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Intersections of Balls and the Ball Hull Mapping*

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We are concerned with some properties of the family of all subsets of a Banach space that can be written as an intersection of balls. A space with the Mazur Intersection Property (MIP) always satisfies those properties, so they can be regarded as weakenings of the MIP in different directions. The 'ball hull' function (mapping a set to the intersection of all closed balls that cover it) is often an effective tool to study those properties.

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1. Introduction

Mazur, in 1933, seems to have been the first to become aware that every closed bounded convex subset of $\mathbf{R}^d$ can be expressed as an intersection of (closed) Euclidean balls [12]. Since balls are the simplest convex sets, if we exclude points themselves, this property can be expected to have geometric consequences of some importance. A Banach space where the analogous statement holds is said to have the *Mazur Intersection Property* (MIP).

The MIP has since attracted the attention of many and is nowadays well known to have a close connection to the geometry of the dual unit ball. For instance, a characterization due to Giles, Gregory and Sims [7] states that $\mathbf{E}$ has the MIP if and only if the set of all weak* denting points of the dual ball is norm dense in the dual unit sphere. The reader is referred to the survey [8], available online, for a very nice introduction to the subject.

It suffices to observe that spaces as familiar as $(\mathbf{R}^2, \|\cdot\|_\infty)$ fail the MIP to realize that, from the point of view of applications, going beyond the MIP is necessary. An approach taking the space $\mathcal{M}$ of all intersections of balls as a subject in its own right was initiated in [9].

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Our own interest is motivated by problems from the theory of random sets, specifically the relationships between the Aumann and Herer expectations of random (closed bounded) sets in a separable Banach space. Under very general conditions, the Aumann expectation is convex, whereas the Herer expectation is an intersection of balls. Spaces with the MIP are the only ones where both expectations can potentially be equal; otherwise, there exist sets which can be Aumann expectations but not Herer expectations. In fact, we show in [19, 20] that *a separable Banach space has the MIP if and only if, for every random closed bounded convex set, its Aumann and Herer expectations are identical*. That solves an open problem in [13].

This paper is devoted to establishing basic and useful results about a number of properties more general than the MIP, some of which are used in the companion paper [20] for an in-depth study of the relationships between Aumann and Herer expectations in spaces without the MIP.

Those properties of $\mathcal{M}$ are trivially satisfied in spaces with the MIP. The main points made in this paper are:

(1) Properties of $\mathcal{M}$ can often be recast as properties of the ball hull mapping, and the latter language is often more useful.
(2) Known results about spaces with the MIP are often particular cases of results about other more general properties.

Our results also give answers to some questions made in the literature [9, 10, 8].

The paper is structured as follows. Sections 2 and 3 are preparatory. The former collects preliminary notions and results from the literature, while the latter collects basic properties of a fairly elementary but essential tool: the notion of hull of a bounded set with respect to the family of all closed balls or to a family of closed halfspaces (ball hull and $\mathcal{H}$-convex hull, respectively). Subsequently, each section presents results about a type of property: sup-stability in Section 4, continuity of the ball hull mapping in Section 5, sum stability and ball stability in Section 6, and representability in Section 7.

2. Preliminary notions

Let $\mathbf{E}$ be a Banach space with norm $\|\cdot\|$. We will denote by B its closed unit ball, by $\mathbf{E}^*$ its dual space, by B^* the closed unit ball of $\mathbf{E}^*$ and by S^* the unit sphere of $\mathbf{E}^*$. The weak* topology of $\mathbf{E}^*$ will be denoted by w^*.

Asplund spaces are a class of Banach spaces which can be characterized by the fact that their duals have the Radon-Nikodým Property. The norm of a dual space has the w^*-*Kadec-Klee* property if weak* convergence plus convergence of the norms imply strong convergence. A subset $\mathcal{H} \subset S^*$ is called *norming* (or 1-norming) if $\|x\| = \sup_{f\in\mathcal{H}} |f(x)|$ for all $x \in \mathbf{E}$.

We adopt the following notation for several families of closed subsets of $\mathbf{E}$:

$\mathcal{F}$: All non-empty closed subsets.

$\mathcal{F}_b$: All non-empty closed bounded subsets.

$\mathcal{F}_{bc}$: All non-empty closed bounded convex subsets.

$\mathcal{K}_c$: All non-empty compact convex subsets.

$\mathcal{B}$: All closed balls $B(a,\varepsilon)$ with $a \in \mathbf{E}$, $\varepsilon > 0$.

For $A, C \in \mathcal{F}$ and $\lambda \geq 0$, we denote by $A + C$ the closure of $\{x + y \mid x \in A, y \in C\}$ and by λA the set $\{\lambda x \mid x \in A\}$. Note that the closure is necessary to make $+$ a well-defined operation in $\mathcal{F}_{bc}$, since the elementwise sum of two closed sets is not closed in general.

The closed convex hull of A will be denoted by $\overline{\text{co}}\, A$, and the quantity $\|A\| = \sup_{x\in A} \|x\|$ is called the *norm* of A. Although $\mathcal{F}$ fails to be a linear space, the mapping $\|\cdot\|$ has all the ordinary properties defining a norm. The *support function* of A is the mapping $s(\cdot, A) : B^* \to \mathbf{R}$ given by $s(f, A) = \sup f(A)$.

We denote by d_H the Hausdorff metric in $\mathcal{F}_b$ given by

$$d_H(A, C) = \max\{\sup_{x\in A} \inf_{y\in C} \|x - y\|, \sup_{y\in C} \inf_{x\in A} \|x - y\|\}.$$

Endowed with this metric, $\mathcal{F}_{bc}$ becomes a complete metric space. Some basic properties of this metric are the following:

$$d_H(A_1 + A_2, C_1 + C_2) \leq d_H(A_1, C_1) + d_H(A_2, C_2),$$

$$d_H(\overline{\text{co}}(A_1 \cup A_2), \overline{\text{co}}(C_1 \cup C_2)) \leq \max\{d_H(A_1, C_1), d_H(A_2, C_2)\}.$$

Recall that we denote by $\mathcal{M}$ the family of all intersections of closed balls in $\mathbf{E}$. Granero, Moreno and Phelps [9, 10] have introduced the subfamily $\mathcal{P} \subset \mathcal{M}$ of *Mazur sets*. A set $A \in \mathcal{F}_b$ is called Mazur if, for every hyperplane H missing A, there is a ball covering A which still misses H. Equivalently, for every $f \in \mathbf{E}^*$ and $\lambda \in \mathbf{R}$, whenever $s(f, A) < \lambda$ there exists a ball $C \supset A$ such that $s(f, C) < \lambda$.

