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Monotone Linear Relations: Maximality and Fitzpatrick Functions

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We analyze and characterize maximal monotonicity of linear relations (set-valued operators with linear graphs). An important tool in our study are Fitzpatrick functions. The results obtained partially extend work on linear and at most single-valued operators by Phelps and Simons and by Bauschke, Borwein and Wang. Furthermore, a description of skew linear relations in terms of the Fitzpatrick family is obtained. We also answer one of Simons' problems by showing that if a maximal monotone operator has a convex graph, then this graph must actually be affine.

Keywords: Adjoint process, Fenchel conjugate, Fitzpatrick family, Fitzpatrick function, linear relation, maximal monotone operator, monotone operator, skew linear relation

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1. Introduction

Linear relations (also known as linear processes) have been considered by many authors for a long time; see [2, 11, 6, 1] and the many references therein. Surprisingly, the class of monotone (in the sense of set-valued analysis) linear relations has been explored much less even though it provides a considerably broader framework for studying monotone linear operators and its members arise frequently in optimization, functional analysis and functional equations.

This paper focuses on monotone linear relations, i.e., on monotone operators with linear graphs. We discuss the relationships of domain, range, and kernel between original and adjoint linear relation as well as Fitzpatrick functions and criteria for maximal mono-

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tonicity. Throughout, X will denote a reflexive real Banach space (with the exception of Section 4), with continuous dual space X^*, and with pairing $\langle\cdot,\cdot\rangle$. The notation used is standard and as in Convex Analysis and Monotone Operator Theory; see, e.g., [16, 18, 19, 20, 25]. Let A be a set-valued operator (also known as multifunction) from X to X^*. Then the *graph* of A is $\operatorname{gra} A := \big\{(x,x^*) \in X \times X^* \mid x^* \in Ax\big\}$, and A is *monotone* if

$$\big(\forall (x,x^*) \in \operatorname{gra} A\big)\big(\forall (y,y^*) \in \operatorname{gra} A\big) \quad \langle x-y, x^*-y^*\rangle \geq 0.$$

A is said to be *maximal monotone* if no proper enlargement (in the sense of graph inclusion) of A is monotone. The *inverse operator* $A^{-1}\colon X^* \rightrightarrows X$ is given by $\operatorname{gra} A^{-1} := \big\{(x^*,x) \in X^* \times X \mid x^* \in Ax\big\}$; the *domain* of A is $\operatorname{dom} A := \big\{x \in X \mid Ax \neq \varnothing\big\}$; its *kernel* is $\ker A := \big\{x \in X \mid 0 \in Ax\big\}$, and its *range* is $\operatorname{ran} A := A(X)$. We say $(x,x^*) \in X \times X^*$ is *monotonically related* to $\operatorname{gra} A$ if $(\forall (y,y^*) \in \operatorname{gra} A)$ $\langle x-y, x^*-y^*\rangle \geq 0$. The *adjoint* of A, written A^*, is defined by

$$\operatorname{gra} A^* := \big\{(x,x^*) \in X \times X^* \mid (x^*,-x) \in (\operatorname{gra} A)^{\perp}\big\},$$

where, for any subset S of a reflexive Banach space Z with continuous dual space Z^*, $S^{\perp} := \big\{z^* \in Z^* \mid z^*|_S \equiv 0\big\}$.

We say A is a maximal monotone linear relation if A is a maximal monotone operator and $\operatorname{gra} A$ is a linear subspace of $X \times X^*$. Finally, if $f\colon X \to \left]-\infty,+\infty\right]$ is proper and convex, we write $f^*\colon x^* \mapsto \sup_{x\in X}\left(\langle x,x^*\rangle - f(x)\right)$, and $\operatorname{dom} f := \big\{x \in X \mid f(x) < +\infty\big\}$, for the *Fenchel conjugate*, and the *domain* of f, respectively.

The outline of the paper is as follows. In Section 2 we provide preliminary results about monotone linear relations. In Section 3, the relationships between domains, ranges, and kernels of A and A^* are discussed. In Section 4, we present a result that states that maximal monotone operators with convex graphs must be affine. Section 5 provides useful relationships among the Fitzpatrick functions F_{A+B}, F_A and F_B. In Section 6, the maximality criteria for monotone linear relations are established; these generalize corresponding results by Phelps and Simons [17] on linear (at most single-valued) monotone operators. The final Section 7 contains a characterization of skew linear operators in terms of the single-valuedness of the Fitzpatrick family associated to the monotone operator. The results in Sections 3, 4 and 7 extend their single-valued counterparts in [4] to monotone linear relations.

2. Auxiliary Results for Monotone Linear Relations

Fact 2.1. *Let $A\colon X \rightrightarrows X^*$ be a linear relation. Then the following hold.*

(*i*) $A0$ *is a linear subspace of* X^*.

(*ii*) $(\forall (x,x^*) \in \operatorname{gra} A)$ $Ax = x^* + A0$.

(*iii*) $(\forall x \in \operatorname{dom} A)(\forall y \in \operatorname{dom} A)(\forall (\alpha,\beta) \in \mathbb{R}^2 \smallsetminus \{(0,0)\})$ $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$.

(*iv*) $(\forall x \in \operatorname{dom} A^*)(\forall y \in \operatorname{dom} A)$ $\langle A^*x, y\rangle = \langle x, Ay\rangle$ *is a singleton.*

(*v*) $\overline{\operatorname{dom} A} = (A^*0)^{\perp}$. *If* $\operatorname{gra} A$ *is closed, then* $(\ker A)^{\perp} = \overline{\operatorname{ran} A^*}$, $\overline{\operatorname{dom} A^*} = (A0)^{\perp}$, *and* $A^{**} = A$.

