Journal of Convex Analysis
Volume 17 (2010), No. 1, 159–171

On the Lower Bounds of Kottman Constants in Orlicz Function Spaces

Z. D. Ren
Department of Mathematics, University of California, Riverside, CA 92521, USA
mailing address: 21240 Pala Foxia Place, Moreno Valley, CA 92557, USA
jqiu98@hotmail.com

Received: December 22, 2006
Revised manuscript received: February 5, 2009

Let $L^{(\Phi)}(\Omega)$ and $L^{\Phi}(\Omega)$ be the Orlicz function spaces defined by an N-function Φ, equipped with the gauge norm and the Orlicz norm respectively, where $\Omega = [0,1]$ or $[0,\infty)$. The Kottman constants $K(L^{(\Phi)}(\Omega))$ and $K(L^{\Phi}(\Omega))$ were discussed in Rao and Ren [8, Ch. 5]. The author obtains some improvments on the lower bounds of these constants in Section 2 (Theorems 2.1 and 2.3). Several examples are given in Section 3 which will be used to make comments upon the papers of Yan [11] as well as Han and Li [4].

Keywords: Orlicz function space, Kottman constant and packing constant

1993 Mathematics Subject Classification: 46B30

1. Preliminaries

In a Banach space X, a sequence of balls with centers $x_1, x_2, \cdots$ and a fixed radius $r > 0$ is said to be packed into the unit ball $B(X)$ if $\|x_i\| \leqslant 1 - r,\ i = 1, 2, \cdots$ and $\|x_i - x_j\| \geqslant 2r,\ i \neq j, i, j = 1, 2, \cdots$.

Definition 1.1. Let X be an infinite dimensional Banach space. The packing constant $P(X)$ of X is defined by $P(X) = \sup\{r > 0$: infinitely many balls of radius r can be packed into $B(X)\}$.

Definition 1.2. For an infinite dimensional Banach space X, the Kottman constant $K(X)$ of X is defined by

$$K(X) = \sup\left\{\inf_{i\neq j} \|x_i - x_j\| : \{x_i\}_1^{\infty} \subset S(X)\right\}, \tag{1}$$

where $S(X)$ is the unit sphere of X.

Clearly, $1 \leqslant K(X) \leqslant 2$. Kottman [6] and Ye [12] found the relationship between $P(X)$ and $K(X)$ as follows.

Proposition 1.3. *For an infinite dimensional Banach space X, one has*

$$P(X) = \frac{K(X)}{2 + K(X)}. \tag{2}$$

ISSN 0944-6532 / $ 2.50

A proof of (2) can be found in Chen [1, p. 145] or Rao and Ren [8, p. 148]. A historical note on $P(X)$ and $K(X)$ is presented in [8, Ch. 5] when X is $L^p(1 < p < \infty)$ or an Orlicz space. In view of (2) we may study the lower bound only for $K(X)$ instead of that of $P(X)$ when X is an Orlicz function space.

Let $\Phi(u) = \int_0^{|u|} \varphi(t)dt$ denote an N-function and let (Ω, Σ, μ) be a σ-finite measure space with μ being nonatomic. The Orlicz function space $L^\Phi(\Omega)$ is defined by $L^\Phi(\Omega) = \{x(t) \in L^0(\Omega) : \rho_\Phi(\lambda x) < \infty$ for some $\lambda > 0\}$, in which $\rho_\Phi(\lambda x) = \int_\Omega \Phi(\lambda|x(t)|)dt$ and $L^0(\Omega)$ is the set of all measurable functions $x(t)$ with $|x(t)| < \infty$, a.e. on Ω. The gauge norm (or Luxemburg norm in some articles) and the Orlicz norm are given respectively by

$$\|x\|_{(\Phi)} = \inf\left\{c > 0 : \rho_\Phi\left(\frac{x}{c}\right) \leqslant 1\right\}, \qquad \|x\|_\Phi = \inf_{k>0} \frac{1}{k}[1 + \rho_\Phi(kx)].$$

As usual, we set $L^{(\Phi)}(\Omega) = (L^\Phi(\Omega),\ \|\cdot\|_{(\Phi)})$ and $L^\Phi(\Omega) = (L^\Phi(\Omega),\ \|\cdot\|_\Phi)$.

Definition 1.4. Let $\Phi(u) = \int_0^{|u|} \varphi(t)dt$ be an N-function.

(i) The first and the second characteristic functions of Φ are given respectively by

$$F_\Phi(t) = \frac{t\varphi(t)}{\Phi(t)}, \quad t > 0; \qquad G_\Phi(u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \quad u > 0, \tag{3}$$

where Φ^{-1} is the inverse of Φ on R^+.

(ii) The first quantitative indices of Φ are provided by

$$A_\Phi = \liminf_{t\to\infty} F_\Phi(t), \qquad B_\Phi = \limsup_{t\to\infty} F_\Phi(t); \tag{4}$$

$$A_\Phi^0 = \liminf_{t\to 0} F_\Phi(t), \qquad B_\Phi^0 = \limsup_{t\to 0} F_\Phi(t); \tag{5}$$

$$\overline{A}_\Phi = \inf_{t>0} F_\Phi(t), \qquad \overline{B}_\Phi = \sup_{t>0} F_\Phi(t). \tag{6}$$

(iii) The second quantitative indices of Φ are given by

$$\alpha_\Phi = \liminf_{u\to\infty} G_\Phi(u), \qquad \beta_\Phi = \limsup_{u\to\infty} G_\Phi(u); \tag{7}$$

$$\alpha_\Phi^0 = \liminf_{u\to 0} G_\Phi(u), \qquad \beta_\Phi^0 = \limsup_{u\to 0} G_\Phi(u); \tag{8}$$

$$\overline{\alpha}_\Phi = \inf_{u>0} G_\Phi(u), \qquad \overline{\beta}_\Phi = \sup_{u>0} G_\Phi(u). \tag{9}$$

Let $\Psi(v) = \int_0^{|v|} \psi(s)ds$ denote the complementary N-function to $\Phi(u)$. Analogously, we can define $A_\Psi,\ B_\Psi,\ A_\Psi^0,\ B_\Psi^0,\ \overline{A}_\Psi,\ \overline{B}_\Psi$ and $\alpha_\Psi,\ \beta_\Psi,\ \alpha_\Psi^0,\ \beta_\Psi^0,\ \overline{\alpha}_\Psi,\ \overline{\beta}_\Psi$.