The family $\mathcal{P}$, unlike $\mathcal{M}$, has the remarkable property of being sum stable ($\mathcal{P}+\mathcal{P} = \mathcal{P}$) [9, Proposition 5.1]. However, $\mathcal{P}$ may be small or even contain just the points and balls of $\mathbf{E}$ [10, Section 3]. A space where $\mathcal{P} = \mathcal{M}$ is called a *Mazur space*.

A variant of the Mazur Intersection Property is the *K-IP* introduced by Vanderwerff [21]. A Banach space is said to have the K-IP when every set $K + \varepsilon B$, for $K \in \mathcal{K}_c$ and $\varepsilon > 0$, is an intersection of balls.

3. Elementary properties of hulls

This section is about two variants of the convex hull which are our basic tool. They are intersections of balls and halfspaces, respectively. Then we collect some relevant information about special subsets of S^*.

Most results in this section are elementary and once stated they will often be used without explicit reference.

Let $A \in \mathcal{F}_b$. The intersection of all closed balls that cover A will be called the *ball hull* of A and denoted by $\beta(A)$. Clearly, $A = \beta(A)$ if and only if $A \in \mathcal{M}$, so the MIP is the same as saying that β is the identity mapping.

Note the following observation.

Lemma 3.1. *Let $A \in \mathcal{F}_b$ and $a \in \mathbf{E}$. Then, $d_H(\beta(A), \{a\}) = d_H(A, \{a\})$.*

Proof. Since $A \subset \beta(A)$, then $d_H(\beta(A), \{a\}) \geq d_H(A, \{a\})$. For the reverse inequality, notice that $\beta(A) = \bigcap_{y \in \mathbf{E}} B(y, d_H(A, \{y\}))$. Then

$$\begin{aligned} d_H(\beta(A), \{a\}) &= \sup\{\|x - a\| \mid x \in \beta(A)\} \\ &= \sup\{\|x - a\| \mid \forall y \in \mathbf{E}, \|x - y\| \leq d_H(A, \{y\})\} \leq d_H(A, \{a\}). \end{aligned}$$

□

With this lemma, it is a simple exercise to prove the following properties of β.

Proposition 3.2. *Let $A, C \in \mathcal{F}_b$, $x \in \mathbf{E}$ and $\lambda \geq 0$. Then,*

(*i*) $\beta(A) + \lambda B \subset \beta(A + \lambda B)$,
(*ii*) $\beta(x + \lambda A) = x + \lambda \beta(A)$,
(*iii*) $\beta(\beta(A)) = \beta(\overline{\text{co}}\, A) = \beta(A)$,
(*iv*) *$\beta(A) \subset \beta(C)$ whenever $A \subset C$,*
(*v*) *$\beta(A)$ is the largest set C such that $d_H(C, \{a\}) = d_H(A, \{a\})$ for all $a \in \mathbf{E}$.*

Another useful property is the following.

Lemma 3.3. *If $A \in \mathcal{F}_{bc}$, then $\beta(A) = \bigcap_{\varepsilon > 0} \beta(A + \varepsilon B)$.*

Proof. Take $x \in \bigcap_{\varepsilon > 0} \beta(A + \varepsilon B)$ and assume, by contradiction, that $x \notin \beta(A)$. Then $x \notin C$ for some ball $C \in \mathcal{B}$ covering A. Since $d(x, C) > 0$, $x \notin C + \varepsilon B$ for some $\varepsilon > 0$. Thus $x \notin \beta(A + \varepsilon B)$, a contradiction. □

Let $\mathcal{H} \subset S^*$. Recall that, as a consequence of the Hahn-Banach theorem,

$$\overline{\text{co}}\, A = \bigcap_{f \in S^*} \{x \in \mathbf{E} \mid f(x) \leq s(f, A)\}.$$

It is then natural to define the $\mathcal{H}$-convex hull [2] of A as

$$\text{co}_{\mathcal{H}} A = \bigcap_{f \in \mathcal{H}} \{x \in \mathbf{E} \mid f(x) \leq s(f, A)\},$$

the intersection of all halfspaces that cover A and are determined by elements of $\mathcal{H}$. Clearly, $\overline{\text{co}}\, A = \text{co}_{S^*} A \subset \text{co}_{\mathcal{H}} A$ for any $\mathcal{H}$.

The following is an easy but useful observation. Its proof is analogous to that of Lemma 3.1.

Proposition 3.4. *Let $A \in \mathcal{F}_b$, $\mathcal{H} \subset \mathcal{S}^*$ and $f \in \mathcal{H}$. Then, $s(f, \text{co}_{\mathcal{H}} A) = s(f, A)$.*

We collect now some elementary properties of the operator $\text{co}_{\mathcal{H}}$.

Proposition 3.5. *Let $\mathcal{H}, \mathcal{H}' \in \mathcal{S}^*$, $A, C \in \mathcal{F}_b$ and $\lambda \geq 0$. Then,*

(*i*) $\text{co}_{\mathcal{H}} A + \lambda B \subset \text{co}_{\mathcal{H}}(A + \lambda B)$,
(*ii*) $\text{co}_{\mathcal{H}} \text{co}_{\mathcal{H}} A = \text{co}_{\mathcal{H}} \overline{\text{co}}\, A = \text{co}_{\mathcal{H}} A$,

(*iii*) $\mathrm{co}_{\mathcal{H}} A \subset \mathrm{co}_{\mathcal{H}} C$ *whenever* $A \subset C$,
(*iv*) $\mathrm{co}_{\mathcal{H}}(x + \lambda A) = x + \lambda\, \mathrm{co}_{\mathcal{H}} A$,
(*v*) $\mathrm{co}_{\mathcal{H}}(\mathrm{co}_{\mathcal{H}} A + \mathrm{co}_{\mathcal{H}} C) = \mathrm{co}_{\mathcal{H}}(A + C)$,
(*vi*) $\mathrm{co}_{\mathcal{H}} A \subset \mathrm{co}_{\mathcal{H}'} A$ *whenever* $\mathcal{H}' \subset \mathcal{H}$,
(*vii*) $\mathrm{co}_{\mathcal{H}} A = \mathrm{co}_{\mathrm{cl}\,\mathcal{H}} A$.

We will use $\mathcal{H}$-convex hulls with $\mathcal{H}$ being one of a number of subsets of S^* having a special geometric significance: weak* denting points, semidenting points, weak* strongly exposed points and extreme points. For the reader's convenience, we collect now their definitions and some properties.