Proof. (*i*): See [11, Corollary I.2.4]. (*ii*): See [11, Proposition I.2.8(a)]. (*iii*): See [11, Corollary I.2.5]. (*iv*): See [11, Proposition III.1.2]. (*v*): See [11, Proposition III.1.4(b)–(d), Theorem III.4.7, and Exercise VIII.1.12]. □

Proposition 2.2. *Let $A\colon X \rightrightarrows X^*$ be a linear relation. Then the following hold.*

(*i*) *Suppose A is monotone. Then $\operatorname{dom} A \subset (A0)^\perp$ and $A0 \subset (\operatorname{dom} A)^\perp$; consequently, if $\operatorname{gra} A$ is closed, then $\operatorname{dom} A \subset \overline{\operatorname{dom} A^*}$ and $A0 \subset A^*0$.*

(*ii*) $(\forall x \in \operatorname{dom} A)(\forall z \in (A0)^\perp)$ *$\langle z, Ax\rangle$ is single-valued.*

(*iii*) $(\forall z \in (A0)^\perp)$ $\operatorname{dom} A \to \mathbb{R}\colon y \mapsto \langle z, Ay\rangle$ *is linear.*

(*iv*) *A is monotone $\Leftrightarrow$ $(\forall x \in \operatorname{dom} A)$ $\langle x, Ax\rangle$ is single-valued and $\langle x, Ax\rangle \geq 0$.*

(*v*) *If $(x, x^*) \in (\operatorname{dom} A) \times X^*$ is monotonically related to $\operatorname{gra} A$ and $x_0^* \in Ax$, then $x^* - x_0^* \in (\operatorname{dom} A)^\perp$.*

Proof. (*i*): Pick $x \in \operatorname{dom} A$. Then there exists $x^* \in X^*$ such that $(x, x^*) \in \operatorname{gra} A$. By monotonicity of A and since $(0,0) \in \operatorname{gra} A$, we have $\langle x, x^*\rangle \geq \langle x, A0\rangle$. Since $A0$ is a linear subspace (Fact 2.1 (*i*)), we obtain $x \perp A0$. This implies $\operatorname{dom} A \subset (A0)^\perp$ and $A0 \subset (\operatorname{dom} A)^\perp$. If $\operatorname{gra} A$ is closed, then Fact 2.1 (*v*) yields $\operatorname{dom} A \subset \overline{\operatorname{dom} A^*}$ and $A0 \subset A^*0$.

(*ii*): Take $x \in \operatorname{dom} A$, $x^* \in Ax$, and $z \in (A0)^\perp$. By Fact 2.1 (*ii*), $\langle z, Ax\rangle = \langle z, x^* + A0\rangle = \langle z, x^*\rangle$.

(*iii*): Take $z \in (A0)^\perp$. By (*ii*), $(\forall y \in \operatorname{dom} A)$ $\langle z, Ay\rangle$ is single-valued. Now let x, y be in $\operatorname{dom} A$, and let α, β be in $\mathbb{R}$. If $(\alpha, \beta) = (0,0)$, then $\langle z, A(\alpha x + \beta y)\rangle = \langle z, A0\rangle = 0 = \alpha\langle z, Ax\rangle + \beta\langle z, Ay\rangle$. And if $(\alpha, \beta) \neq (0,0)$, then Fact 2.1 (*iii*) yields $\langle z, A(\alpha x + \beta y) = \langle z, \alpha Ax + \beta Ay\rangle = \alpha\langle z, Ax\rangle + \beta\langle z, Ay\rangle$. This verifies the linearity.

(*iv*): "$\Rightarrow$": This follows from (*i*), (*ii*), and the fact that $(0,0) \in \operatorname{gra} A$. "$\Leftarrow$": If x and y belong to $\operatorname{dom} A$, then Fact 2.1 (*iii*) yields $\langle x - y, Ax - Ay\rangle = \langle x - y, A(x - y)\rangle \geq 0$.

(*v*): Let $(x, x^*) \in (\operatorname{dom} A) \times X^*$ be monotonically related to $\operatorname{gra} A$, and take $x_0^* \in Ax$. For every $(v, v^*) \in \operatorname{gra} A$, we have that $x_0^* + v^* \in A(x + v)$ (by Fact 2.1 (*iii*)); hence, $\langle x - (x + v), x^* - (x_0^* + v^*)\rangle \geq 0$ and thus $\langle v, v^*\rangle \geq \langle v, x^* - x_0^*\rangle$. Now take $\lambda > 0$ and replace (v, v^*) in the last inequality by $(\lambda v, \lambda v^*)$. Then divide by λ and let $\lambda \to 0^+$ to see that $0 \geq \langle \operatorname{dom} A,\ x^* - x_0^*\rangle$. Since $\operatorname{dom} A$ is linear, it follows that $x^* - x_0^* \in (\operatorname{dom} A)^\perp$. $\square$

For $A\colon X \rightrightarrows X^*$ it will be convenient to define (as in, e.g., [4])

$$(\forall x \in X) \quad q_A(x) := \begin{cases} \frac{1}{2}\langle x, Ax\rangle, & \text{if } x \in \operatorname{dom} A; \\ \infty, & \text{otherwise.} \end{cases}$$

Proposition 2.3. *Let $A\colon X \rightrightarrows X^*$ be a linear relation, let x and y be in $\operatorname{dom} A$, and let $\lambda \in \mathbb{R}$. Then*

$$\begin{aligned} &\lambda q_A(x) + (1-\lambda) q_A(y) - q_A(\lambda x + (1-\lambda) y) \\ &= \lambda(1-\lambda) q_A(x - y) = \tfrac{1}{2}\lambda(1-\lambda)\langle x - y, Ax - Ay\rangle. \end{aligned} \tag{1}$$

Moreover, A is monotone $\Leftrightarrow$ q_A is single-valued and convex.

Proof. Proposition 2.2 (*i*) & (*ii*) shows that q_A is single-valued on $\operatorname{dom} A$. Combining with Proposition 2.2 (*ii*), we obtain (1). The characterization now follows from Proposition 2.2 (*iv*). $\square$

Proposition 2.4. *Let $A\colon X \rightrightarrows X^*$ be a maximal monotone linear relation. Then $(\operatorname{dom} A)^\perp = A0$ and hence $\overline{\operatorname{dom} A} = (A0)^\perp$.*

Proof. Since $A+N_{\operatorname{dom} A} = A+(\operatorname{dom} A)^\perp$, where $N_{\operatorname{dom} A}$ denotes the normal cone operator for $\operatorname{dom} A$, is a monotone extension of A and A is maximal monotone, we must have $A + (\operatorname{dom} A)^\perp = A$. Then $A0 + (\operatorname{dom} A)^\perp = A0$. As $0 \in A0$, $(\operatorname{dom} A)^\perp \subset A0$. The reverse inclusion follows from Proposition 2.2 (*i*). □

3. Domain, Range, Kernel, and Adjoint

In this section, we study relationships among domains, ranges and kernels of a maximal monotone linear relation and its adjoint.