Lemma 1.5 (Rao and Ren [8, p. 163]). *Let $\Phi(u) = \int_0^{|u|} \varphi(t)dt$ and $\Psi(v) = \int_0^{|v|} \psi(s)ds$ denote a pair of complementary N-functions. For $F_\Phi(t) = t\varphi(t)/\Phi(t),\ t > 0$ and $F_\Psi(s) = s\psi(s)/\Psi(s),\ s > 0$ if both φ and ψ are continuous on R^+, then we have*

$$\frac{1}{F_\Phi(t)} + \frac{1}{F_\Psi(s)} = 1, \quad s = \varphi(t) > 0. \tag{10}$$

Lemma 1.6 (Rao and Ren [8, p. 11]). *For a pair* $(\Phi,\ \Psi)$ *of complementary* N*-functions we have*

$$2\alpha_\Phi\beta_\Psi = 1 = 2\alpha_\Psi\beta_\Phi, \qquad 2\alpha_\Phi^0\beta_\Psi^0 = 1 = 2\alpha_\Psi^0\beta_\Phi^0, \qquad 2\overline{\alpha}_\Phi\overline{\beta}_\Psi = 1 = 2\overline{\alpha}_\Psi\overline{\beta}_\Phi. \tag{11}$$

Lemma 1.7 (Rao and Ren [8, p. 93]). *Let* $\Phi(u)$, $F_\Phi(t)$ *and* $G_\Phi(u)$ *be as in Definition* 1.4(i). *If* $F_\Phi(t)$ *is decreasing (increasing) on* $(0,\infty)$, *then* $G_\Phi(u)$ *is also decreasing (increasing) on* $(0,\infty)$.

Lemma 1.8 (Ren [9]). *Let* $\Phi(u)$, $F_\Phi(t)$ *and* $G_\Phi(u)$ *be as in Definition* 1.4(i).

(*i*) *If* $C_\Phi = \lim_{t\to\infty} F_\Phi(t)$ *exists* ($C_\Phi \leqslant \infty$), *then* $\gamma_\Phi = \lim_{u\to\infty} G_\Phi(u)$ *exists also and*

$$\gamma_\Phi = 2^{-1/C_\Phi}. \tag{12}$$

(*ii*) *If* $C_\Phi^0 = \lim_{t\to 0} F_\Phi(t)$ *exists* ($C_\Phi^0 \leqslant \infty$), *then* $\gamma_\Phi^0 = \lim_{u\to 0} G_\Phi(u)$ *exists also and*

$$\gamma_\Phi^0 = 2^{-1/C_\Phi^0}. \tag{13}$$

Definition 1.9.

(i) An N-function $\Phi(u)$ is said to satisfy the Δ_2-condition for large u (for small u, or for all $u \geqslant 0$), written often as $\Phi \in \Delta_2(\infty)(\Phi \in \Delta_2(0)$, or $\Phi \in \Delta_2)$, if there exist $C > 2$ and $u_0 > 0$ such that $\Phi(2u) \leqslant C\Phi(u)$ for $u \geqslant u_0$ (for $0 \leqslant u \leqslant u_0$, or for $u \geqslant 0$).

(ii) $\Phi(u)$ is said to obey the ∇_2-condition for large u (for small u, or for all $u \geqslant 0$), in symbol $\Phi \in \nabla_2(\infty)(\Phi \in \nabla_2(0)$, or $\Phi \in \nabla_2)$, if there are $l > 1$ and $u_0 > 0$ such that $2l\Phi(u) \leqslant \Phi(lu)$ for $u \geqslant u_0$ (for $0 \leqslant u \leqslant u_0$, or for $u \geqslant 0$).

Proposition 1.10. *For an* N*-function* Φ *with its quantitative indices as in Definition* 1.4, *we have the following assertions.*

(*i*) $\Phi \in \Delta_2(\infty)(\Phi \in \Delta_2(0))[\Phi \in \Delta_2]$ *if and only if* $B_\Phi < \infty(B_\Phi^0 < \infty)[\overline{B}_\Phi < \infty]$, *if and only if* $\beta_\Phi < 1(\beta_\Phi^0 < 1)[\overline{\beta}_\Phi < 1]$.

(*ii*) $\Phi \in \nabla_2(\infty)(\Phi \in \nabla_2(0))[\Phi \in \nabla_2]$ *if and only if* $A_\Phi > 1(A_\Phi^0 > 1)[\overline{A}_\Phi > 1]$, *if and only if* $\alpha_\Phi > 1/2(\alpha_\Phi^0 > 1/2)[\overline{\alpha}_\Phi > 1/2]$.

A proof of Proposition 1.10 can be found in Rao and Ren [7, Ch. 1] and [8, Ch. 1].

Remark 1.11. (i) Hudzik [5] proved that if X is a nonreflexive Banach lattice, then $P(X) = 1/2$, so that $K(X) = 2$ by (2). Since Orlicz spaces are Banach lattices, we see that $K(L^{(\Phi)}(\Omega)) = 2 = K(L^\Phi(\Omega))$ if $\Phi\overline{\in}\Delta_2(\infty)\cap\nabla_2(\infty)$ when $\mu(\Omega) < \infty$ (if $\Phi\overline{\in}\Delta_2\cap\nabla_2$ when $\mu(\Omega) = \infty$).