A *weak* slice* of a subset $A \subset \mathbf{E}^*$ is a set

$$S(A, x, \delta) = \{f \in A \mid f(x) > \sup_{g \in A} g(x) - \delta\},$$

where $x \in \mathbf{E}, \delta > 0$. A *weak* denting* (or *w^*-denting*) point of A is a point $f \in A$ such that for all $\varepsilon > 0$ there exists a weak* slice $S(A, x, \delta)$ that contains f and has diameter at most ε.

$f \in A$ is called a *semidenting* point of A [4] if for all $\varepsilon > 0$ there exists a weak* slice $S(A, x, \delta)$ such that the diameter of $S(A, x, \delta) \cup \{f\}$ is at most ε.

$f \in A$ is called a *weak* strongly exposed* point of A if there exists some $x \in \mathbf{E}$ such that $f(x) > g(x)$ for all $g \in A \backslash \{f\}$, and $f_n(x) \to f(x)$ implies $f_n \to f$ in the dual norm whenever $\{f_n\}_n \subset A$.

$f \in A$ is called a *extreme* point of A if $f = 2^{-1}(g + h)$ with $g, h \in A$ implies $f = g = h$.

We adopt the following notation:

$\mathcal{W}^*\mathcal{D}$: the set of all weak* denting points of B^*,

$\mathcal{SD}$: the set of all semidenting points of B^*,

$\mathcal{W}^*\mathcal{S}$: the set of all weak* strongly exposed points of B^*,

$\mathcal{E}^*$: the set of all extreme points of B^*.

Then, we have the inclusions $\mathcal{W}^*\mathcal{S} \subset \mathcal{W}^*\mathcal{D} \subset \mathcal{E}^*$ and $\mathrm{cl}\,\mathcal{W}^*\mathcal{D} \subset \mathcal{SD}$. The following are useful characterizations of $\mathcal{W}^*\mathcal{D}$, $\mathcal{SD}$ and $\mathcal{E}^*$.

Lemma 3.6 (Chen–Lin [4, 5]). *Let $f \in B^*$. Then,*

(*i*) *$f \in \mathcal{W}^*\mathcal{D}$ if and only if for all $A \in \mathcal{F}_b$ and $\lambda \in \mathbf{R}$ such that $s(f, A) < \lambda$, there exists $C \in \mathcal{B}$ covering A such that $s(f, C) < \lambda$.*
(*ii*) *$f \in \mathcal{SD}$ if and only if for all $A \in \mathcal{F}_b$ and $x \in \mathbf{E}$ such that $s(f, A) < f(x)$, there exists $C \in \mathcal{B}$ covering A such that $x \notin C$.*
(*iii*) *$f \in \mathcal{E}^*$ if and only if for all $A \in \mathcal{K}_c$ and $\lambda \in \mathbf{R}$ such that $s(f, A) < \lambda$, there exists $C \in \mathcal{B}$ covering A such that $s(f, C) < \lambda$.*

4. Sup-stability properties

In this section, we present a natural stability property for $\mathcal{M}$. The space $\mathcal{F}_{bc}$ is a sup-semilattice with the join operation $A \vee C = \overline{\mathrm{co}}(A \cup C)$. Although stability with respect to that operation was mentioned in the introduction to [9], no result about it was presented there. The purpose of this section is to show that this type of property has implications on the structure of the family $\mathcal{M}$.

We define, similarly, *ball sup-stability* as the property that $A \in \mathcal{M}$ implies that $A \vee C$ is in $\mathcal{M}$ for every ball C. Note that $\beta(A \vee C) = \beta(A \cup C)$, because elements of $\mathcal{M}$ are always closed and convex.

Sup-stability of $\mathcal{M}$ can be rewritten as the property that β is a sup-semilattice homomorphism.

Proposition 4.1. *The following hold:*

(*i*) *$\mathcal{M}$ is sup-stable if and only if $\beta(A \vee C) = \beta(A) \vee \beta(C)$ for all $A, C \in \mathcal{F}_{bc}$.*

(*ii*) *$\mathcal{M}$ is ball sup-stable if and only if $\beta(A \vee C) = \beta(A) \vee C$ for every $A \in \mathcal{F}_{bc}, C \in \mathcal{B}$.*

Proof. Since $\beta(A), \beta(C) \in \mathcal{M}$, we have

$$\beta(A \vee C) \subset \beta(\beta(A) \vee \beta(C)) = \beta(A) \vee \beta(C)$$

by the sup-stability of $\mathcal{M}$. But

$$\beta(A) \vee \beta(C) = \overline{\text{co}}(\beta(A) \cup \beta(C)) \subset \overline{\text{co}}\, \beta(A \cup C) = \beta(A \cup C) = \beta(A \vee C).$$

We deduce $\beta(A \vee C) = \beta(A) \vee \beta(C)$.

The converse is easy: if $A, C \in \mathcal{M}$, then

$$A \vee C = \beta(A) \vee \beta(C) = \beta(A \vee C),$$

so $A \vee C \in \mathcal{M}$.

The proof of part (*ii*) is analogous. □

Remark 4.2. In order to check ball sup-stability, it suffices to consider just B instead of all $C \in \mathcal{B}$.

The next proposition shows that ball sup-stability of $\mathcal{M}$ has definite consequences. In particular, it contains geometric information about $\mathbf{E}^*$: by [21, Theorem 2.1], the K-IP is equivalent to the w^*-density of $\mathcal{E}^*$ in S^*.

Proposition 4.3. *If $\mathcal{M}$ is ball sup-stable, then $\mathbf{E}$ has the K-IP.*

Proof. Take any $K \in \mathcal{K}_c$. It suffices to prove that $K + B \in \mathcal{M}$. For any $\varepsilon > 0$, there exists a finite sequence $\{x_i\}_{i=1}^k \subset K$ such that $K \subset \bigcup_i (x_i + \varepsilon B)$ (for clarity, we omit the dependence of k and x_i on ε). Hence,

$$K + B \subset \bigcup_i (x_i + (1+\varepsilon)B) \subset \bigvee_i (x_i + (1+\varepsilon)B) \subset K + (1+\varepsilon)B.$$

Applying iteratively Proposition 4.1, we prove $\bigvee_i (x_i + (1+\varepsilon)B) \in \mathcal{M}$.