Fact 3.1 (Brézis-Browder, [7, Theorem 2]). *Let $A\colon X \rightrightarrows X^*$ be a monotone linear relation such that* $\operatorname{gra} A$ *is closed. Then the following are equivalent.*

(*i*) *A is maximal monotone.*
(*ii*) *A^* is maximal monotone.*
(*iii*) *A^* is monotone.*

The next result generalizes [4, Proposition 3.1] from linear operators to linear relations.

Theorem 3.2. *Let $A\colon X \rightrightarrows X^*$ be a maximal monotone linear relation. Then the following hold.*

(*i*) $\ker A = \ker A^*$.
(*ii*) $\overline{\operatorname{ran} A} = \overline{\operatorname{ran} A^*}$.
(*iii*) $(\operatorname{dom} A^*)^\perp = A^*0 = A0 = (\operatorname{dom} A)^\perp$.
(*iv*) $\overline{\operatorname{dom} A^*} = \overline{\operatorname{dom} A}$.

Proof. By Fact 3.1, A^* is maximal monotone.

(*i*): Let $x \in \ker A$, $y \in \operatorname{dom} A$, and $\alpha \in \mathbb{R}$. Then

$$0 \le \langle \alpha x + y, A(\alpha x + y)\rangle = \alpha^2\langle x, Ax\rangle + \alpha\langle x, Ay\rangle + \alpha\langle y, Ax\rangle + \langle y, Ay\rangle. \tag{2}$$

Since $0 \in Ax$, Fact 2.1 (*ii*) yields $Ax = A0$. By Proposition 2.2 (*i*), $\langle x, Ax\rangle = 0$ and $\alpha\langle y, Ax\rangle = 0$. Hence, in view of (2), $0 \le \alpha\langle x, Ay\rangle + \langle y, Ay\rangle$. It follows that $\langle x, Ay\rangle = 0$. Hence $(0, -x) \in (\operatorname{gra} A)^\perp$, i.e., $0 \in A^*x$. Therefore, $\ker A \subset \ker A^*$. On the other hand, applying this line of thought to A^*, we obtain $\ker A^* \subset \ker A^{**} = \ker A$. Altogether, $\ker A = \ker A^*$.

(*ii*): Combine (*i*) and Fact 2.1 (*v*).

(*iii*): As A^* is maximal monotone, it follows from Proposition 2.4 that $(\operatorname{dom} A^*)^\perp = A^*0$. In view of Fact 2.1 (*v*) and the maximal monotonicity of A, we have $(\operatorname{dom} A^*)^\perp = A^*0 = A0 = (\operatorname{dom} A)^\perp$, thus $(\operatorname{dom} A^*)^\perp = (\operatorname{dom} A)^\perp$.

(*iv*): Apply $\perp$ to (*iii*). □

Corollary 3.3. *Let $A\colon X \rightrightarrows X^*$ be a maximal monotone linear relation such that $\overline{\operatorname{dom} A} = X$. Then both A and A^* are single-valued and linear on their domains.*

Corollary 3.4. *Let $A\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a maximal monotone linear relation. Then* $\ker A = \ker A^*$*,* $\operatorname{ran} A = \operatorname{ran} A^*$*, and* $\operatorname{dom} A = \operatorname{dom} A^* = (A0)^\perp = (A^*0)^\perp$.

Remark 3.5. Consider Theorem 3.2 (*ii*). The Volterra operator illustrates that $\operatorname{ran} A$ and $\operatorname{ran} A^*$ are not comparable in general (see [4, Example 3.3]). Considering the inverse of the Volterra operator, we obtain an analogous negative statement for the domain.

4. Maximal Monotone Operators with Convex Graphs

In this section, X is not assumed to be reflexive. In [10, Corollary 4.1], Butnariu and Kassay discuss monotone operators with closed convex graphs; see also [20, Section 46]. We shall show that if the graph of a maximal monotone operator is convex, then the graph must in fact be affine (i.e., a translate of a linear subspace).

Proposition 4.1. *Let $A\colon X \rightrightarrows X^*$ be maximal monotone such that $\operatorname{gra} A$ is a convex cone. Then $\operatorname{gra} A$ is a linear subspace of $X \times X^*$.*

Proof. Take $(x, x^*) \in \operatorname{gra} A$ and also $(y, y^*) \in \operatorname{gra} A$. As $\operatorname{gra} A$ is a convex cone, we have $(x, x^*) + (y, y^*) = (x + y,\ x^* + y^*) \in \operatorname{gra} A$. Since $(0, 0) \in \operatorname{gra} A$, we obtain $0 \leq \langle x + y,\ x^* + y^*\rangle = \langle (-x) - y,\ (-x^*) - y^*\rangle$. From the maximal monotonicity of A, it follows that $-(x, x^*) \in \operatorname{gra} A$. Therefore,

$$-\operatorname{gra} A \subset \operatorname{gra} A. \tag{3}$$

A result due to Rockafellar (see [18, Theorem 2.7], which is stated in Euclidean space but the proof of which works without change in our present setting) completes the proof. □

Theorem 4.2. *Let $A\colon X \rightrightarrows X^*$ be maximal monotone such that $\operatorname{gra} A$ is convex. Then $\operatorname{gra} A$ is actually affine.*