(ii) In order to construct the Rademacher sequence of functions we consider $\Omega = [0,1]$ or $[0,\infty)$ with the usual Lebesque measure μ in the next section. In this paper by $a \leqslant \{b, c\}$ we denote that $a \leqslant b$ and that $a \leqslant c$. For reflexive Orlicz function spaces $L^{(\Phi)}[0,1]$ and $L^\Phi[0,1]$(i.e., $\Phi \in \Delta_2(\infty)\cap\nabla_2(\infty)$) we have

$$\sqrt{2} \leqslant \left\{K(L^{(\Phi)}[0,1]),\ K(L^\Phi[0,1])\right\} < 2. \tag{14}$$

Similarly, (14) is true for $L^{(\Phi)}[0,\infty)$ and $L^\Phi[0,\infty)$ if $\Phi \in \Delta_2\cap\nabla_2$ (cf. Rao and Ren [8, Ch. 5]).

2. Lower Bounds for $K(L^{(\Phi)}[0,1])$ and $K(L^{\Phi}[0,1])$

Now we refine some results on the lower bounds for the Kottman constants $K(L^{(\Phi)}[0,1])$ and $K(L^{\Phi}[0,1])$ given in Rao and Ren [8, Ch. 5.3].

Theorem 2.1. *Let (Φ, Ψ) be a pair of complementary N-functions. For the Orlicz spaces $L^{(\Phi)}[0,1]$ and $L^{\Phi}[0,1]$ we have*

$$\max\left(\frac{1}{\alpha_\Phi}, 2\beta^*_\Phi\right) \leqslant K(L^{(\Phi)}[0,1]) \tag{15}$$

and

$$\max\left(2\beta_\Psi, \frac{1}{\alpha^*_\Psi}\right) \leqslant K(L^{\Phi}[0,1]), \tag{16}$$

where α_Φ and β_Ψ are given in Definition 1.4(iii), and

$$\beta^*_\Phi = \sup_{u\geqslant 1} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \qquad \alpha^*_\Psi = \inf_{v\geqslant 1} \frac{\Psi^{-1}(v)}{\Psi^{-1}(2v)}. \tag{17}$$

Proof. For (15) we have proved that $(\alpha_\Phi)^{-1} \leqslant K(L^{(\Phi)}[0,1])$ in Rao and Ren [8, pp. 167–169]. Now we will show

$$2\beta^*_\Phi \leqslant K(L^{(\Phi)}[0,1]), \tag{18}$$

which will complete the proof of (15). For any given $u \geqslant 1$ we construct the Rademacher functions $\{R_i(t), i \geqslant 1\}$ on the interval $[0, 1/u) \subset [0,1]$ as follows

$$R_i(t) = \sum_{k=1}^{2^i} (-1)^{k+1} \chi_{G_k^{(i)}}(t), \qquad G_k^{(i)} = \left[\frac{k-1}{2^i u}, \frac{k}{2^i u}\right), \quad 1 \leqslant k \leqslant 2^i,$$

where $\chi_{G_k^{(i)}}$ is the characteristic function of $G_k^{(i)}$. Let us define

$$x_i(t) = \Phi^{-1}(u) R_i(t), \quad i \geqslant 1. \tag{19}$$

It is seen that $|x_i(t)| = \Phi^{-1}(u)\chi_{[0,1/u)}(t)$ since $U_{k=1}^{2^i} G_k^{(i)} = [0, 1/u)$, so that $\|x_i\|_{(\Phi)} = 1, i \geqslant 1$. Further, $|x_i(t) - x_j(t)| = 2\Phi^{-1}(u)\chi_{G_{i,j}}(t)$ with $\mu(G_{i,j}) = 1/2u$ for $i \neq j$. Thus, we obtain for $i \neq j$

$$\|x_i - x_j\|_{(\Phi)} = \frac{2\Phi^{-1}(u)}{\Phi^{-1}\left(\dfrac{1}{\mu(G_{i,j})}\right)} = \frac{2\Phi^{-1}(u)}{\Phi^{-1}(2u)},$$

which proves (18) by (1) and (17).

Next we will prove (16). In Rao and Ren [8, p. 170] it was shown that

$$2\beta_\Psi \leqslant K(L^{\Phi}[0,1]). \tag{20}$$

Now we will prove the inequality

$$\frac{1}{\alpha_\Psi^*} \leqslant K(L^\Phi[0,1]), \tag{21}$$

which will finish the proof of (16). For any given $u \geqslant 1$ we define

$$y_i(t) = \frac{u}{\Psi^{-1}(u)} R_i(t), \quad i \geqslant 1, \tag{22}$$

where $R_i(t)$ is as in (19). Then $\|y_i\|_\Phi = 1$ and if $i \neq j$ we see that $|y_i(t) - y_j(t)| = [2u/\Psi^{-1}(u)]\chi_{G_{i,j}}(t)$ with $\mu(G_{i,j}) = 1/2u$ and that

$$\|y_i - y_j\|_\Phi = \frac{2u}{\Psi^{-1}(u)} \mu(G_{i,j}) \Psi^{-1}\left(\frac{1}{\mu(G_{i,j})}\right) = \frac{\Psi^{-1}(2u)}{\Psi^{-1}(u)},$$

implying (21) by (1) and (17) since $u \geqslant 1$ is arbitrary. $\square$

Remark 2.2. (i) Inequality (18) is the refinement of that $2\beta_\Phi \leqslant K(L^{(\Phi)}[0,1])$ given in Rao and Ren [8, p. 167] because

$$2\beta_\Phi \leqslant 2\beta_\Phi^*. \tag{23}$$

For an N-function $\Phi(u) = \int_0^{|u|} \varphi(t)dt$ if $F_\Phi(t)(= t\varphi(t)/\Phi(t),\ t > 0)$ is strictly decreasing on $(0, \infty)$, then there is a strict inequality in (23). For instance, consider $\Phi(u) = u^2 \ln(1 + |u|)$. Clearly, $F_\Phi(t) = 2 + t/(1+t)\ln(1+t),\ t > 0,\ C_\Phi^0 = \lim_{t\to 0} F_\Phi(t) = 3,\ C_\Phi = \lim_{t\to\infty} F_\Phi(t) = 2$, and $F_\Phi(t)$ is strictly decreasing on $(0,\infty)$. For any $t > 0$ we have that $3 > t\varphi(t)/\Phi(t) > 2$, so that

$$\int_{t_1}^{t_2} \frac{3}{t} dt > \int_{t_1}^{t_2} \frac{\varphi(t)}{\Phi(t)} dt > \int_{t_1}^{t_2} \frac{2}{t} dt$$

if $0 < t_1 < t_2 < \infty$. By letting $t_1 = \Phi^{-1}(1)$ and $t_2 = \Phi^{-1}(2)$ we obtain from Lemma 1.8 that

$$\sqrt{2} = 2\beta_\Phi < 2\beta_\Phi^* = \frac{2\Phi^{-1}(1)}{\Phi^{-1}(2)} < 2\beta_\Phi^0 = 2^{2/3}.$$