By the definition of $\mathcal{M}$, the intersection of all the sets $\bigvee_{i=1}^k (x_i + (1+\varepsilon)B)$ over $\varepsilon > 0$ is also in $\mathcal{M}$. But that intersection is sandwiched between $K + B$ and $K + (1+\varepsilon)B$ for each $\varepsilon > 0$, whence it is $K + B$. □

Corollary 4.4. *If the dual norm has the w^*-Kadec-Klee property (in particular, if* $\mathbf{E}$ *is finite-dimensional), then* $\mathbf{E}$ *has the MIP if and only if* $\mathcal{M}$ *is ball sup-stable.*

Proof. Under the MIP, $\mathcal{M} = \mathcal{F}_{bc}$ so one half is trivial. For the converse, notice that the K-IP plus w^*-Kadec-Klee imply the MIP [21, Remark 2.4]. □

This is a partial answer to a natural question: under what conditions some structural properties of $\mathcal{M}$ do imply the MIP? In other words, what properties of $\mathcal{M}$ do enforce the identity $\mathcal{M} = \mathcal{F}_{bc}$?

5. Continuity properties of the ball hull mapping

This section is motivated by the following result [16, Proposition 3.2].

Proposition 5.1 (Moreno–Schneider [16]). *The mapping β is d_H-continuous at every element of $\mathcal{F}_{bc}$ having non-empty interior.*

Moreno and Schneider also show that β fails to be continuous in general. We will present several variations on the continuity theme.

Observe that

$$\beta(A) = \{x \in \mathbf{E} \mid \forall a \in \mathbf{E}, \|x - a\| \le d_H(A, \{a\})\}.$$

With this in mind, we introduce the mapping $\beta^- : \mathcal{F}_{bc} \to \mathcal{F}_{bc}$ given by

$$\beta^-(A) = \operatorname{cl} \bigcup_{\varepsilon > 0} \{x \in \mathbf{E} \mid \forall a \in \mathbf{E}, \|x - a\| \le d_H(A, \{a\}) - \varepsilon\} \subset \beta(A).$$

We also need the definitions of some modes of convergence for sets. A sequence $\{A_n\}_n \subset \mathcal{F}_{bc}$ is said to converge to A in the *Mosco sense* if

$$w\text{-}\limsup_n A_n \subset A \subset \liminf_n A_n,$$

where $\liminf_n A_n$ denotes the set of all limits of convergent sequences $a_n \in A_n$ and $w\text{-}\limsup_n A_n$ denotes the set of all weak limits of convergent subsequences $a_{n_k} \in A_{n_k}$. The *gap* between two sets A, C is the quantity $D(A, C) = \inf_{x \in A, y \in C} \|x - y\|$. The Δ*-gap topology* is the weak topology generated by the mappings $D(\cdot, C)$ for C in some family Δ. If $\Delta = \mathcal{F}_{bc}$, it is called the *slice topology*; if Δ is the set of all singletons, it is called the *Wijsman topology*.

An alternative to Proposition 5.1 is as follows.

Proposition 5.2. *Let $A_n, A \in \mathcal{F}_{bc}$ be such that $A_n \to A$ in d_H. If $\beta^-(A) = \beta(A)$, then*

- *(i)* $\beta(A_n) \to \beta(A)$ *in the Mosco sense.*
- *(ii)* *If the norm is Fréchet differentiable and* $\mathbf{E}$ *has the Radon-Nikodým property, or if the dual norm has the w^*-Kadec-Klee property, then $\beta(A_n) \to \beta(A)$ in the slice topology.*
- *(iii)* *If A_n, A are compact (in particular, if* $\mathbf{E}$ *is finite-dimensional), then $\beta(A_n) \to \beta(A)$ in d_H.*

Proof. For any fixed $\varepsilon > 0$, eventually $A_n \subset A + \varepsilon B$, so also $\beta(A_n) \subset \beta(A+\varepsilon B)$ and thus w-$\limsup_n \beta(A_n) \subset \beta(A+\varepsilon B)$. That yields

$$w\text{-}\limsup_n \beta(A_n) \subset \bigcap_{\varepsilon>0} \beta(A+\varepsilon B) = \beta(A)$$

by Lemma 3.3.

For the liminf part, fix $\varepsilon > 0$ again. Eventually, $A \subset A_n + \varepsilon B$, so

$$\begin{aligned} &\{x \mid \forall a, \|x-a\| \le d_H(A,\{a\}) - \varepsilon\} \\ &\subset \{x \mid \forall a, \|x-a\| \le d_H(A_n+\varepsilon B,\{a\}) - \varepsilon\} = \beta(A_n) \end{aligned}$$

since $d_H(A_n+\varepsilon B,\{a\}) = d_H(A_n,\{a\}) + \varepsilon$. Thus, for each $\varepsilon > 0$,

$$\{x \mid \forall a, \|x-a\| \le d_H(A,\{a\}) - \varepsilon\} \subset \liminf_n \beta(A_n)$$

and

$$\beta(A) = \operatorname{cl} \bigcup_{\varepsilon>0} \{x \mid \forall a, \|x-a\| \le d_H(A,\{a\}) - \varepsilon\} \subset \liminf_n \beta(A_n)$$

as well. That completes the proof of (i).

For part (ii), notice that Mosco convergence of closed convex sets implies convergence in the Wijsman topology [1], and, under any of the two assumptions, Wijsman convergence to a closed convex bounded set is equivalent to slice convergence [3, Corollary 3.13.(a) and Theorem 3.2].

As to part (iii), first note that $\beta(A_n)$ and $\beta(A)$ are compact, since a family of balls covers a closed bounded set if and only if it covers its ball hull. It suffices to notice then that Mosco convergence of compact convex sets to a compact limit is equivalent to d_H-convergence. □

The assumption in Proposition 5.2 can be strictly weaker than that in Proposition 5.1.

Proposition 5.3. *Let* $\mathbf{E}$ *be finite-dimensional. If* $A \in \mathcal{F}_{bc}$ *has non-empty interior, then* $\beta^-(A) = \beta(A)$.

Proof. If $A \in \mathcal{F}_{bc}$ has non-empty interior, then the set

$$A^{-\varepsilon} = \{x \in A \mid B(x,\varepsilon) \subset A\}$$

is non-empty for sufficiently small $\varepsilon > 0$, and clearly $A = \operatorname{cl} \bigcup_{\varepsilon>0} A^{-\varepsilon}$. Since

$$s(f,A) = \sup_{\varepsilon>0} s(f,A^{-\varepsilon})$$

for each $f \in \mathbf{E}^*$, and B^* is compact, by Dini's lemma $s(\cdot,A^{-\varepsilon}) \to s(\cdot,A)$ uniformly, namely $d_H(A^{-\varepsilon},A) \to 0$.