Proof. Let $(x_0, x_0^*) \in \operatorname{gra} A$ and $B\colon X \rightrightarrows X^*$ be such that $\operatorname{gra} B = \operatorname{gra} A - (x_0, x_0^*)$. Thus $\operatorname{gra} B$ is convex with $(0, 0) \in \operatorname{gra} B$, and B is maximal monotone. Take $\alpha \geq 0$ and $(x, x^*) \in \operatorname{gra} B$. In view of Proposition 4.1, it suffices to show that $\alpha(x, x^*) \in \operatorname{gra} B$. If $\alpha \leq 1$, then the convexity of $\operatorname{gra} B$ yields $\alpha(x, x^*) = \alpha(x, x^*) + (1 - \alpha)(0, 0) \in \operatorname{gra} B$. Thus assume that $\alpha > 1$ and let $(y, y^*) \in \operatorname{gra} B$. Using the previous reasoning, we deduce that $\frac{1}{\alpha}(y, y^*) \in \operatorname{gra} B$. Thus, $\langle \alpha x - y, \alpha x^* - y^*\rangle = \alpha^2\langle x - \frac{1}{\alpha}y, x^* - \frac{1}{\alpha}y^*\rangle \geq 0$. Since B is maximal monotone, $\alpha(x, x^*) \in \operatorname{gra} B$. □

Remark 4.3. Theorem 4.2 provides a complete answer to [20, Problem 46.4 on page 183]. The proof of Theorem 4.2 appeared first in [24]. After submission of the original version of this manuscript, B. Svaiter informed us that he and Marques Alves also obtained this result, see [13, Lemma 3.1]. For a recent, more general version of Theorem 4.2, see Voisei and Zălinescu's [23, Proposition 5]. Note also that for every nonzero closed proper subspace L of X, the normal cone operator $N_L = \partial\iota_L$ is a maximal monotone linear relation; however, neither N_L nor its inverse is an affine mapping.

5. The Fitzpatrick Function of a Sum

Fitzpatrick functions – introduced first by Fitzpatrick [12] in 1988 (see also [8, 14, 15]) – have turned out to be immensely useful in the study of maximal monotone operators; see, e.g., [20] and the references therein.

Definition 5.1. Let $A\colon X \rightrightarrows X^*$. The *Fitzpatrick function* of A is

$$F_A\colon (x,x^*) \mapsto \sup_{(y,y^*)\in \operatorname{gra} A} \big(\langle x,y^*\rangle + \langle y,x^*\rangle - \langle y,y^*\rangle\big). \tag{4}$$

The following partial inf-convolution, introduced by Simons and Zălinescu [21] (see also Burachik and Svaiter's [9]), plays an important role in the study of the maximal monotonicity of the sum of two maximal monotone operators.

Definition 5.2. Let $F_1, F_2\colon X \times X^* \to \left]-\infty,+\infty\right]$. Then the *partial inf-convolution* $F_1\Box_2 F_2$ is the function defined on $X\times X^*$ by

$$F_1\Box_2 F_2\colon (x,x^*) \mapsto \inf_{y^*\in X^*} \big(F_1(x,x^*-y^*) + F_2(x,y^*)\big).$$

Let $A, B : X \rightrightarrows X^*$ be maximal monotone operators. It is not hard to see that $F_{A+B} \le F_A\Box_2 F_B$; moreover, equality may fail [5, Proposition 4.2 and Example 4.7]. In [4, Corollary 5.6], it was shown that $F_{A+B} = F_A\Box_2 F_B$ when A, B are continuous linear monotone operators and some constraint qualification holds. In this section, we substantially generalize this result to maximal monotone linear relations. Following [15], it will be convenient to set $F^{\intercal}\colon X^*\times X\colon (x^*,x)\mapsto F(x,x^*)$, when $F\colon X\times X^* \to \left]-\infty,+\infty\right]$, and similarly for a function defined on $X^*\times X$.

We start with some basic properties about Fitzpatrick functions.

Proposition 5.3. *Let $A\colon X \rightrightarrows X^*$ be monotone linear relation. Then the following hold.*

(*i*) $\operatorname{gra}(-A^*) = (\operatorname{gra} A^{-1})^{\perp}$.

(*ii*) $F_A|_{\operatorname{gra}(-A^*)} \equiv 0$.

Proof. (*i*): Take $(x,x^*)\in X\times X^*$. Then $(x,x^*)\in (\operatorname{gra} A^{-1})^{\perp} \Leftrightarrow (x^*,x)\in(\operatorname{gra} A)^{\perp} \Leftrightarrow (x,-x^*)\in \operatorname{gra} A^* \Leftrightarrow (x,x^*)\in\operatorname{gra}(-A^*)$.

(*ii*): Take $(x,x^*)\in\operatorname{gra}(-A^*)$. By (*i*), $(x^*,x)\in(\operatorname{gra} A)^{\perp}$. Since $(0,0)\in\operatorname{gra} A$ and A is monotone, we have $F_A(x,x^*)\ge 0$ and $\langle y,y^*\rangle \ge 0$ for every $(y,y^*)\in\operatorname{gra} A$. This yields

$$F_A(x,x^*) = \sup_{(y,y^*)\in\operatorname{gra} A}\big(\langle x^*,y\rangle + \langle x,y^*\rangle - \langle y^*,y\rangle\big) = \sup_{(y,y^*)\in\operatorname{gra} A}\big(0-\langle y^*,y\rangle\big) \le 0.$$

Altogether, we have $F_A(x,x^*)=0$. $\square$

It turns out to be convenient to define

$$P_X\colon X\times X^*\to X\colon (x,x^*)\mapsto x.$$

We shall need the following facts for later proofs.

