\in\mathbb{E}_c(S)} E(\varepsilon, x)$ for $\varepsilon \geq 0$ and $x \in X$. For an arbitrary representative function h_S one can consider the following enlargement of S (see [12, 13]): $S_{h_S} : \mathbb{R}_+ \times X \rightrightarrows X^*$, $S_{h_S}(\varepsilon, x) := \{x^* \in X^* : h_S(x, x^*) \leq \varepsilon + \langle x^*, x\rangle\}$. It follows immediately from the definitions above that $S_{\varphi_S} = S^e$. It was proved (see [12]) that for a maximal monotone operator S, $S_{h_S} \in \mathbb{E}_c(S)$ and actually there exists a one-to-one correspondence between $\mathbb{E}_c(S)$ and the set

$$\left\{ h : X \times X^* \to \overline{\mathbb{R}} : \begin{array}{l} h \text{ is convex and lower semicontinuous in the strong topology,} \\ h \geq c \text{ and } G(S) \subseteq \{(x, x^*) \in X \times X^* : h(x, x^*) = \langle x^*, x\rangle\} \end{array} \right\},$$

which has been denoted by $\mathcal{H}(S)$ (moreover, this correspondence is an isomorphism with respect to some suitable operations, see [13]). Hence, in case S is a maximal monotone operator, there exists a unique function belonging to $\mathcal{H}(S)$ such that $S^{se} = S_{h_S}$ (see [12, Theorem 3.6]) and in fact $S^{se} = S_{\sigma_S}$, where $\sigma_S : X \times X^* \to \overline{\mathbb{R}}, \sigma_S(x, x^*) = \operatorname{cl}\operatorname{co}(c + \delta_{G(S)})(x, x^*)$ (see [12, relation (35)]). Let us mention that $\varphi_S(x, x^*) = \sigma_S^{*\top}(x, x^*)$ for all $(x, x^*) \in X \times X^*$.

Remark 2.7. If $S = \partial f$, where f is a proper, convex and lower semicontinuous function, then

$$\partial f(x) \subseteq \partial_\varepsilon f(x) \subseteq \partial^\varepsilon f(x) := (\partial f)^e(\varepsilon, x),$$

and the inclusions can be strict (see [10, 20]). Moreover, taking $h : X \times X^* \to \overline{\mathbb{R}}$, $h(x, x^*) = f(x) + f^*(x^*)$ for all $(x, x^*) \in X \times X^*$, which is a representative function of ∂f, we see that $(\partial f)_h(\varepsilon, x) = \partial_\varepsilon f(x)$.

Let us recall in the following the notion of *full enlargeability* introduced in [8].

Definition 2.8 ([8]). Let $S : X \rightrightarrows X^*$ be a maximal monotone operator and consider an element $E \in \mathbb{E}(S)$. We say that:

(i) the enlargement E fully enlarges S at the point $x \in D(S)$ if and only if for all $\varepsilon > 0$ there exists $\delta = \delta(x, \varepsilon) > 0$ such that $S(x) + B(0, \delta) \subseteq E(\varepsilon, x)$ ($B(0, \delta)$ is the closed ball centered at the origin with radius δ);

(ii) E is a full enlargement of S when property (i) holds for all $x \in D(S)$.

The operators which are fully enlargeable by S^e are characterized in [8, Theorem 3.2]. The question posed in [8] concerning the characterization of the operators that are fully enlargeable by S^{se} was left as an open problem. We give below an answer to this question. Actually we provide a characterization of those operators S which are fully enlargeable by S_{h_S}, where h_S is an arbitrary representative function of S. To this end for an operator $S : X \rightrightarrows X^*$ we introduce, as in [8], the function $\beta_S : X \times X^* \to \overline{\mathbb{R}}$, $\beta_S(x, x^*) = h_S(x, x^*) - \langle x, x^* \rangle$ and for $x^* \in X^*$ and $U \subseteq X^*$ consider the *metric distance* from x^* to U, that is $d(x^*, U) = \inf_{u^* \in U} \|u^* - x^*\|$.