Now Proposition 5.1 gives $\beta(A^{-\varepsilon}) \to \beta(A)$. Since $A^{-\varepsilon} + \varepsilon B \subset A$,

$$d_H(A^{-\varepsilon},\{a\}) \le d_H(A,\{a\}) - \varepsilon$$

and we readily have $\beta(A^{-\varepsilon}) \subset \beta^-(A)$. A sandwich argument completes the proof. □

Example 5.4. Let $\mathbf{E} = \mathbf{R}^2$ with the sum norm. If A is the segment $\mathrm{co}\{(0,0),(0,1)\}$, then

$$\{x \in \mathbf{E} \mid \forall a \in \mathbf{E}, \|x - a\| \le d_H(A, \{a\}) - \varepsilon\} = \mathrm{co}\{(0,\varepsilon),(0,1-\varepsilon)\}$$

for every $\varepsilon \in (0, 1/2)$, whence $\beta^-(A) = \beta(A)$, even if $\mathrm{int}\, A = \emptyset$.

We present now a sufficient condition for the continuity of β.

Proposition 5.5. *If $\mathcal{M}$ is ball sup-stable, then β is d_H-continuous.*

Proof. Let $A_n \to A$ and fix $\varepsilon > 0$. We assume without loss of generality that $0 \in A$. Since d_H-convergence is the same as uniform convergence of the distance functions, $d(0, A_n) \to 0$. Then there exists $\{x_n\}_n$ converging to 0 such that each $x_n \in A_n$. Therefore,

$$\begin{aligned} &d_H((A_n - x_n) \vee (\varepsilon/4)B, A \vee (\varepsilon/4)B) \\ &\le \max\{d_H(A_n, A) + d_H(\{-x_n\}, 0), d_H((\varepsilon/4)B, (\varepsilon/4)B)\} \\ &= d_H(A_n, A) + \|x_n\| \to 0. \end{aligned}$$

By Proposition 5.1, $\beta((A_n - x_n) \vee (\varepsilon/4)B) \to \beta(A \vee (\varepsilon/4)B)$. Proposition 4.1 yields then

$$\beta(A_n - x_n) \vee (\varepsilon/4)B \to \beta(A) \vee (\varepsilon/4)B.$$

Then

$$\begin{aligned} &d_H(\beta(A_n), \beta(A)) \\ &\le d_H(\beta(A_n), \beta(A_n) - x_n) + d_H(\beta(A_n) - x_n, \beta(A_n - x_n) \vee (\varepsilon/4)B) \\ &\qquad + d_H(\beta(A_n - x_n) \vee (\varepsilon/4)B, \beta(A) \vee (\varepsilon/4)B) + d_H(\beta(A) \vee (\varepsilon/4)B, \beta(A)). \end{aligned}$$

We look at each summand separately now. For the first one,

$$\begin{aligned} &d_H(\beta(A_n), \beta(A_n) - x_n) \\ &\le d_H(\beta(A_n), \beta(A_n)) + d_H(\{0\}, \{-x_n\}) = \|x_n\| \to 0. \end{aligned}$$

For the second one, since $x_n \in A_n$,

$$\begin{aligned} &d_H(\beta(A_n) - x_n, \beta(A_n - x_n) \vee (\varepsilon/4)B) \\ &= d_H(\beta(A_n - x_n) \vee \{0\}, \beta(A_n - x_n) \vee (\varepsilon/4)B) \le \varepsilon/4. \end{aligned}$$

The fourth one is bounded analogously by $\varepsilon/4$, and the third was shown above to converge to 0. Accordingly, for all sufficiently large n we have

$$d_H(\beta(A_n), \beta(A)) \le 4 \cdot \frac{\varepsilon}{4} = \varepsilon.$$

□

Since β is continuous on elements of $\mathcal{F}_{bc}$ with non-empty interior and every $A \in \mathcal{F}_{bc}$ can be 'outer approximated' by the sequence $A + n^{-1}B$, it is natural to ask whether β has a continuity property for decreasing sequences.

Proposition 5.6. *If* $A_n \searrow A$ *and* $d_H(A_n, A) \to 0$*, then* $d_H(\beta(A_n), \beta(A)) \to 0$.

Proof. Fix $\varepsilon > 0$. Applying the hypothesis and Lemma 3.3, we find $\delta > 0$ such that $\beta(A + \delta B) \subset \beta(A) + \varepsilon B$ and also $A_n \subset A + \delta B$ for all sufficiently large n. Since $\beta(A) \subset \beta(A_n)$, we have

$$d_H(\beta(A_n), \beta(A)) = \inf\{\nu > 0 \mid \beta(A_n) \subset \beta(A) + \nu B\} \leq \varepsilon$$

and, by the arbitrariness of ε, we are done. □

In Proposition 5.6, if some A_k is compact then $A_n \searrow A$ already implies $d_H(A_n, A) \to 0$.

6. Sum stability properties

We call $\mathcal{M}$ *sum stable* if $A, C \in \mathcal{M}$ implies $A + C \in \mathcal{M}$ (in short, $\mathcal{M} + \mathcal{M} = \mathcal{M}$). Stability properties appear implicitly in [21], and [9] begins a systematic study of them. Please notice that + is denoted $\hat{+}$ in [9], where + denotes (maybe non-closed) elementwise addition. Both notions of sum stability of $\mathcal{M}$ coincide if and only if **E** is reflexive (combine [13, Theorem 1.24, p. 160] and [8, Remark 5.2]).

A variant of sum stability is *ball stability*, namely the property that $\mathcal{M} + \mathcal{B} \subset \mathcal{M}$. At some instances we will speak about even weaker stability properties, particularly $\mathcal{C} + \mathcal{B} \subset \mathcal{M}$ for some subset $\mathcal{C} \subset \mathcal{M}$.

Here we present characterizations of sum stability and ball stability, in terms of properties of the operator β. We begin by showing that β is always superlinear.

Proposition 6.1. *Let* $A, C \in \mathcal{F}_b$*. Then,* $\beta(A) + \beta(C) \subset \beta(A + C)$*. Equivalently,* $\beta(\beta(A) + \beta(C)) = \beta(A + C)$.

Proof. Let $A', C' \in \mathcal{F}_b$, and $x \in A'$. Since $x + \beta(C') = \beta(x + C') \subset \beta(A' + C')$, we have $A' + \beta(C') \subset \beta(A' + C')$. By using this inclusion twice,

$$\beta(A) + \beta(C) \subset \beta(A + \beta(C)) \subset \beta(\beta(A + C)) = \beta(A + C).$$

But then $\beta(A + C) \subset \beta(\beta(A) + \beta(C)) \subset \beta(A + C)$. □

Corollary 6.2. $\mathcal{M}$ *is sum stable if and only if* β *is linear (equivalently, subadditive).*

Proof. Taking into account Proposition 6.1, linearity is equivalent to the property that $\beta(\beta(A) + \beta(C)) = \beta(A) + \beta(C)$ for all $A, C \in \mathcal{F}_b$. This is the same as saying that $A + C \in \mathcal{M}$ whenever $A, C \in \mathcal{M}$. □

Remark 6.3. Similarly, $\mathcal{M}$ is ball stable if and only if $\beta(A + C) = \beta(A) + C$ for all $A \in \mathcal{F}_b$, $C \in \mathcal{B}$. Or, equivalently, $\beta(A + B) = \beta(A) + B$ for any $A \in \mathcal{F}_b$.