Fact 5.4 (Fitzpatrick). *Let $A\colon X\rightrightarrows X^*$ be maximal monotone. Then F_A is proper lower semicontinuous and convex, and $F_A^{*\intercal}\ge F_A\ge\langle\cdot,\cdot\rangle$.*

Proof. See [12, Corollary 3.9 and Proposition 4.2]. $\square$

Proposition 5.5. *Let $A\colon X \rightrightarrows X^*$ be a monotone linear relation such that its graph is closed. Then $F_A^*\colon (x^*, x) \mapsto \iota_{\operatorname{gra} A^{-1}}(x^*, x) + \langle x, x^*\rangle$.*

Proof. Define $G\colon X^* \times X \to \left]-\infty, +\infty\right] : (x^*, x) \mapsto \iota_{\operatorname{gra} A}(x, x^*) + \langle x, x^*\rangle$. By Proposition 2.2 (*iv*), $\langle x, x^*\rangle = \langle x, Ax\rangle$ for every $(x, x^*) \in \operatorname{gra} A$; then by Proposition 2.3, G is a convex function. As $\operatorname{gra} A$ is closed, G is lower semicontinuous. Thus, G is a proper lower semicontinuous convex function. By definition of F_A, $F_A = G^*$. Therefore, we have $F_A^* = G^{**} = G$. $\square$

We also set, for any set S in a real vector space,

$$\operatorname{cone} S := \bigcup_{\lambda>0} \lambda S = \big\{\lambda s \mid \lambda > 0 \text{ and } s \in S\big\}.$$

Fact 5.6 (Simons-Zălinescu-Voisei). *Let $A\colon X \rightrightarrows X^*$ be maximal monotone. Then the following hold.*

(*i*) $\operatorname{dom} A \subset P_X\big(\operatorname{dom} F_A^{*\intercal}\big) \subset P_X(\operatorname{dom} F_A) \subset \overline{\operatorname{dom} A}$.

(*ii*) *Suppose that $A, B\colon X \rightrightarrows X^*$ are maximal monotone linear relations and that $\operatorname{dom} A - \operatorname{dom} B$ is closed. Then $A + B$ is maximal monotone.*

Proof. (*i*): Combine [20, Theorem 31.2] and [21, Lemma 5.3(a)]. (*ii*): See [21, Theorem 5.5], and [22] for a version that holds even when X is nonreflexive. $\square$

Fact 5.7 (Simons-Zălinescu). *Let $F_1, F_2\colon X \times X^* \to \left]-\infty, +\infty\right]$ be proper, lower semicontinuous, and convex. Assume that for every $(x, x^*) \in X \times X^*$,*

$$(F_1 \Box_2 F_2)(x, x^*) > -\infty$$

and that $\operatorname{cone}\big(P_X \operatorname{dom} F_1 - P_X \operatorname{dom} F_2\big)$ is a closed subspace of X. Then for every $(x, x^*) \in X \times X^*$,

$$(F_1 \Box_2 F_2)^*(x^*, x) = \min_{y^* \in X^*} \big(F_1^*(x^* - y^*, x) + F_2^*(y^*, x)\big).$$

Proof. See [21, Theorem 4.2]. $\square$

Lemma 5.8. *Let $A, B\colon X \rightrightarrows X^*$ be maximal monotone, and suppose that $\operatorname{cone}\big(\operatorname{dom} A - \operatorname{dom} B\big)$ is a closed subspace of X. Then*

$$\begin{aligned} &\operatorname{cone}\big(P_X \operatorname{dom} F_A - P_X \operatorname{dom} F_B\big) \\ =\ &\operatorname{cone}\big(\operatorname{dom} A - \operatorname{dom} B\big) = \operatorname{cone}\big(P_X \operatorname{dom} F_A^{*\intercal} - P_X \operatorname{dom} F_B^{*\intercal}\big). \end{aligned}$$

Proof. Using Fact 5.6 (*i*), we see that

$$\operatorname{cone}\big(\operatorname{dom} A - \operatorname{dom} B\big) \subset \operatorname{cone}\big(P_X \operatorname{dom} F_A - P_X \operatorname{dom} F_B\big) \subset \operatorname{cone}\big(\overline{\operatorname{dom} A} - \overline{\operatorname{dom} B}\big). \quad (5)$$

On the other hand, we have

$$(\forall \lambda > 0) \quad \lambda\big(\overline{\operatorname{dom} A} - \overline{\operatorname{dom} B}\big) \subset \overline{\lambda\big(\operatorname{dom} A - \operatorname{dom} B\big)} \subset \overline{\operatorname{cone}\big(\operatorname{dom} A - \operatorname{dom} B\big)}. \quad (6)$$

Thus, by (6) and the hypothesis,

$$\operatorname{cone}\big(\overline{\operatorname{dom} A} - \overline{\operatorname{dom} B}\big) \subset \overline{\operatorname{cone}\big(\operatorname{dom} A - \operatorname{dom} B\big)} = \operatorname{cone}\big(\operatorname{dom} A - \operatorname{dom} B\big). \tag{7}$$

Hence, by (5) and (7), $\operatorname{cone}\big(P_X \operatorname{dom} F_A - P_X \operatorname{dom} F_B\big) = \operatorname{cone}\big(\operatorname{dom} A - \operatorname{dom} B\big)$. In a similar fashion, Fact 5.6 (i) implies that $\operatorname{cone}\big(P_X \operatorname{dom} F_A^{*\intercal} - P_X \operatorname{dom} F_B^{*\intercal}\big) = \operatorname{cone}\big(\operatorname{dom} A - \operatorname{dom} B\big)$. $\square$

Proposition 5.9. *Let $A, B\colon X \rightrightarrows X^*$ be maximal monotone and suppose that $\operatorname{cone}\big(\operatorname{dom} A - \operatorname{dom} B\big)$ is a closed subspace of X. Then $F_A \square_2 F_B$ is proper, lower semicontinuous, and convex, and the partial infimal convolution is exact everywhere.*

Proof. Take $(x, x^*) \in X \times X^*$. By Fact 5.4, $(F_A \square_2 F_B)(x, x^*) \geq \langle x, x^*\rangle > -\infty$. Lemma 5.8 implies that

$$\operatorname{cone}\big(P_X \operatorname{dom} F_A - P_X \operatorname{dom} F_B\big) = \operatorname{cone}\big(\operatorname{dom} A - \operatorname{dom} B\big) \text{ is a closed subspace.}$$

Using Fact 5.7, we see that

$$\big(F_A \square_2 F_B\big)^{*\intercal}(x, x^*) = \min_{y^* \in X^*} \big(F_A^*(x^* - y^*, x) + F_B^*(y^*, x)\big) = \big(F_A^{*\intercal} \square_2 F_B^{*\intercal}\big)(x, x^*). \tag{8}$$

By Fact 5.4,

$$\big(F_A^{*\intercal} \square_2 F_B^{*\intercal}\big)(x, x^*) \geq \langle x, x^*\rangle > -\infty.$$