Theorem 2.9. *Let $S : X \rightrightarrows X^*$ be a maximal monotone operator and h_S be a representative function of S. Then the following statements are equivalent:*

(i) S_{h_S} is a full enlargement of S;

(ii) for all $x \in D(S)$ $h_S(x, \cdot)$ is continuous at every $x^* \in S(x)$, uniformly in $S(x)$.

Proof. We give first the proof of the implication $(i) \Rightarrow (ii)$, which is similar to the proof of implication $(a) \Rightarrow (b)$ in [8, Theorem 3.2]. Let be $x \in D(S)$. Taking into consideration the definition of the function β_S, the continuity of $h_S(x, \cdot)$ is equivalent to the continuity of $\beta_S(x, \cdot)$. For $x^* \in S(x)$ we fix $\varepsilon > 0$ and consider $\delta > 0$ such that $S(x) + B(0, \delta) \subseteq S_{h_S}(\varepsilon, x)$, which exists by the definition of full enlargeability. Take $y^* \in X^*$ such that $d(y^*, S(x)) < \delta$. Consequently, there exists $u^* \in S(x)$ such that $y^* - u^* \in B(0, \delta)$. Hence $y^* = u^* + (y^* - u^*) \in S(x) + B(0, \delta) \subseteq S_{h_S}(\varepsilon, x)$, that is $\beta_S(x, y^*) \le \varepsilon$. We obtain $|\beta_S(x, y^*) - \beta_S(x, x^*)| = \beta_S(x, y^*) \le \varepsilon$ for all $x^* \in S(x)$. As δ depends only on x and ε, (ii) holds.

Assume now that (ii) holds and fix $x \in D(S)$ and $\varepsilon > 0$. Since the function $\beta_S(x, \cdot)$ is uniformly continuous on $S(x)$, there exists $\delta > 0$ (which depends on x and ε) fulfilling

$$\beta_S(x, y^*) \le \varepsilon, \quad \text{for all } y^* \in X^* \text{ such that } d(y^*, S(x)) < \delta. \tag{2}$$

We claim that for $\overline{\delta} := (1/2)\delta$ we have $S(x) + B(0, \overline{\delta}) \subseteq S_{h_S}(\varepsilon, x)$. Indeed, take $x^* \in S(x)$ and $v^* \in B(0, \overline{\delta})$. Then $d(x^* + v^*, S(x)) = \inf_{u^* \in S(x)} \|x^* + v^* - u^*\| \le \|v^*\| \le \overline{\delta} < \delta$. Combining this inequality with (2) we get $\beta_S(x, x^* + v^*) \le \varepsilon$, which is nothing else than $x^* + v^* \in S_{h_S}(\varepsilon, x)$ and the claim is proved. Hence (i) is fulfilled and the proof is complete. □

Remark 2.10. Taking $h_S := \varphi_S$ in Theorem 2.9 we obtain exactly the equivalence between (a) and (b) in [8, Theorem 3.2]. In this case a further equivalent characterization of full enlargeability of S by S^e can be also given (see [8, Theorem 3.2 (c)]).

Although the function σ_S is not a representative function of the maximal monotone operator S (since it is lower semicontinuous with respect to the strong topology of $X \times X^*$ and not necessarily in the strong-weak* topology), one can prove the following characterization of the operators which are fully enlargeable by S^{se} with the same technique as in the proof of Theorem 2.9.

Corollary 2.11. *Let $S : X \rightrightarrows X^*$ be a maximal monotone operator. Then the following statements are equivalent:*

(i) *S^{se} is a full enlargement of S;*

(ii) *for all $x \in D(S)$ $\sigma_S(x, \cdot)$ is continuous at every $x^* \in S(x)$, uniformly in $S(x)$.*

3. The weak* closedness of the set $S_{h_S}(\varepsilon_1, x) + T_{h_T}(\varepsilon_2, x)$

In this section we provide a weak *generalized interior-point regularity condition* which guarantees that for $S, T : X \rightrightarrows X^*$ maximal monotone operators with representative functions h_S and h_T, respectively, and $\varepsilon_1, \varepsilon_2 \geq 0$ and $x \in X$ the set $S_{h_S}(\varepsilon_1, x)+T_{h_T}(\varepsilon_2, x)$ is weak* closed. Some comments regarding the link to similar results given in the literature are also made.