Proposition 6.4. $\mathcal{M}$ *is ball stable if and only if* β *is non-expansive.*

Proof. *Sufficiency.* Let $A, C \in \mathcal{F}_b$. By the definition of the Hausdorff metric, $A \subset C + d_H(A, C)B$. Then

$$\beta(A) \subset \beta(C + d_H(A, C)B) \subset \beta(\beta(C) + d_H(A, C)B) = \beta(C) + d_H(A, C)B,$$

and analogously $\beta(C) \subset \beta(A) + d_H(A,C)B$. This yields $d_H(\beta(A), \beta C) \leq d_H(A,C)$.

Necessity. Assume that $d_H(\beta(A), \beta(C)) \leq d_H(A,C)$ for all $A, C \in \mathcal{F}_b$. Let $D \in \mathcal{M}$ and $\lambda > 0, x \in \mathbf{E}$, it suffices to check that $\beta(D + x + \lambda B) = D + x + \lambda B$.

From non-expansiveness, $d_H(\beta(D+x+\lambda B), \beta(D+x)) \leq d_H(D+x+\lambda B, D+x) = \lambda$. We deduce that

$$\beta(D + x + \lambda B) \subset \beta(D + x) + \lambda B = D + x + \lambda B,$$

and we are done. □

We give an answer now to a question posed by Granero, Moreno and Phelps in [9, p. 193] (see also [10, pp. 412, 417], [8, p. 82] and [6]). Namely, are there normed spaces where $\mathcal{M}$ is not ball stable? A positive answer relies on results of Vanderwerff [21] (see also [16] for an independent argument).

A Banach space is said to have the *K-MIP* if every compact convex set is an intersection of balls. We immediately see that the K-IP can be recast as the stability property that $(\mathcal{M} \cap \mathcal{K}_c) + \mathcal{B} \subset \mathcal{M}$, in spaces with the K-MIP. Therefore, every space with the K-MIP but without the K-IP is a space where $\mathcal{M}$ fails to be ball stable.

Remark 6.5. Hence, the reverse inclusion to Proposition 3.2 (i) may fail even if we assume that A is compact and is an intersection of balls.

In order to give examples, we slightly adapt [21, Theorem 2.6].

Proposition 6.6 (Vanderwerff). *Let* $\mathbf{E}$ *be a non-reflexive Banach space admitting a norm whose dual is locally uniformly rotund. Then* $\mathbf{E}$ *can be renormed to simultaneously have the K-MIP and fail the K-IP.*

In particular, non-reflexive Banach spaces with separable dual can be renormed so that $\mathcal{M}$ is not ball stable.

Granero, Moreno and Phelps [9, Proposition 4.4] have shown that $(\mathbf{R}^3, \|\cdot\|_1)$ is ball stable but not sum stable. In fact, Moreno, Papini and Phelps [15, Proposition 5.2] even show the rather stronger result that A being *diametrically maximal*, i.e. $A = \bigcap_{x \in A} B(x, \mathrm{diam}(A)) \in \mathcal{M}$, does not ensure $A + (-A) \in \mathcal{M}$.

We present now a very short proof, based on β, of an important and surprising result: the *converse* to sum stability always holds [9, Proposition 7.1]. The proof in [9] relies on the Bishop–Phelps Theorem.

Theorem 6.7. *Let* $A, C \in \mathcal{F}_{bc}$. *If* $A + C \in \mathcal{M}$, *then* $A, C \in \mathcal{M}$.

Proof. If $A + C \in \mathcal{M}$, then Proposition 6.1 gives

$$A + C \subset \beta(A) + C \subset \beta(A) + \beta(C) \subset \beta(A + C) = A + C.$$

Hence we have the identity $A + C = \beta(A) + C$, from which [18, Lemma 1] yields $A = \beta(A)$. By symmetry, also $C = \beta(C)$. □

The following is an interesting consequence of Theorem 6.7.

Proposition 6.8. *Either $\mathcal{M}$ is ball stable or there exists some $A \in \mathcal{M}$ such that, for every $C \in \mathcal{F}_{bc}$ and $\lambda > 0$, we have $A + C + \lambda B \notin \mathcal{M}$.*

Proof. Since every set of the form $D + \lambda B$ decomposes as $(D + \lambda' B) + (\lambda - \lambda')B$ for any $\lambda' \in (0, \lambda)$, Theorem 6.7 yields

$$D + \lambda B \in \mathcal{M} \Longrightarrow D + \lambda' B \in \mathcal{M} \text{ for every } \lambda' \in (0, \lambda]$$

and thus the set $\{\lambda \geq 0 \mid D + \lambda B \in \mathcal{M}\}$ is an interval.

To show that it contains its upper end-point λ_0, notice that $D + \lambda_0 B$ has non-empty interior and $D + \lambda B$ d_H-converges to $D + \lambda_0 B$ as $\lambda \to \lambda_0$, so Proposition 5.1 implies $D + \lambda B \to \beta(D + \lambda_0 B)$. By the uniqueness of the limit, $D + \lambda_0 B \in \mathcal{M}$.

Accordingly, either $\mathcal{M}$ is ball stable or there will be some $D \in \mathcal{M}$ and $\lambda_0 > 0$ such that

$$D + \lambda B \in \mathcal{M} \Longleftrightarrow \lambda \in [0, \lambda_0].$$

Taking $A = D + \lambda_0 B$, we see that indeed $A + \lambda B$ is not in $\mathcal{M}$ whatever $\lambda > 0$ may be. But then, again by Theorem 6.7, neither will $A + C + \lambda B$ for any $C \in \mathcal{F}_{bc}$. □

Thus, in the absence of stability properties, $\mathcal{M}$ contains sets with behaviour almost as 'bad' as those not in $\mathcal{M}$. It is not hard to show, from these results, that $\mathcal{M}$ has empty d_H-interior whenever it is not the whole $\mathcal{F}_{bc}$. Finer results on the porosity of $\mathcal{M}$ can be found in [11].