(ii) Inequality (20) is the improvement of that $2\alpha_\Psi \leqslant K(L^\Phi[0,1])$ (or equivalently by (2), $[1 + (\alpha_\Psi)^{-1}]^{-1} \leqslant P(L^\Phi[0,1])$ given in Cleaver [2]).

(iii) It should be also noted that (21) refines both that $(\alpha_\Psi)^{-1} \leqslant K(L^\Phi[0,1])$ given by Rao and Ren [8, p. 170] and that $\Psi^{-1}(2)/\Psi^{-1}(1) \leqslant K(L^\Phi[0,1])$ given by Cleaver [2] since

$$\left\{\frac{1}{\alpha_\Psi}, \frac{\Psi^{-1}(2)}{\Psi^{-1}(1)}\right\} \leqslant \frac{1}{\alpha_\Psi^*}.$$

A special situation of Theorem 2.1 is as follows.

Theorem 2.3. *Let $\Phi(u) = \int_0^{|u|} \varphi(t)dt$ and $\Psi(v) = \int_0^{|v|} \psi(s)ds$ denote a pair of complementary N-functions with both φ and ψ being continuous on R^+. We assume that $\Phi \in \Delta_2(\infty) \cap \nabla_2(\infty)$, i.e., $L^{(\Phi)}[0,1]$ and $L^{\Phi}[0,1]$ are reflexive.*

(*i*) *If $F_\Phi(t)(= t\varphi(t)/\Phi(t),\ t > 0)$ is decreasing on $(0,\infty)$, then* (15) *reduces to*

$$\max\left(2^{1/C_\Phi},\ \frac{2\Phi^{-1}(1)}{\Phi^{-1}(2)}\right) \leqslant K(L^{(\Phi)}[0,1]) \tag{24}$$

and (16) *reduces to*

$$\max\left(2^{1/C_\Phi},\ \frac{\Psi^{-1}(2)}{\Psi^{-1}(1)}\right) \leqslant K(L^{\Phi}[0,1]), \tag{25}$$

where $C_\Phi = \lim_{t\to\infty} F_\Phi(t)$.

(*ii*) *If $F_\Phi(t)$ is increasing on $(0,\infty)$, then* (15) *and* (16) *reduce to*

$$\max(2^{1/C_\Phi},\ 2^{1-1/C_\Phi}) \leqslant \{K(L^{(\Phi)}[0,1]),\ K(L^{\Phi}[0,1])\}. \tag{26}$$

Proof. (*i*) If $F_\Phi(t)$ is decreasing on $(0,\infty)$, then $F_\Psi(s)(= s\varphi(s)/\Psi(s),\ s > 0)$ is increasing on $(0,\infty)$ in view of (10) in Lemma 1.5. Thus, $G_\Phi(u)(= \Phi^{-1}(u)/\Phi^{-1}(2u),\ u > 0)$ is decreasing and $G_\Psi(v)(= \Psi^{-1}(v)/\Psi^{-1}(2v),\ v > 0)$ is increasing on $(0,\infty)$ by Lemma 1.7. From (17) we have

$$\beta_\Phi^* = G_\Phi(1) = \frac{\Phi^{-1}(1)}{\Phi^{-1}(2)}, \qquad \alpha_\Psi^* = G_\Psi(1) = \frac{\Psi^{-1}(1)}{\Psi^{-1}(2)}. \tag{27}$$

On the other hand, since both limits $C_\Phi = \lim_{t\to\infty} F_\Phi(t)$ and $C_\Psi = \lim_{s\to\infty} F_\Psi(s)$ exist one has from (10) that

$$\frac{1}{C_\Phi} + \frac{1}{C_\Psi} = 1. \tag{28}$$

It follows from (12) in Lemma 1.8 and (28) that

$$\alpha_\Phi = \gamma_\Phi = 2^{-1/C_\Phi}, \qquad 2\beta_\Psi = 2\gamma_\Psi = 2^{1-1/C_\Psi} = 2^{1/C_\Phi}. \tag{29}$$

Thus, (24) and (25) follow from (15), (16), (27) and (29).

(*ii*) If $F_\Phi(t)$ is inereasing on $(0,\infty)$, then $F_\Psi(s)$ is decreasing on $(0,\infty)$ again by (10) in Lemma 1.5, so that $G_\Phi(u)$ is increasing and $G_\Psi(v)$ is decreasing on $(0,\infty)$ by Lemma 1.7. Also, both G_Φ and C_Ψ exist, so that (29) holds. In this case we have from (12) and (28).

$$2\beta_\Phi^* = 2\gamma_\Phi = 2^{1-1/C_\Phi}, \qquad \alpha_\Psi^* = \gamma_\Psi = 2^{-1/C_\Psi} = 2^{-1+1/C_\Phi}. \tag{30}$$

Finally, (26) follows from (15), (16), (29) and (30). □

At the end of this section we recall a known result concerning with the lower bounds of $K(L^{(\Phi)}[0,\infty))$ and $K(L^{\Phi}[0,\infty))$.