In view of Lemma 5.8,

$$\operatorname{cone}\big(P_X \operatorname{dom} F_A^{*\intercal} - P_X \operatorname{dom} F_B^{*\intercal}\big) = \operatorname{cone}\big(\operatorname{dom} A - \operatorname{dom} B\big) \text{ is a closed subspace.}$$

Therefore, using Fact 5.7 and (8),

$$\begin{aligned}
\big(F_A \square_2 F_B\big)^{**}(x, x^*) = \big(F_A \square_2 F_B\big)^{*\intercal *}(x^*, x) &= \min_{y^* \in X^*} \big(F_A^{*\intercal *}(x^* - y^*, x) + F_B^{*\intercal *}(y^*, x)\big) \\
&= \min_{y^* \in X^*} \big(F_A(x, x^* - y^*) + F_B(x, y^*)\big) \\
&= \big(F_A \square_2 F_B\big)(x, x^*).
\end{aligned}$$

Hence $F_A \square_2 F_B$ is proper, lower semicontinuous, and convex, and the partial infimal convolution is exact. $\square$

We are now ready for the main result of this section.

Theorem 5.10 (Fitzpatrick function of the sum). *Let $A, B\colon X \rightrightarrows X^*$ be maximal monotone linear relations, and suppose that $\operatorname{dom} A - \operatorname{dom} B$ is closed. Then $F_{A+B} = F_A \square_2 F_B$.*

Proof. Lemma 5.8 implies that

$$\begin{aligned}
&\operatorname{cone}\big(P_X \operatorname{dom} F_A - P_X \operatorname{dom} F_B\big) \\
= \ &\operatorname{cone}\big(\operatorname{dom} A - \operatorname{dom} B\big) = \operatorname{dom} A - \operatorname{dom} B \text{ is a closed subspace.}
\end{aligned}$$

Take $(x, x^*) \in X \times X^*$. Then, by Fact 5.4, $(F_A \Box_2 F_B)(x, x^*) \geq \langle x, x^* \rangle > -\infty$. Using Fact 5.7, Fact 5.6 (*ii*), and Proposition 5.5, we deduce that

$$\begin{aligned}(F_A \Box_2 F_B)^*(x^*, x) &= \min_{y^* \in X^*} \big(F_A^*(x^* - y^*, x) + F_B^*(y^*, x)\big)\\ &= \min_{y^* \in X^*} \big(\iota_{\operatorname{gra} A}(x, x^* - y^*) + \langle x^* - y^*, x\rangle + \iota_{\operatorname{gra} B}(x, y^*) + \langle y^*, x\rangle\big)\\ &= \iota_{\operatorname{gra}(A+B)}(x, x^*) + \langle x^*, x\rangle = F_{A+B}^*(x^*, x).\end{aligned}$$

Taking Fenchel conjugates and applying Proposition 5.9 now yields the result. □

6. Maximal Monotonicity

In this section, we shall obtain criteria for maximal monotonicity of linear relations. These criteria generalize some of the results by Phelps and Simons [17] (or [20, Theorem 47.1]) which also form the base of our proofs. The following concept of the halo (see [17, Definition 2.2]) is very useful.

Definition 6.1. Let $A\colon X \rightrightarrows X^*$ be a monotone linear relation. A vector $x \in X$ belongs to the *halo* of A, written

$$x \in \operatorname{halo} A \quad \Leftrightarrow \quad \big(\exists\, M \geq 0\big)\big(\forall (y, y^*) \in \operatorname{gra} A\big)\ \ \langle y^*, x - y\rangle \leq M\|x - y\|. \tag{9}$$

Proposition 6.2. *Let $A\colon X \rightrightarrows X^*$ be a monotone linear relation. Then* $\operatorname{dom} A \subset \operatorname{halo} A \subset (A0)^\perp$.

Proof. We start by proving the left inclusion. Let $x \in \operatorname{dom} A$, let $x^* \in Ax$, set $M := \|x^*\|$, and take $(y, y^*) \in \operatorname{gra} A$. By monotonicity of A, $0 \leq \langle x^* - y^*, x - y\rangle$; hence, $\langle y^*, x - y\rangle \leq \langle x^*, x - y\rangle \leq \|x^*\|\,\|x - y\| = M\|x - y\|$. Therefore, $x \in \operatorname{halo} A$. The right inclusion is seen to be true by taking $y = 0$ in (9). □

The next two results generalize Phelps and Simons' [17, Lemma 2.3 and Theorem 2.5]; we follow their proofs.

Proposition 6.3. *Let $A\colon X \rightrightarrows X^*$ be a monotone linear relation. Then*

$$\operatorname{halo} A = P_X\Bigg(\bigcup_{B \text{ is a monotone extension of } A} \operatorname{gra} B\Bigg).$$

Proof. "⇐": Let $(x, x^*) \in X \times X^*$ belong to some monotone extension of A. Then

$$\big(\forall (y, y^*) \in \operatorname{gra} A\big) \quad \langle y^*, x - y\rangle \leq \langle x^*, x - y\rangle \leq \|x^*\|\|x - y\|.$$

Hence (9) holds with $M = \|x^*\|$.

"⇒": Take $x \in \operatorname{halo} A$. Then there exists $M \geq 0$ such that

$$\big(\forall (y, y^*) \in \operatorname{gra} A\big) \quad \langle y^*, x - y\rangle \leq M\|x - y\|. \tag{10}$$

Now set

$$C := \big\{(y, \lambda) \in X \times \mathbb{R} \mid \lambda \geq M\|x - y\|\big\}$$

and

$$D := \big\{(y,\lambda) \in (\operatorname{dom} A) \times \mathbb{R} \mid \lambda \le \langle Ay, x-y\rangle\big\}.$$