7. Representability of $\mathcal{M}$

We call $\mathcal{M}$ *representable by a subset* $\mathcal{H} \subset S^*$ if the representation

$$A = \bigcap_{f \in \mathcal{H}} f^{-1}[\inf f(A), \sup f(A)]$$

holds for $A \in \mathcal{F}_b$ if and only if $A \in \mathcal{M}$. We will say that $\mathcal{M}$ is *representable* if it is so by some $\mathcal{H}$.

Granero, Moreno and Phelps [9, p. 205] have asked whether $\mathcal{M}$ is representable in every Banach space, and proved that this is the case in spaces where the unit ball of each finite-dimensional subspace is a polytope [9, Corollary 4.2].

The answer to their question is negative, and the characterization of such spaces is the main objective of this section.

Theorem 7.1. *Let $\mathcal{H}$ be a symmetric subset of S^*. Then, the following conditions are equivalent:*

(*i*) *$\mathcal{M}$ is representable by $\mathcal{H}$,*
(*ii*) *$\mathcal{H} \subset \mathcal{SD}$ and $\mathcal{H}$ is norming.*

In particular, $\mathcal{M}$ is representable if and only if $\mathcal{SD}$ is norming.

If $\mathcal{H}$ is symmetric, then the representability of $\mathcal{M}$ by $\mathcal{H}$ can be rewritten as the property that '$A = \text{co}_{\mathcal{H}} A$ if and only if $A \in \mathcal{M}$'.

The proof of Theorem 7.1 is contained in the following propositions, which have independent interest (see [20]).

Proposition 7.2. *Let $\mathcal{H}$ be a symmetric subset of S^*. Then, $\mathcal{M}$ is representable by $\mathcal{H}$ if and only if $\beta = \mathrm{co}_{\mathcal{H}}$.*

Proof. Notice that $\mathrm{co}_{\mathcal{H}} A = \mathrm{co}_{\mathcal{H}} \mathrm{co}_{\mathcal{H}} A$ for every $A \in \mathcal{F}_b$. Hence, by the representability, $\mathrm{co}_{\mathcal{H}} A \in \mathcal{M}$; in other words, $\beta(\mathrm{co}_{\mathcal{H}} A) = \mathrm{co}_{\mathcal{H}} A$. Analogously, since $\beta(A) \in \mathcal{M}$, it also implies $\mathrm{co}_{\mathcal{H}} \beta(A) = \beta(A)$. Taking into account these identities,

$$\mathrm{co}_{\mathcal{H}} A \subset \mathrm{co}_{\mathcal{H}} \beta(A) \subset \mathrm{co}_{\mathcal{H}} \beta(\mathrm{co}_{\mathcal{H}} A) = \beta(\mathrm{co}_{\mathcal{H}} A) = \mathrm{co}_{\mathcal{H}} A,$$

whence $\mathrm{co}_{\mathcal{H}} A = \mathrm{co}_{\mathcal{H}} \beta(A) = \beta(A)$.

Conversely, if $\beta(A) = \mathrm{co}_{\mathcal{H}} A$ then $A = \mathrm{co}_{\mathcal{H}} A$ is the same thing as $A = \beta(A)$ i.e. $A \in \mathcal{M}$. Thence $\mathcal{M}$ is representable by $\mathcal{H}$. □

Proposition 7.3. *Let $\mathcal{H} \subset S^*$. Then, the following conditions are equivalent:*

(*i*) $\beta \subset \mathrm{co}_{\mathcal{H}}$,
(*ii*) $\mathrm{co}_{\mathcal{H}} \circ \beta = \mathrm{co}_{\mathcal{H}}$,
(*iii*) $\beta \circ \mathrm{co}_{\mathcal{H}} = \mathrm{co}_{\mathcal{H}}$,
(*iv*) $\mathcal{H} \subset \mathcal{SD}$.

Proof. Let $A \in \mathcal{F}_b$. If (i) holds, then

$$\mathrm{co}_{\mathcal{H}} A \subset \mathrm{co}_{\mathcal{H}} \beta(A) \subset \mathrm{co}_{\mathcal{H}} \mathrm{co}_{\mathcal{H}} A = \mathrm{co}_{\mathcal{H}} A.$$

Conversely, it is clear that (ii) implies (i).

If (ii) holds, then using also (i) we have

$$\beta(\mathrm{co}_{\mathcal{H}} A) = \beta(\mathrm{co}_{\mathcal{H}} \beta(A)) \subset \mathrm{co}_{\mathcal{H}} \mathrm{co}_{\mathcal{H}} \beta(A) = \mathrm{co}_{\mathcal{H}} \mathrm{co}_{\mathcal{H}} A = \mathrm{co}_{\mathcal{H}} A \subset \beta(\mathrm{co}_{\mathcal{H}} A).$$

The converse is analogous.

Now (i) is the same as saying that $x \notin \mathrm{co}_{\mathcal{H}} A$ implies $x \notin \beta(A)$. Namely, for all $x \in \mathbf{E}$ and all $A \in \mathcal{F}_b$, if there exists some $f \in \mathcal{H}$ such that $f(x) > s(f, A)$, then there exists a ball $C \subset \mathcal{B}$ such that $A \subset C$ but $x \notin A$. That is exactly saying that every $f \in \mathcal{H}$ is a semidenting point of B^*. Hence (i) and (iv) are equivalent too. □

Proposition 7.4. *Let $\mathcal{H}$ be a symmetric subset of S^*. Then, the following conditions are equivalent:*

(*i*) $\mathrm{co}_{\mathcal{H}} \subset \beta$,
(*ii*) $\beta \circ \mathrm{co}_{\mathcal{H}} = \beta$,
(*iii*) $\mathrm{co}_{\mathcal{H}} \circ \beta = \beta$,
(*iv*) $\mathrm{co}_{\mathcal{H}} B = B$,
(*v*) *$\mathcal{H}$ is norming.*

Proof. The proof of $(i) \Rightarrow (ii) \Leftrightarrow (iii)$ is similar to Proposition 7.3. Moreover, clearly (ii) implies (i) and (i) implies (iv). Now assume (iv), then $\|x\| > 1$ implies $f(x) > 1$ for some $f \in \mathcal{H}$. Define

$$x_n = \frac{1 + n^{-1}(\|x\| - 1)}{\|x\|} x.$$

Since $\|x_n\| > 1$, we can find a sequence $\{f_n\}_n \subset \mathcal{H}$ such that $f_n(x_n) > 1$. Hence

$$\|x\| \geq f_n(x) > \frac{\|x\|}{1 + n^{-1}(\|x\| - 1)} \to \|x\|$$

and we deduce that $\|x\| = \sup_{f \in \mathcal{H}} f(x)$ for all x with norm larger than 1. But this extends immediately to arbitrary $x \in \mathbf{E}$.