Theorem 2.4. *Let (Φ, Ψ) be a pair of complementary N-functions. We assume that $\Phi \in \Delta_2 \cap \nabla_2$. Then one has*

$$\max\left(\frac{1}{\overline{\alpha}_\Phi},\ 2\overline{\beta}_\Phi\right) \leqslant \{K(L^{(\Phi)}[0,\infty)),\ K(L^\Phi[0,\infty))\}, \tag{31}$$

where $\overline{\alpha}_\Phi$ and $\overline{\beta}_\Phi$ are given by (9) *in Definition* 1.4(iii).

Proof. For $K(L^{(\Phi)}[0,\infty))$, (31) was proved in Rao and Ren [8, p. 168]. For $K(L^\Phi[0,\infty))$, it was shown in [8, p. 170] that

$$\max\left(2\overline{\beta}_\Psi, \frac{1}{\overline{\alpha}_\Psi}\right) \leqslant K(L^\Phi[0,\infty)),$$

which proves (31) by the third one in (11) of Lemma 1.6. □

3. Examples

To illustrate the theorems obtained in Section 2 and to make comments upon some related papers dealing with the Kottman and packing constants of Orlicz function spaces we now present several examples for computations on the lower bounds of $K(L^{(\Phi)}(\Omega))$ and $K(L^\Phi(\Omega))$ when $\Omega = [0,1]$ or $[0,\infty)$.

Example 3.1. Consider N-function $\Phi_p(u) = |u|^p \ln(1+|u|)$ with $3/2 \leqslant p \leqslant 2$. We assert that there exists a constant $0 < \lambda \leqslant 1/2$ such that

$$K(L^{(\Phi_p)}[0,1]) > 2^{1/p}, \quad p \in (2-\lambda, 2]. \tag{32}$$

Proof. Since $\varphi_p(t) = \Phi_p'(t) = pt^{p-1}\ln(1+t) + t^p/(1+t),\ t > 0$, the function

$$F_{\Phi_p}(t) = \frac{t\varphi_p(t)}{\Phi_p(t)} = p + \frac{t}{(1+t)\ln(1+t)}, \quad t > 0 \tag{33}$$

is decreasing on $(0,\infty)$, $C^0_{\Phi_p} = \lim_{t\to 0} F_{\Phi_p}(t) = p+1$ and $C_{\Phi_p} = \lim_{t\to\infty} F_{\Phi_p}(t) = p$. By letting $t_1 = \Phi_p^{-1}(1)$ and $t_2 = \Phi_p^{-1}(2)$ we have from (33)

$$\begin{aligned} \ln 2 = \int_{t_1}^{t_2} \frac{\varphi_p(t)}{\Phi_p(t)} dt &= \int_{t_1}^{t_2} \frac{p}{t} dt + \int_{t_1}^{t_2} \frac{1}{(1+t)\ln(1+t)} dt \\ &= \ln\left[\frac{\Phi_p^{-1}(2)}{\Phi_p^{-1}(1)}\right]^p + \ln f(p), \end{aligned}$$

where $f(p) = \ln(1+\Phi_p^{-1}(2))/\ln(1+\Phi_p^{-1}(1))$. Let $C = \min\{f(p) : 3/2 \leqslant p \leqslant 2\} - 1$. Then $C > 0$ since $f(p) > 1$ for every $p \in [3/2, 2]$. It follows from the above that $2^{1/p} \geqslant [\Phi_p^{-1}(2)/\Phi_p^{-1}(1)](1+C)^{1/p}$, or

$$\frac{2\Phi_p^{-1}(1)}{\Phi_p^{-1}(2)} \geqslant (1+C)^{1/p} 2^{1-1/p}, \quad p \in [3/2, 2]. \tag{34}$$

Let us choose $\lambda = \min\{\log_2(1+C), 1/2\}$. Then $0 < \lambda \leqslant 1/2$, and condition $p \in (2-\lambda, 2]$ implies that

$$(1 + C)^{1/p} 2^{1-1/p} > 2^{1/p}. \tag{35}$$

Finally, (32) follows from (24) in Theorem 2.3(i), (34) and (35). □

Example 3.2. Gribanov [3] introduced N-function

$$\Phi(u) = u^2 e^{-1/|u|} \tag{36}$$

with $\Phi(0) = 0$. For the corresponding Orlicz function spaces $L^{(\Phi)}[0, 1]$ and $L^{\Phi}[0, 1]$ we assert that

$$\left\{K(L^{(\Phi)}[0, 1]),\ K(L^{\Phi}[0, 1])\right\} > \sqrt{2}. \tag{37}$$

Proof. Clearly, $\varphi(t) = \Phi'(t) = (2t + 1)e^{-1/t}$ if $t > 0$ and the function

$$F_\Phi(t) = \frac{t\varphi(t)}{\Phi(t)} = 2 + \frac{1}{t}, \quad t > 0 \tag{38}$$

is strictly decreasing on $(0, \infty)$. Note that $\Phi \in \Delta_2(\infty) \cap \nabla_2(\infty)$ since $C_\Phi = \lim_{t\to\infty} F_\Phi(t) = 2$. By letting $t_1 = \Phi^{-1}(1)$ and $t_2 = \Phi^{-1}(2)$ we have from (38)

$$\ln 2 = \int_{t_1}^{t_2} \frac{\varphi(t)}{\Phi(t)} dt = \int_{t_1}^{t_2} \left(\frac{2}{t} + \frac{1}{t^2}\right) dt = \ln\left[\frac{\Phi^{-1}(2)}{\Phi^{-1}(1)}(1 + C)\right]^2, \tag{39}$$

where

$$C = \exp\left\{\frac{1}{2}\left(\frac{1}{\Phi^{-1}(1)} - \frac{1}{\Phi^{-1}(2)}\right)\right\} - 1 > 0.$$

By Theorem 2.3(i) and (39) we obtain

$$K(L^{(\Phi)}[0, 1]) \geqslant \frac{2\Phi^{-1}(1)}{\Phi^{-1}(2)} = (1 + C)\sqrt{2} > \sqrt{2}. \tag{40}$$