(Note that $\langle Ay, x-y\rangle$ is single-valued by Proposition 2.2 (*ii*) and Proposition 6.2.) Clearly, C is convex with nonempty interior. Proposition 2.2 (*iii*), Proposition 2.3, and Proposition 6.2 imply that D is convex and nonempty. By (10), $(\operatorname{int} C) \cap D = \varnothing$. The Separation Theorem guarantees the existence of $\alpha \in \mathbb{R}$ and of $(x^*,\mu) \in X \times \mathbb{R}$ such that $(x^*,\mu) \neq (0,0)$ and

$$\big(\forall (y,\lambda) \in C\big) \quad \langle y, x^*\rangle + \lambda\mu \ge \alpha \tag{11}$$

$$\big(\forall (y,\lambda) \in D\big) \quad \langle y, x^*\rangle + \lambda\mu \le \alpha. \tag{12}$$

Since $(\forall \lambda \le 0)$ $(0,\lambda) \in D$ by Proposition 6.2, (12) implies that $\mu \ge 0$. If $\mu = 0$, then (11) yields $\inf\langle x^*, X\rangle \ge \alpha$, which implies $x^* = 0$ and hence $(x^*,\mu) = (0,0)$, a contradiction. Therefore, $\mu > 0$. Dividing the inequalities (11) and (12) by μ thus yields

$$\langle \tfrac{x^*}{\mu}, x\rangle \ge \tfrac{\alpha}{\mu} \quad \text{and} \quad \big(\forall y \in \operatorname{dom} A\big)\ \langle \tfrac{x^*}{\mu}, y\rangle + \langle Ay, x-y\rangle \le \tfrac{\alpha}{\mu}.$$

Therefore,

$$\big(\forall (y,y^*) \in \operatorname{gra} A\big) \quad \langle \tfrac{x^*}{\mu} - y^*, x-y\rangle \ge 0.$$

Hence $(x, \frac{x^*}{\mu})$ is monotonically related to $\operatorname{gra} A$. □

Remark 6.4. It is interesting to note that Proposition 6.3 can also be proved by Simons' M-technique. To see this, take $x \in \operatorname{halo} A$ and denote the dual closed unit ball by B^*. Then x is characterized by $\inf_{y\in\operatorname{dom} A}(M\|x-y\| - \langle Ay, x-y\rangle) \ge 0$; equivalently, by

$$\inf_{y\in\operatorname{dom} A}\ \max_{b^*\in MB^*} \big(\langle b^*, x-y\rangle - \langle Ay, x-y\rangle\big) \ge 0.$$

As the function $(y,b^*) \mapsto \langle b^*, x-y\rangle - \langle Ay, x-y\rangle$ is convex in y, and concave and upper semicontinuous in b^*, the Minimax Theorem [20, Theorem 3.2] results in

$$\max_{b^*\in MB^*}\ \inf_{y\in\operatorname{dom} A} \big(\langle b^*, x-y\rangle - \langle Ay, x-y\rangle\big) = \inf_{y\in\operatorname{dom} A}\ \max_{b^*\in MB^*} \big(\langle b^*, x-y\rangle - \langle Ay, x-y\rangle\big) \ge 0.$$

Hence there exists $c^* \in MB^*$ such that

$$\inf_{y\in\operatorname{dom} A} \big(\langle c^*, x-y\rangle - \langle Ay, x-y\rangle\big) \ge 0.$$

Therefore, (x,c^*) is monotonically related to $\operatorname{gra} A$.

Theorem 6.5 (Maximality). *Let $A\colon X \rightrightarrows X^*$ be a monotone linear relation. Then*

$$A \text{ is maximal monotone} \quad \Leftrightarrow \quad (\operatorname{dom} A)^\perp = A0 \text{ and } \operatorname{halo} A = \operatorname{dom} A.$$

Proof. "$\Rightarrow$": By Proposition 2.4, $(\operatorname{dom} A)^\perp = A0$. Proposition 6.3 yields $\operatorname{dom} A \subset \operatorname{halo} A$. Now take $x \in \operatorname{halo} A$. By Proposition 6.3, there exists x^* such that (x,x^*) is monotonically related to $\operatorname{gra} A$. Since A is maximal monotone, $(x,x^*) \in \operatorname{gra} A$, so $x \in \operatorname{dom} A$. Thus, $\operatorname{halo} A = \operatorname{dom} A$.

"$\Leftarrow$": Suppose $(x,x^*) \in X \times X^*$ is monotonically related to A. By Proposition 6.3, $x \in \operatorname{halo} A$. Thus $x \in \operatorname{dom} A$ and we pick $x_0^* \in Ax$. By Proposition 2.2 (*v*) and Fact 2.1 (*ii*), we have $x^* \in x_0^* + (\operatorname{dom} A)^\perp = x_0^* + A0 = Ax$. Therefore, A is maximal monotone. □

Corollary 6.6. *Let $A\colon X \rightrightarrows X^*$ be a monotone linear relation, and suppose that* $\operatorname{dom} A$ *is closed. Then A is maximal monotone* $\Leftrightarrow$ $(\operatorname{dom} A)^\perp = A0$.

Proof. "$\Rightarrow$": Apply Theorem 6.5. "$\Leftarrow$": Since $\operatorname{dom} A$ is closed, the hypothesis yields $\operatorname{dom} A = (A0)^\perp$. By Proposition 6.2, $\operatorname{dom} A = \operatorname{halo} A$. Once again, apply Theorem 6.5. $\square$

Corollary 6.7. *Let $A\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a monotone linear relation. Then A is maximal monotone* $\Leftrightarrow$ $(\operatorname{dom} A)^\perp = A0$.

7. Characterization of Skew Monotone Linear Relations

As an application of Theorem 6.5, we shall characterize skew linear relations. Theorem 7.6 below extends [4, Theorem 2.9] from monotone linear operators to monotone linear relations.

Definition 7.1 (Skew linear relation). Let $A\colon X \rightrightarrows X^*$ be a linear relation. We say that A is *skew* if $A^* = -A$.

Proposition 7.2. *Let $A\colon X \rightrightarrows X^*$ be a skew linear relation. Then both A and A^* are maximal monotone.*

Proof. By Fact 2.1 (*iv*), $(\forall x \in \operatorname{dom} A)$ $\langle Ax, x\rangle = 0$. Thus, using Proposition 2.2 (*iv*) and Fact 2.1 (*iv*), we see that both A and A^* are monotone. By Fact 3.1 and Fact 2.1 (*v*), A and A^* are maximal monotone. $\square$

Definition 7.3 (Fitzpatrick family). Let $A\colon X \rightrightarrows X^*$ be a maximal monotone linear relation. The associated *Fitzpatrick family* $\mathcal{F}_A$ consists of all functions $F\colon X \times X^* \to \left]-\infty, +\infty\right]$ that are lower semicontinuous and convex, and that satisfy $F \geq \langle \cdot, \cdot \rangle$, and $F = \langle \cdot, \cdot\rangle$ on $\operatorname{gra} A$.