We prove first a preliminary result which will be useful when proving the main theorem of the section.

Lemma 3.1. *Let X and Y be separated locally convex spaces and $\Phi : X \times Y \to \overline{\mathbb{R}}$ a proper convex and lower semicontinuous function. Then for all $x \in \mathrm{pr}_X(\mathrm{dom}(\Phi))$ this yields*

$$\mathrm{pr}_{Y^*}(\mathrm{dom}(\Phi^*)) \subseteq \mathrm{dom}\left((\Phi(x, \cdot))^*\right) \subseteq \mathrm{cl}_{\omega(Y^*,Y)}\left(\mathrm{pr}_{Y^*}(\mathrm{dom}(\Phi^*))\right).$$

Proof. Let $x \in \mathrm{pr}_X(\mathrm{dom}(\Phi))$ be fixed and define $\Psi : Y \times X \to \overline{\mathbb{R}}$ as being $\Psi(y, u) = \Phi(x + u, y)$. The function Ψ is proper convex and lower semicontinuous and fulfills $\Psi(y, 0) = \Phi(x, y)$ for all $y \in Y$. Since $x \in \mathrm{pr}_X(\mathrm{dom}(\Phi))$ one has that $0 \in \mathrm{pr}_X(\mathrm{dom}(\Psi))$.

According to [6, Theorem 1] (see also [23]) we have that

$$(\Psi(\cdot, 0))^* = \mathrm{cl}_{\omega(Y^*,Y)}\left(\inf\nolimits_{x^* \in X^*} \Psi^*(\cdot, x^*)\right).$$

Consequently,

$$\begin{aligned}\mathrm{dom}\left(\inf\nolimits_{x^* \in X^*} \Psi^*(\cdot, x^*)\right) &\subseteq \mathrm{dom}\left(\mathrm{cl}_{\omega(Y^*,Y)}\left(\inf\nolimits_{x^* \in X^*} \Psi^*(\cdot, x^*)\right)\right)\\ &\subseteq \mathrm{cl}_{\omega(Y^*,Y)}\left(\mathrm{dom}\left(\inf\nolimits_{x^* \in X^*} \Psi^*(\cdot, x^*)\right)\right).\end{aligned}$$

Further we have that $v^* \in \mathrm{dom}\left(\inf_{x^* \in X^*} \Psi^*(\cdot, x^*)\right)$ if and only if there exists $x^* \in X^*$ such that $\Psi^*(v^*, x^*) < +\infty$. Since

$$\begin{aligned}\Psi^*(v^*, x^*) &= \sup_{v \in Y, u \in X} \{\langle v^*, v\rangle + \langle x^*, u\rangle - \Phi(x + u, v)\}\\ &= \sup_{v \in Y, t \in X} \{\langle v^*, v\rangle + \langle x^*, t - x\rangle - \Phi(t, v)\} = -\langle x^*, x\rangle + \Phi^*(x^*, v^*),\end{aligned}$$

this is the same with having that $\Phi^*(x^*, v^*) < +\infty$ or, equivalently, $(x^*, v^*) \in \mathrm{dom}(\Phi^*)$. Therefore $\mathrm{dom}\left(\inf_{x^* \in X^*} \Psi^*(\cdot, x^*)\right) = \mathrm{pr}_{Y^*}(\mathrm{dom}(\Phi^*))$ and the conclusion follows. $\square$

In the following we assume that the nonzero real Banach space X is endowed with the strong topology, while its topological dual X^* is endowed with the weak* topology $\omega(X^*, X)$. Thus for a given function $f : X^* \to \overline{\mathbb{R}}$ its conjugate function $f^* : X \to \overline{\mathbb{R}}$ is $f^*(x) = \sup_{x^* \in X^*}\{\langle x^*, x\rangle - f(x^*)\}$, while for $h : X \times X^* \to \overline{\mathbb{R}}$, $h^* : X^* \times X \to \overline{\mathbb{R}}$ is defined as $h^*(z^*, z) = \sup_{x \in X, x^* \in X^*}\{\langle z^*, x\rangle + \langle x^*, z\rangle - h(x, x^*)\}$. A second preliminary result follows.