Next we show (37) for the space $L^{\Phi}[0, 1]$ equipped with the Orlicz norm. Let $\Psi(v)$ be the complementary N-function to $\Phi(u)$. Then $F_\Psi(s)(= s\Psi'(s)/\Psi(s),\ s > 0)$ is strictly increasing on $(0,\ \infty)$ by Lemma 1.5, $C_\Psi = \lim_{s\to\infty} F_\Psi(s) = 2$ in view of (28) and $G_\Psi(v)(= \Psi^{-1}(v)/\Psi^{-1}(2v),\ v > 0)$ is strictly increasing on $(0,\ \infty)$ by Lemma 1.7. Therefore, we see that

$$\frac{\Psi^{-1}(1)}{\Psi^{-1}(2)} = G_\Psi(1) < \lim_{v\to\infty} G_\Psi(v) = \gamma_\Psi = 2^{-1/C_\Psi} = \frac{1}{\sqrt{2}}.$$

The above and Theorem 2.3(i) imply

$$K(L^{\Phi}[0, 1]) \geqslant \frac{\Psi^{-1}(2)}{\Psi^{-1}(1)} > \sqrt{2}. \tag{41}$$

Finally, (37) follows from (40) and (41). □

Example 3.3. Consider $\Phi_p(u) = |u|^p \ln(1+|u|)$ as in Example 3.1 but $2 \leqslant p < \infty$. We assert that

$$\{K(L^{(\Phi_p)}[0,1]),\ K(L^{\Phi_p}(0,1])\} > 2^{1-1/p}, \quad 2 \leqslant p < \infty. \tag{42}$$

Proof. Since $F_{\Phi_p}(t)$ is decreasing on $(0,\infty)$ and $C_{\Phi_p} = p$ (cf. (33) in Example 3.1), by Theorem 2.3(i) and Lemmas 1.7 and 1.8 we have

$$K(L^{(\Phi_p)}[0,1]) \geqslant \frac{2\Phi_p^{-1}(1)}{\Phi_p^{-1}(2)} > 2 \lim_{u\to\infty} G_{\Phi_p}(u) = 2\gamma_{\Phi_p} = 2^{1-1/p}. \tag{43}$$

Let $\Psi_p(v)$ be the complementary N-function to $\Phi_p(u)$. Then $F_{\Psi_p}(s) (= s\Psi_p'(s)/\Psi_p(s),\ s > 0)$ is increasing on $(0,\infty)$ and $C_{\Psi_p} = \lim_{s\to\infty} F_{\Psi_p}(s) = p/(p-1)$ by Lemma 1.5. Since $G_{\Psi_p}(v) (= \Psi_p^{-1}(u)/\Psi_p^{-1}(2u),\ u > 0)$ is increasing on $(0,\infty)$, one has

$$\frac{\Psi_p^{-1}(1)}{\Psi_p^{-1}(2)} = G_{\Psi_p}(1) < \lim_{v\to\infty} G_{\Psi_p}(v) = 2^{-1/C_{\Psi_p}} = 2^{-1+1/p}.$$

It follows from Theorem 2.3(i) and the above that

$$K(L^{\Phi_p}[0,1]) \geqslant \frac{\Psi_p^{-1}(2)}{\Psi_p^{-1}(1)} > 2^{1-1/p},$$

which proves (42) together with (43). □

Example 3.4. Let $M(u) = u^2(1 + e^{-1/|u|})$. We claim that

$$\{K(L^{(M)}[0,\infty)),\ K(L^M[0,\infty))\} \geqslant \frac{2M^{-1}(u_0)}{M^{-1}(2u_0)}, \tag{44}$$

where u_0 is the solution of equation $G_M'(u) = 0$ in which $G_M(u) = M^{-1}(u)/M^{-1}(2u)$, $u > 0$.

Proof. The function

$$F_M(t) = \frac{tM'(t)}{M(t)} = 2 + \frac{1/t}{1+e^{1/t}}, \quad t > 0$$

is not monotonic on $(0,\infty)$ because $C_M^0 = \lim_{t\to 0} F_M(t) = 2 = \lim_{t\to\infty} F_M(t) = C_M$ and $F_M(t) > 2$ for each $t > 0$. Given any $u > 0$, if $t_1 = M^{-1}(u)$ and $t_2 = M^{-1}(2u)$, then we have that

$$\ln 2 = \int_{t_1}^{t_2} \frac{M'(t)}{M(t)} dt > \int_{t_1}^{t_2} \frac{2}{t} dt = \ln\left[\frac{M^{-1}(2u)}{M^{-1}(u)}\right]^2,$$

so that $2M^{-1}(u)/M^{-1}(2u) > \sqrt{2}$. Note that $\overline{\alpha}_M = \alpha_M^0 = \alpha_M = 1/\sqrt{2}$. Finally, it follows from Theorem 2.4 that

$$\{K(L^{(M)}[0,\infty)),\ K(L^M[0,\infty))\} \geqslant \max\left(\frac{1}{\overline{\alpha}_M},\ 2\overline{\beta}_M\right) = 2\overline{\beta}_M,$$

where $2\overline{\beta}_M = \sup_{u>0} 2M^{-1}(u)/M^{-1}(2u) = 2M^{-1}(u_0)/M^{-1}(2u_0)$, proving (44). □

4. Comments on Related Papers

Yan [11] as well as Han and Li [4] announced that they found the exact values of the Kottman constant and the packing constant of the Orlicz function spaces $L^{(\Phi)}[0,1]$ and $L^{\Phi}[0,1]$ defined by some special N-functions. To make comments upon their papers we start with the following.