Fact 7.4 (Fitzpatrick). *Let $A\colon X \rightrightarrows X^*$ be a maximal monotone linear relation. Then for every $(x, x^*) \in X \times X^*$,*

$$\begin{aligned} F_A(x,x^*) &= \min\big\{F(x,x^*) \mid F \in \mathcal{F}_A\big\} \quad \textit{and} \\ F_A^{*\intercal}(x,x^*) &= \max\big\{F(x,x^*) \mid F \in \mathcal{F}_A\big\}. \end{aligned} \tag{13}$$

Proof. See [12, Theorem 3.10]. $\square$

Example 7.5. Let $A\colon X \rightrightarrows X^*$ be a skew linear relation. Then $F_A = F_A^{*\intercal} = \iota_{\operatorname{gra} A}$.

Proof. Since $(\forall x \in \operatorname{dom} A)$ $\langle Ax, x\rangle = 0$, Proposition 5.5 implies that $F_A^{*\intercal} = \iota_{\operatorname{gra} A}$. Moreover,

$$F_A = \big(F_A^{*\intercal}\big)^{*\intercal} = \big(\iota_{\operatorname{gra} A}\big)^{*\intercal} = \big(\iota^{\intercal}_{\operatorname{gra} A}\big)^{*} = \big(\iota_{\operatorname{gra} A^{-1}}\big)^{*} = \iota_{(\operatorname{gra} A^{-1})^\perp} = \iota_{\operatorname{gra}(-A^*)} = \iota_{\operatorname{gra} A},$$

by Proposition 5.3 (*i*). Therefore, $F_A = F_A^{*\intercal} = \iota_{\operatorname{gra} A}$. $\square$

We now characterize skew linear relations in terms of the Fitzpatrick family. Note that the Fitzpatrick family is in this case as small as possible, i.e., a singleton. (For a related discussion concerning subdifferential operators, see [3, Section 5].)

Theorem 7.6. *Let $A\colon X \rightrightarrows X^*$ be a maximal monotone linear relation. Then*

$$A \text{ is skew} \quad \Leftrightarrow \quad \operatorname{dom} A = \operatorname{dom} A^* \text{ and } \mathcal{F}_A \text{ is a singleton,}$$

in which case $\mathcal{F}_A = \{\iota_{\operatorname{gra} A}\}$.

Proof. "$\Rightarrow$": Combine Example 7.5 with Fact 7.4.

"$\Leftarrow$": Fact 7.4 and Proposition 5.5 yield

$$F_A = F_A^{*\intercal} = \iota_{\operatorname{gra} A} + \langle\cdot,\cdot\rangle. \tag{14}$$

By Proposition 5.3 (ii), for every $(y,y^*) \in \operatorname{gra}(-A^*)$, we have $F_A(y,y^*) = 0$; hence, in view of (14), $(y,y^*) \in \operatorname{gra} A$ and $\langle y,y^*\rangle = 0$. Thus

$$\operatorname{gra} -A^* \subset \operatorname{gra} A \quad \text{ and } \quad \big(\forall (y,y^*) \in \operatorname{gra} A^*\big)\ \ \langle y^*,y\rangle = 0. \tag{15}$$

Since A is monotone (by hypothesis), so is $-A^*$. We wish to show that $-A^*$ is maximal monotone. To this end, take $x \in \operatorname{halo}(-A^*)$. According to Proposition 6.3, there exist $x^* \in X^*$ such that (x,x^*) is monotonically related to $\operatorname{gra}(-A^*)$, i.e., $(\forall (y,y^*) \in \operatorname{gra} A^*)$ $\langle x-y, x^*+y^*\rangle \geq 0$; equivalently,

$$\big(\forall (y,y^*) \in \operatorname{gra} A^*\big) \quad \langle x^*,x\rangle + \langle y^*,x\rangle - \langle x^*,y\rangle - \langle y^*,y\rangle \geq 0. \tag{16}$$

Using (15), this in turn is equivalent to

$$\big(\forall (y,y^*) \in \operatorname{gra} A^*\big) \quad \langle x^*,x\rangle \geq -\langle y^*,x\rangle + \langle x^*,y\rangle,$$

and – since $\operatorname{gra} A^*$ is a linear subspace of $X \times X^*$ – also to

$$\big(\forall (y,y^*) \in \operatorname{gra} A^*\big) \quad 0 = -\langle y^*,x\rangle + \langle x^*,y\rangle = \langle (x^*,-x),(y,y^*)\rangle.$$

Thus, $(x,x^*) \in \operatorname{gra} A^{**} = \operatorname{gra} A$ (Fact 2.1 (v)) and in particular $x \in \operatorname{dom} A$. As $\operatorname{dom} A = \operatorname{dom} A^*$, we have $x \in \operatorname{dom} A^* = \operatorname{dom}(-A^*)$. Therefore, $\operatorname{halo}(-A^*) \subset \operatorname{dom}(-A^*)$. The opposite inclusion is clear from Proposition 6.2. Altogether,

$$\operatorname{dom}(-A^*) = \operatorname{halo}(-A^*). \tag{17}$$

By Fact 3.1, A^* is maximal monotone; hence, Theorem 6.5 yields $(\operatorname{dom} A^*)^\perp = A^*0$. Since $\operatorname{dom} A^* = \operatorname{dom}(-A^*)$ and $A^*0 = -A^*0$, we have

$$\big(\operatorname{dom}(-A^*)\big)^\perp = -A^*0. \tag{18}$$

Using (18), (17), and Theorem 6.5, we conclude that $-A^*$ is maximal monotone. Since A is maximal monotone, the inclusion in (15) implies that $A = -A^*$. Therefore, A is skew. $\square$

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