Theorem 3.2. *Let $f, g : X^* \to \overline{\mathbb{R}}$ be two proper convex and weak* lower semicontinuous functions such that $0 \in \mathrm{dom}(f^*) \cap \mathrm{dom}(g^*)$ and consider the sets $F := \{x^* \in X^* : f(x^*) \leq 0\}$ and $G := \{x^* \in X^* : g(x^*) \leq 0\}$. If $0 \in {}^{ic}(\mathrm{dom}(f^*) - \mathrm{dom}(g^*))$, then $F + G$ is weak* closed.*

Proof. The sets F and G are both convex and weak* closed. If $F + G$ is empty, then there is nothing to be proved. Therefore we assume that $F+G$ is not empty and consider $\sigma_F, \sigma_G : X \to \overline{\mathbb{R}}$ the *support functions* of F and G defined by $\sigma_F(x) = \sup_{x^* \in F}\langle x^*, x\rangle$ and $\sigma_G(x) = \sup_{x^* \in G}\langle x^*, x\rangle$, respectively. Both functions σ_F and σ_G are proper convex and lower semicontinuous (both in the $\omega(X, X^*)$ and strong topologies) and $0 \in \mathrm{dom}(\sigma_F) \cap \mathrm{dom}(\sigma_G)$. Therefore whenever $0 \in {}^{ic}(\mathrm{dom}(\sigma_F) - \mathrm{dom}(\sigma_G))$ one has (see, for example, [34, Theorem 2.8.7 (*vii*)]) that $\sigma^*_{F+G} = (\sigma_F + \sigma_G)^* = \sigma^*_F \Box \sigma^*_G = \delta_F \Box \delta_G = \delta_{F+G}$ and this guarantees that $F + G$ is weak* closed.

To obtain the conclusion we show that if $0 \in {}^{ic}(\mathrm{dom}(f^*) - \mathrm{dom}(g^*))$, then we have $0 \in {}^{ic}(\mathrm{dom}(\sigma_F) - \mathrm{dom}(\sigma_G))$. To this aim we consider the functions $h, k : X \to \overline{\mathbb{R}}$ as being $h := \inf_{\lambda > 0}(\lambda f)^*$ and $k := \inf_{\mu > 0}(\mu g)^*$. Since the mapping

$$(x, \lambda) \mapsto \begin{cases} (\lambda f)^*(x), & \text{if } x \in X, \lambda > 0, \\ +\infty, & \text{otherwise} \end{cases}$$

is convex, one has that h is convex and, consequently, that $\mathrm{cl}(h)$ is convex, too. Moreover, as one can easily see, $\mathrm{cl}(h)$ is a proper function. The same applies for $\mathrm{cl}(k)$ and, by [34, Theorem 2.3.4] using also that $\mathrm{cl}(h)$ and $\mathrm{cl}(k)$ are proper convex and lower semicontinuous, we get that $\mathrm{cl}(h) = h^{**} = \sigma_F$ and $\mathrm{cl}(k) = k^{**} = \sigma_G$. Thus $\mathrm{dom}(h) \subseteq \mathrm{dom}(\sigma_F) \subseteq \mathrm{cl}(\mathrm{dom}(h))$ and $\mathrm{dom}(k) \subseteq \mathrm{dom}(\sigma_G) \subseteq \mathrm{cl}(\mathrm{dom}(k))$.

On the other hand, $x \in \mathrm{dom}(h)$ if and only if there exists $\lambda > 0$ such that $(\lambda f)^*(x) = \lambda f^*((1/\lambda)x) < +\infty$. This is further equivalent to the fact that there exists $\lambda > 0$ such that $x \in \lambda\,\mathrm{dom}(f^*)$ or, in other words, to $x \in \cup_{\lambda>0}\lambda\,\mathrm{dom}(f^*)$. Consequently, $\mathrm{dom}(h) = \cup_{\lambda>0}\lambda\,\mathrm{dom}(f^*)$ and, analogously, $\mathrm{dom}(k) = \cup_{\lambda>0}\lambda\,\mathrm{dom}(g^*)$. Thus one automatically has one hand that