Theorem 2.3 in Yan [11]. *Let $\Phi(u) = \int_0^{|u|} \varphi(t)$ be an N-function and let $F_\Phi(t) = t\varphi(t)/\Phi(t),\ t > 0$.*

(i) If $F_\Phi(t)$ is decreasing on $(0,\infty)$ and $1 < C_\Phi < 2$, where $C_\Phi = \lim_{t\to\infty} F_\Phi(t)$, then

$$K(L^{(\Phi)}[0,1]) = K(L^{\Phi}[0,1]) = 2^{1/C_\Phi}. \tag{45}$$

(ii) If $F_\Phi(t)$ is increasing on $(0,\infty)$ and $C_\Phi > 2$, then

$$K(L^{(\Phi)}[0,1]) = K(L^{\Phi}[0,1]) = 2^{1-1/C_\Phi}.$$

Comment 4.1. At least, (i) of the above theorem is incorrect for $L^{(\Phi)}[0,1]$.

Proof. Consider N-function $\Phi_p(u) = |u|^p \ln(1+|u|)$ with $1 < p < 2$. Then $F_{\Phi_p}(t)$ is obviously decreasing on $(0,\infty)$ and $C_{\Phi_p} = p$ (cf. (33)). If (45) were true, then $K(L^{(\Phi_p)}[0,1]) = 2^{1/p}$ as given in Yan [11, Example 2.8]. This contradicts (32) in Example 3.1 in Section 3. $\square$

To make comment on the paper of Han and Li [4] we have to deal with the $M_\Delta(\infty)$-condition.

Definition 4.2 (Salehov [10]). An N-function $\Phi(u)$ is said to satisfy the M_Δ-condition for large u, in symbol $\Phi \in M_\Delta(\infty)$, if

$$\lim_{u\to\infty} \frac{\ln \Phi(u)}{\ln u} = p < \infty. \tag{46}$$

Remark 4.3. (i) Salehov [10] showed that if $\Phi \in M_\Delta(\infty)$ then $\Phi \in \Delta_2(\infty)$. The converse is not true in general.

(ii) If $\Phi \in \Delta_2(\infty)$, $\Phi'(u)$ is continuous for large u and the limit $C_\Phi = \lim_{t\to\infty} t\Phi'(t)/\Phi(t)$ exists, then $\Phi \in M_\Delta(\infty)$. For, by L'Hôpital's rule we have

$$\lim_{u\to\infty} \frac{\ln \Phi(u)}{\ln u} = \lim_{u\to\infty} \frac{(\ln \Phi(u))'}{(\ln u)'} = \lim_{u\to\infty} \frac{u\Phi'(u)}{\Phi(u)} = C_\Phi$$

and $C_\Phi < \infty$ since $\Phi \in \Delta_2(\infty)$(cf. Proposition 1.10 in Section 1).

(iii) Let $\Psi(v)$ be the complementary N-function to $\Phi(u)$. Then $\Phi \in M_\Delta(\infty)$ if and only if

$$\lim_{v\to\infty} \frac{\ln \Psi(v)}{\ln v} = q > 1. \tag{47}$$

We can write $\Psi \in M_\nabla(\infty)$ if (47) holds. The proof is as follows. If $\Phi \in M_\Delta(\infty)$ and $p = 1$ in (46), then for any given $\varepsilon > 0$ there is a $u_0 > 0$ such that $\ln \Phi(u)/\ln u \leqslant 1+\varepsilon$

or $\Phi(u) \leqslant u^{1+\varepsilon}$ if $u \geqslant u_0$. By Proposition 2 in Rao and Ren [7, p. 15] there is a $v_0 > 0$ such that $\Psi(v) \geqslant v^{(1+\varepsilon)/\varepsilon}$ if $v \geqslant v_0$ since $v^{(1+\varepsilon)/\varepsilon}$ is complementary to $u^{1+\varepsilon}$. Thus, $\lim_{v\to\infty} \ln \Psi(v)/\ln v = \infty$ since ε is arbitrary. If $\Phi \in M_\Delta(\infty)$ and $p > 1$ in (46), then for any given $0 < \varepsilon < p-1$ there is a $u_1 > 0$ such that $p - \varepsilon \leqslant \ln \Phi(u)/\ln u \leqslant p+\varepsilon$ or $u^{p-\varepsilon} \leqslant \Phi(u) \leqslant u^{p+\varepsilon}$ if $u \geqslant u_1$, which implies that there exists a $v_1 > 0$ such that

$$v^{(p-\varepsilon)/(p-\varepsilon-1)} \geqslant \Psi(v) \geqslant v^{(p+\varepsilon)/(p+\varepsilon-1)}, \quad v \geqslant v_1,$$

proving (47) with $q = p/(p-1)$. Similarly, (47) implies (46).

In view of Proposition 1.3 the main result in the paper of Han and Li [4] can be stated as follows.

Theorem 5 in [4]. *For an N-function $\Phi(u)$, if $\Phi \in M_\Delta(\infty)$ and $\lim_{u\to\infty} \ln \Phi(u)/\ln u = p > 1$, then*

$$K(L^\Phi[0,1]) = \begin{cases} 2^{1/p}, & 1 < p \leqslant 2; \\ 2^{1-1/p}, & 2 < p < \infty. \end{cases} \tag{48}$$

Comment 4.4. Formula (48) fails of success.

Proof. Consider $\Phi(u) = u^2 e^{-1/|u|}$ as given in Example 3.2. Then $\Phi(u)$ satisfies the conditions of the above theorem since $\lim_{u\to\infty} \ln \Phi(u)/\ln u = 2$. If (48) were true, then $K(L^\Phi[0,1]) = \sqrt{2}$, which contradicts (37) of Example 3.2 in Section 3.

Next we consider $\Phi_p(u) = |u|^p \ln(1+|u|)$ with $2 < p < \infty$. It is seen that $\Phi_p \in M_\Delta(\infty)$ since $\lim_{u\to\infty} \ln \Phi_p(u)/\ln u = p$. If (48) were true, then $K(L^{\Phi_p}[0,1]) = 2^{1-1/p}$, which is impossible by (42) of Example 3.3 in Section 3. □

Go further, we need the following.