Let us prove now the implication $(v) \Rightarrow (iv)$. It suffices to show that $\mathrm{co}_{\mathcal{H}} B \subset B$. Let $x \notin B$, since $\mathcal{H}$ is norming, $|f(x)| > 1$ for some $f \in \mathcal{H}$. Since $\mathcal{H} = -\mathcal{H}$, we can assume without loss of generality that $f(x) > 1 = s(f, B)$. Hence $x \notin \mathrm{co}_{\mathcal{H}} B$.

Finally, if (iv) holds then let $A \in \mathcal{F}_b$. In order to prove that $\mathrm{co}_{\mathcal{H}} A \subset \beta(A)$, it suffices to check that $A \subset x + \lambda B \in \mathcal{B}$ implies $\mathrm{co}_{\mathcal{H}} A \subset x + \lambda B$. But

$$\mathrm{co}_{\mathcal{H}} A \subset \mathrm{co}_{\mathcal{H}}(x + \lambda B) = x + \lambda \,\mathrm{co}_{\mathcal{H}} B = x + \lambda B.$$

□

The useful Chen–Lin characterization of spaces with the MIP appears as a particular case of Theorem 7.1.

Corollary 7.5 (Chen–Lin [4]). $\mathbf{E}$ *has the MIP if and only if* $\mathcal{SD}$ *is dense in* S^* *(equivalently,* $\mathcal{SD} = S^*$*).*

Proof. Since the MIP amounts to saying that $\mathcal{M}$ is representable by S^*, Theorem 7.1 gives $\mathcal{SD} = S^*$. Conversely, if $\mathcal{SD}$ is dense, then it is norming and by the implication $(ii) \Longrightarrow (i)$ in Theorem 7.1, $\mathcal{M}$ is representable by $\mathcal{SD}$. But, by density,

$$A = \mathrm{co}_{\mathcal{SD}} A = \beta(A) \in \mathcal{M}$$

for every $A \in \mathcal{F}_b$. That proves the MIP. □

This shows that the MIP is not only a particular case of representability but also both notions allow characterizations via semidenting points.

We also note that $(iii) \Leftrightarrow (v)$ in Proposition 7.4 improves [9, Lemma 8.1] and shows that being a James boundary is not essential in its context.

It was proved in [9] that polyhedral Banach spaces have $\mathcal{M}$ representable; but it should be noted that representable spaces include many spaces which are neither polyhedral nor have the MIP.

Corollary 7.6. *If* $\mathbf{E}$ *is Asplund, then* $\mathcal{M}$ *is representable by* $\mathcal{W}^*\mathcal{S}$.

Proof. If $\mathbf{E}$ is Asplund, then B^* is the weak* closure of the convex hull of $\mathcal{W}^*\mathcal{S}$ (e.g. [17]). It is not hard to show then that $\mathcal{W}^*\mathcal{S}$ is norming. Since $\mathcal{W}^*\mathcal{S} \subset \mathcal{W}^*\mathcal{D} \subset \mathcal{SD}$, Theorem 7.1 applies. □

It is interesting to point out the following variant of Theorem 7.1, which gives a positive answer to a relaxed form of the question we are considering. Let us say that $\mathcal{M}$ is representable in $\mathcal{C}$ by $\mathcal{H}$, where $\mathcal{C} \subset \mathcal{F}_{bc}$, if the same representation

$$A = \bigcap_{f \in \mathcal{H}} f^{-1}[\inf f(A), \sup f(A)]$$

holds for $A \in \mathcal{C}$ if and only if $A \in \mathcal{M} \cap \mathcal{C}$.

Theorem 7.7. *In every Banach space, $\mathcal{M}$ is representable in $\mathcal{K}_c$ by $\mathcal{E}^*$.*

Proof. The same proof of Proposition 7.2 allows one to show that $\mathcal{M}$ is representable in $\mathcal{C}$ by $\mathcal{H}$ if and only if $\beta = \mathrm{co}_{\mathcal{H}}$ on $\mathcal{C}$.

By Proposition 7.4, $\mathrm{co}_{\mathcal{E}^*} \subset \beta$ on $\mathcal{F}_{bc}$. In order to prove the converse for $A \in \mathcal{K}$, let $x \notin \mathrm{co}_{\mathcal{E}^*} A$. Then $f(x) > s(f, A)$ for some $f \in \mathcal{E}^*$. By the Chen–Lin characterization of extreme points (Lemma 3.6), there is a ball $C \in \mathcal{B}$ such that $A \subset C$ and $f(x) > s(f, C)$. Therefore $x \notin C$ and we have $x \notin \beta(A)$. □

Semidenting points admit the following characterization in terms of support functions.

Proposition 7.8. *Let $f \in S^*$. Then, $f \in \mathcal{SD}$ if and only if $s(f, A) = s(f, \beta(A))$ for each $A \in \mathcal{F}_{bc}$.*

Proof. By Proposition 7.3, if $f \in \mathcal{SD}$ then $\beta \subset \mathrm{co}_{\{f\}}$. Thus, for all $A \in \mathcal{F}_b$,

$$s(f, A) \le s(f, \beta(A)) \le s(f, \mathrm{co}_{\{f\}} A) = s(f, A).$$

Conversely, if $f(x) > s(f, A) = s(f, \beta(A))$, then $x \notin \beta(A)$, namely $A \subset C$ and $x \notin C$ for some $C \in \mathcal{B}$. Since this is valid for all x, A, indeed f is semidenting. □

Remark 7.9. If $\mathbf{E}$ is a Mazur space, it follows that $\mathcal{W}^*\mathcal{D} = \mathcal{SD}$ and we obtain [10, Corollary 2.4]. Also, recalling Corollary 7.5, $\mathbf{E}$ is Mazur and has the MIP if and only if $\mathcal{W}^*\mathcal{D} = S^*$ and we obtain [9, Proposition 5.3].

It has been asked whether the identity $\mathcal{SD} = \mathrm{cl}\,\mathcal{W}^*\mathcal{D}$ characterizes Mazur spaces [9, p. 413]. Indeed, that fails as the space $(\mathbf{R}^3, \|\cdot\|_1)$ shows: it is not a Mazur space [10, Proposition 3.4] but $\mathcal{W}^*\mathcal{D} = \mathcal{E}^* = \mathcal{SD}$ as every $f \in S^* \backslash \mathcal{E}^*$ has an associated hyperplane missing the Euclidean unit ball but hitting its ball hull.

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