$$\bigcup_{\lambda>0} \lambda(\mathrm{dom}(f^*) - \mathrm{dom}(g^*)) \subseteq \bigcup_{\lambda>0} \lambda\,\mathrm{dom}(f^*) - \bigcup_{\mu>0} \mu\,\mathrm{dom}(g^*) \subseteq \mathrm{dom}(\sigma_F) - \mathrm{dom}(\sigma_G)$$

and, on the other hand, that

$$\begin{aligned} &\mathrm{dom}(\sigma_F) - \mathrm{dom}(\sigma_G) \\ \subseteq\ &\mathrm{cl}\left(\bigcup_{\lambda>0} \lambda\,\mathrm{dom}(f^*)\right) - \mathrm{cl}\left(\bigcup_{\mu>0} \mu\,\mathrm{dom}(g^*)\right) \subseteq \mathrm{cl}\left(\bigcup_{\lambda>0} \lambda\,\mathrm{dom}(f^*) - \bigcup_{\mu>0} \mu\,\mathrm{dom}(g^*)\right). \end{aligned}$$

For $\lambda, \mu > 0$, $x \in \operatorname{dom}(f^*)$ and $y \in \operatorname{dom}(g^*)$, using that $0 \in \operatorname{dom}(f^*) \cap \operatorname{dom}(g^*)$ and the convexity of $\operatorname{dom}(f^*)$ and $\operatorname{dom}(g^*)$, it yields

$$\begin{aligned}\lambda x - \mu y &= (\lambda+\mu)\left(\lambda/(\lambda+\mu)(x-0) + \mu/(\lambda+\mu)(0-y)\right)\\ &\in (\lambda+\mu)(\operatorname{dom}(f^*) - \operatorname{dom}(g^*))\\ &\subseteq \cup_{\lambda>0}\lambda(\operatorname{dom}(f^*) - \operatorname{dom}(g^*)).\end{aligned}$$

Thus

$$\underset{\lambda>0}{\cup}\lambda(\operatorname{dom}(f^*) - \operatorname{dom}(g^*)) \subseteq \operatorname{dom}(\sigma_F) - \operatorname{dom}(\sigma_G) \subseteq \operatorname{cl}\left(\underset{\lambda>0}{\cup}\lambda(\operatorname{dom}(f^*) - \operatorname{dom}(g^*))\right)$$

and from here

$$\begin{aligned}\underset{\lambda>0}{\cup}\lambda(\operatorname{dom}(f^*) - \operatorname{dom}(g^*)) &\subseteq \underset{\lambda>0}{\cup}\lambda(\operatorname{dom}(\sigma_F) - \operatorname{dom}(\sigma_G))\\ &\subseteq \operatorname{cl}\left(\underset{\lambda>0}{\cup}\lambda(\operatorname{dom}(f^*) - \operatorname{dom}(g^*))\right).\end{aligned}$$

By the hypothesis we have that $\cup_{\lambda>0}\lambda(\operatorname{dom}(f^*) - \operatorname{dom}(g^*))$ is a closed linear subspace and this means that the inclusions in the relation above are fulfilled as equalities. This has as consequence the fact that $\cup_{\lambda>0}\lambda(\operatorname{dom}(\sigma_F) - \operatorname{dom}(\sigma_G))$ is a closed linear subspace, too, or, equivalently, $0 \in {}^{ic}(\operatorname{dom}(\sigma_F) - \operatorname{dom}(\sigma_G))$, which concludes the proof. □

We come now to the proof of the main result of this section.

Theorem 3.3. *Let $S, T : X \rightrightarrows X^*$ be two maximal monotone operators with representative functions h_S and h_T, respectively. If*

$$0 \in {}^{ic}\left(\operatorname{pr}_X(\operatorname{dom}(h_S^*)) - \operatorname{pr}_X(\operatorname{dom}(h_T^*))\right),$$

then for all $\varepsilon_1, \varepsilon_2 \geq 0$ and $x \in X$ the set $S_{h_S}(\varepsilon_1, x) + T_{h_T}(\varepsilon_2, x)$ is weak closed.*

Proof. Let $\varepsilon_1, \varepsilon_2 \geq 0$ and $x \in X$ be fixed. Assume that $S_{h_S}(\varepsilon_1, x) + T_{h_T}(\varepsilon_2, x)$ is nonempty. Thus $x \in \operatorname{pr}_X(\operatorname{dom}(h_S)) \cap \operatorname{pr}_X(\operatorname{dom}(h_T))$. Consider the functions $f, g : X^* \to \overline{\mathbb{R}}$ defined by $f(x^*) = h_S(x, x^*) - \langle x^*, x\rangle - \varepsilon_1$ and $g(x^*) = h_T(x, x^*) - \langle x^*, x\rangle - \varepsilon_2$, respectively. The functions f and g are proper convex and weak* lower semicontinuous. Since $\inf_{x^*\in X^*} f(x^*) \geq -\varepsilon_1 > -\infty$ and $\inf_{x^*\in X^*} g(x^*) \geq -\varepsilon_2 > -\infty$, it yields $0 \in \operatorname{dom}(f^*) \cap \operatorname{dom}(g^*)$. Moreover, $S_{h_S}(\varepsilon_1, x) = \{x^* \in X^* : f(x^*) \leq 0\}$ and $T_{h_T}(\varepsilon_2, x) = \{x^* \in X^* : g(x^*) \leq 0\}$.

By the hypothesis, the set $\cup_{\lambda>0}\lambda\left(\operatorname{pr}_X(\operatorname{dom}(h_S^*)) - \operatorname{pr}_X(\operatorname{dom}(h_T^*))\right)$ is a closed linear subspace. Taking into consideration Lemma 3.1 this yields

$$\operatorname{pr}_X(\operatorname{dom}(h_S^*)) \subseteq \operatorname{dom}\left((h_S(x,\cdot))^*\right) \subseteq \operatorname{cl}\left(\operatorname{pr}_X(\operatorname{dom}(h_S^*))\right)$$

and, similarly,

$$\operatorname{pr}_X(\operatorname{dom}(h_T^*)) \subseteq \operatorname{dom}\left((h_T(x,\cdot))^*\right) \subseteq \operatorname{cl}\left(\operatorname{pr}_X(\operatorname{dom}(h_T^*))\right).$$

Consequently,

$$\operatorname{pr}_X(\operatorname{dom}(h_S^*)) - \operatorname{pr}_X(\operatorname{dom}(h_T^*)) \subseteq \operatorname{dom}\left((h_S(x,\cdot))^*\right) - \operatorname{dom}\left((h_T(x,\cdot))^*\right)$$

$$\subseteq \mathrm{cl}\,(\mathrm{pr}_X(\mathrm{dom}(h_S^*))) - \mathrm{cl}\,(\mathrm{pr}_X(\mathrm{dom}(h_T^*))) \subseteq \mathrm{cl}\,(\mathrm{pr}_X(\mathrm{dom}(h_S^*)) - \mathrm{pr}_X(\mathrm{dom}(h_T^*)))$$

and from here one has

$$\begin{aligned} &\bigcup_{\lambda>0} \lambda\,(\mathrm{pr}_X(\mathrm{dom}(h_S^*)) - \mathrm{pr}_X(\mathrm{dom}(h_T^*))) \\ \subseteq\ &\bigcup_{\lambda>0} \lambda\,(\mathrm{dom}\,((h_S(x,\cdot))^*) - \mathrm{dom}\,((h_T(x,\cdot))^*)) \\ \subseteq\ &\mathrm{cl}\left(\bigcup_{\lambda>0} \lambda\,(\mathrm{pr}_X(\mathrm{dom}(h_S^*)) - \mathrm{pr}_X(\mathrm{dom}(h_T^*)))\right). \end{aligned}$$

This means that $\cup_{\lambda>0} \lambda\,(\mathrm{dom}\,((h_S(x,\cdot))^*) - \mathrm{dom}\,((h_T(x,\cdot))^*))$ is a closed linear subspace, too, or, equivalently, $0 \in {}^{ic}\,(\mathrm{dom}\,((h_S(x,\cdot))^*) - \mathrm{dom}\,((h_T(x,\cdot))^*))$.

For all $u \in X$ it holds $f^*(u) = \sup_{x^* \in X^*}\{\langle x^*, u+x\rangle - h_S(x,x^*)\} + \varepsilon_1 = (h_S(x,\cdot))^*(u+x) + \varepsilon_1$ and therefore $\mathrm{dom}(f^*) = \mathrm{dom}\,((h_S(x,\cdot))^*) - x$. Analogously, we obtain $\mathrm{dom}(g^*) = \mathrm{dom}\,((h_T(x,\cdot))^*) - x$, which implies that $0 \in {}^{ic}(\mathrm{dom}(f^*) - \mathrm{dom}(g^*))$. Finally, Theorem 3.2 guarantees that the set $S_{h_S}(\varepsilon_1, x) + T_{h_T}(\varepsilon_2, x)$ is weak* closed. □

The following theorem follows as a direct consequence of the result above by considering as representative functions of S and T their *Fitzpatrick functions*.

Theorem 3.4. *Let $S, T : X \rightrightarrows X^*$ be two maximal monotone operators. If*

$$0 \in {}^{ic}\,(\mathrm{pr}_X(\mathrm{dom}(\varphi_S^*)) - \mathrm{pr}_X(\mathrm{dom}(\varphi_T^*))),$$

then for all $\varepsilon_1, \varepsilon_2 \geq 0$ and $x \in X$ the set $S^e(\varepsilon_1, x) + T^e(\varepsilon_2, x)$ is weak* closed.

In the following we formulate two further regularity conditions which are sufficient for obtaining the same conclusion as in the theorem above, but they are expressed via the domains of the two operators involved. To this end we recall a result given in [29, Lemma 5.3], the proof of which uses techniques taken from [27, p. 57–62 and p. 87–88], in case X is a reflexive Banach space and the representative functions are exactly the Fitzpatrick functions. It can be proved in an analogous way that the result remains valid in a general Banach space and when considering arbitrary representative functions.

Lemma 3.5. *Let $S, T : X \rightrightarrows X^*$ be two maximal monotone operators with representative functions h_S and h_T, respectively. The following statements are true:*

(*a*) *If F is a closed subspace of X, $w \in X$ and $D(S) \subseteq F + w$ then $\mathrm{pr}_X(\mathrm{dom}(h_S)) \subseteq F + w$;*

(*b*) $\cup_{\lambda>0}\lambda\,(\mathrm{pr}_X(\mathrm{dom}(h_S)) - \mathrm{pr}_X(\mathrm{dom}(h_T))) \subseteq \mathrm{cl}\,(\mathrm{lin}\,(D(S) - D(T)))$.

Remark 3.6. It follows easily from Proposition 2.2 and Lemma 3.5 that for $S, T : X \rightrightarrows X^*$ maximal monotone operators the following inclusions hold

$$\begin{aligned} &\bigcup_{\lambda>0} \lambda\,(D(S) - D(T)) \subseteq \bigcup_{\lambda>0} \lambda\,(\mathrm{co}(D(S)) - \mathrm{co}(D(T))) \\ \subseteq\ &\bigcup_{\lambda>0} \lambda\,(\mathrm{pr}_X(\mathrm{dom}(\varphi_S^*)) - \mathrm{pr}_X(\mathrm{dom}(\varphi_T^*))) \subseteq \bigcup_{\lambda>0} \lambda\,(\mathrm{pr}_X(\mathrm{dom}(\varphi_S)) - \mathrm{pr}_X(\mathrm{dom}(\varphi_T))) \\ \subseteq\ &\mathrm{cl}\,(\mathrm{lin}(D(S) - D(T))) \subseteq \mathrm{cl}\,(\mathrm{lin}(\mathrm{co}(D(S)) - \mathrm{co}(D(T)))) \\ \subseteq\ &\mathrm{cl}\,(\mathrm{lin}(\mathrm{pr}_X(\mathrm{dom}(\varphi_S^*)) - \mathrm{pr}_X(\mathrm{dom}(\varphi_T^*)))) \\ \subseteq\ &\mathrm{cl}\,(\mathrm{lin}(\mathrm{pr}_X(\mathrm{dom}(\varphi_S)) - \mathrm{pr}_X(\mathrm{dom}(\varphi_T)))) \subseteq \mathrm{cl}\,(\mathrm{lin}(D(S) - D(T)))\,, \end{aligned}$$

thus

$$\begin{aligned}\operatorname{cl}(\operatorname{lin}(D(S)-D(T))) &= \operatorname{cl}(\operatorname{lin}(\operatorname{co}(D(S))-\operatorname{co}(D(T))))\\ &= \operatorname{cl}(\operatorname{lin}(\operatorname{pr}_X(\operatorname{dom}(\varphi_S^*))-\operatorname{pr}_X(\operatorname{dom}(\varphi_T^*))))\\ &= \operatorname{cl}(\operatorname{lin}(\operatorname{pr}_X(\operatorname{dom}(\varphi_S))-\operatorname{pr}_X(\operatorname{dom}(\varphi_T)))).\end{aligned}$$

The remark above allows us to formulate the following result.

Theorem 3.7. *Let $S, T : X \rightrightarrows X^*$ be two maximal monotone operators. If*

$$0 \in {}^{ic}(\operatorname{co}(D(S)) - \operatorname{co}(D(T))),$$

then for all $\varepsilon_1, \varepsilon_2 \geq 0$ and $x \in X$ the set $S^e(\varepsilon_1, x) + T^e(\varepsilon_2, x)$ is weak closed.*

For a particular instance of Theorem 3.7, when $0 \in \operatorname{core}(\operatorname{co}(D(S)) - \operatorname{co}(D(T)))$ and $\varepsilon_1 = \varepsilon_2 = 0$, we refer the reader to [32, Corollary 2.3].

Remark 3.8. In case X is a reflexive Banach space the weak generalized interior-point regularity conditions stated in Theorem 3.4 and Theorem 3.7 for the weak* closedness of $S^e(\varepsilon_1, x) + T^e(\varepsilon_2, x)$, when $\varepsilon_1, \varepsilon_2 \geq 0$ and $x \in X$, are equivalent. More than that, they are further equivalent to (see [35])

$$0 \in {}^{ic}(D(S) - D(T))$$

and to

$$0 \in {}^{ic}(\operatorname{pr}_X(\operatorname{dom}(\varphi_S)) - \operatorname{pr}_X(\operatorname{dom}(\varphi_T))).$$

In case X is a general Banach space and the operators S and T are *strongly-representable*, then whenever $0 \in {}^{ic}(\operatorname{pr}_X(\operatorname{dom}(\varphi_S)) - \operatorname{pr}_X(\operatorname{dom}(\varphi_T)))$ or, equivalently (see [33, Theorem 16]), $0 \in {}^{ic}(\operatorname{co}(D(S)) - \operatorname{co}(D(T)))$ or $0 \in {}^{ic}(D(S) - D(T))$, then we also have that for $\varepsilon_1, \varepsilon_2 \geq 0$ and $x \in X$ the set $S^e(\varepsilon_1, x) + T^e(\varepsilon_2, x)$ is weak* closed.

Remark 3.9. In [16, Theorem 3.7] the authors prove by using tools from the functional analysis that in case X is a Banach space and $S, T : X \rightrightarrows X^*$ are two maximal monotone operators such that $0 \in \operatorname{core}(\operatorname{pr}_X(\operatorname{dom}(\varphi_S)) - \operatorname{pr}_X(\operatorname{dom}(\varphi_T)))$ one has for all $\varepsilon \geq 0$ and $x \in X$ that $S^e(\varepsilon, x) + T^e(\varepsilon, x)$ is weak* closed (in fact, the result works even for $\varepsilon_1 \neq \varepsilon_2$). When X is reflexive or when S and T are strongly-representable, the regularity conditions given in Remark 3.8 turn out to be weaker than the one in [16]. Nevertheless, it is still an open question whether the condition $0 \in {}^{ic}(\operatorname{pr}_X(\operatorname{dom}(\varphi_S)) - \operatorname{pr}_X(\operatorname{dom}(\varphi_T)))$ is in general sufficient for the weak* closedness of $S^e(\varepsilon_1, x) + T^e(\varepsilon_2, x)$.

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