Definition 4.5 (Cleaver [2]). Let $M(u)$ be an N-function and let $M_0(u) = u^2$. If $0 \leqslant s \leqslant 1$ we define $\Phi_s(u)$ to be the inverse of

$$\Phi_s^{-1}(u) = [M^{-1}(u)]^{1-s}[M_0^{-1}(u)]^s = [M^{-1}(u)]^{1-s}u^{s/2}, \quad u \geqslant 0. \tag{49}$$

It was proved that $\Phi_s^{-1}(u)$ has inverse $\Phi_s(u)$ and $\Phi_s(u)$ is an N-function if $\Phi_s(-u) = \Phi_s(u)$ for $u < 0$. Cf. Rao and Ren [7, p. 223]. Clearly, $\Phi_s(u)|_{s=0} = M(u)$ and $\Phi_s(u)|_{s=1} = u^2$. The author proved the following.

Theorem 4.6 (Ren [9]). *For any N-function $M(u)$, let $\Phi_s(u)$ be defined by its inverse $\Phi_s^{-1}(u)$ in (49). If $0 < s \leqslant 1$, then $\Phi_s \in \Delta_2 \cap \nabla_2$.*

A proof of Theorem 4.6 can be found also in Rao and Ren [8, p. 40]. For convenience and clearness we use the following technical term.

Definition 4.7. N-function $\Phi(u)$ is said to be an intermediate N-function if there exist an N-function $M(u)$ and a constant $0 < s < 1$ such that $\Phi^{-1}(u) = [M^{-1}(u)]^{1-s}u^{s/2} (u \geqslant 0)$, i.e., $\Phi^{-1}(u) = \Phi_s^{-1}(u)$ with $\Phi_s^{-1}(u)$ being in (49).

Example 4.8. Let $1 < p < \infty$ and let $\Phi_p(u)$ be the inverse of

$$\Phi_p^{-1}(u) = [\ln(1+u)]^{1/2p} u^{1/4}, \quad u \geqslant 0.$$

Then $\Phi_p(u)$ is an intermediate N-function between $M_p(u) = e^{|u|^p} - 1$ and u^2 with $s = 1/2$, since $M_p^{-1}(u) = [\ln(1+u)]^{1/p} (u \geqslant 0)$. For the intermediate Orlicz spaces $L^{(\Phi_p)}(\Omega)$ and $L^{\Phi_p}(\Omega)$, by Theorem 9 in Rao and Ren [8, p. 174] we have

$$K(L^{(\Phi_p)}(\Omega)) = K(L^{\Phi_p}(\Omega)) = 2^{3/4},$$

where $\Omega = [0,1]$ or $[0,\infty)$.

By Definitions 4.5 and 4.7, a theorem in Yan [11] can be stated as follows.

Theorem 2.2 in Yan [11]. *Let $\Phi(u) = \int_0^{|u|} \varphi(t)dt$ denote an N-function and let $F_\Phi(t) = t\varphi(t)/\Phi(t)$, $t > 0$. Then $\Phi(u)$ is an intermediate N-function if one of the following conditions is satisfied:*

(*i*) *$F_\Phi(t)$ is decreasing on $(0,\infty)$ and $1 < C_\Phi < 2$, where $C_\Phi = \lim_{t\to\infty} F_\Phi(t)$;*
(*ii*) *$F_\Phi(t)$ is increasing on $(0,\infty)$ and $C_\Phi > 2$.*

Comment 4.9. The above theorem is wrong.

Proof. (a) Consider $\Phi(u) = |u|^{3/2} e^{-1/|u|}$ with $\Phi(0) = 0$. Clearly, $\Phi(u)$ is an N-function since $\varphi(t) = \Phi'(t) > 0$ and $\varphi'(t) > 0$ if $t > 0$. Note that $F_\Phi(t) = 3/2 + 1/t (t > 0)$, is decreasing on $(0,\infty)$ and $C_\Phi = 3/2$, i.e., $\Phi(u)$ satisfies condition (*i*). If $\Phi(u)$ were an intermediate N-function, then $\Phi \in \Delta_2 \cap \nabla_2$ by Theorem 4.6. A contradiction since $\Phi \overline{\in} \Delta_2(0)$ because $C_\Phi^0 = \lim_{t\to 0} F_\Phi(t) = \infty$.

(b) Let $\Psi(v)$ be the complementary N-function to $\Phi(u)$. Then $F_\Psi(s) (= s\Psi'(s)/\Psi(s),\ s > 0)$ is increasing on $(0,\infty)$ and $C_\Psi = \lim_{s\to\infty} F_\Psi(s) = 3$ by Lemma 1.5, i.e., $\Psi(v)$ satisfies condition (*ii*). Since $\Psi \overline{\in} \nabla_2(0)$, $\Psi(v)$ does not an intermediate N-function again by Theorem 4.6. □

We conclude this paper by the following.

Remark 4.10. For a given N-function $\Phi(u)$, in order that $\Phi(u)$ becomes an intermediate N-function (cf. Definition 4.7), it is necessary but not sufficient that $\Phi \in \Delta_2 \cap \nabla_2$. For instance, we consider N-function

$$\Phi_p(u) = |u|^{2p} + 2|u|^p, \quad 2 < p < \infty. \tag{50}$$

Clearly, $\Phi_p \in \Delta_2 \cap \nabla_2$. We assert that $\Phi_p(u)$ does not an intermediate N-function for large p. Suppose to the contrary, there exist an N-function $M(u)$ and a constant $0 < s < 1$ such that $\Phi_p^{-1}(u) = [M^{-1}(u)]^{1-s} u^{s/2}$, $u \geqslant 0$. Then

$$M^{-1}(u) = \frac{[\Phi_p^{-1}(u)]^{1/(1-s)}}{u^{s/2(1-s)}} = \frac{\left(\sqrt{u+1} - 1\right)^{1/p(1-s)}}{u^{s/2(1-s)}}, \quad u > 0,$$

which implies that for $p > 2/s$

$$\lim_{u\to 0}\frac{M^{-1}(u)}{M^{-1}(2u)} = 2^{(ps-2)/2p(1-s)} > 1$$

and

$$\lim_{u\to \infty}\frac{M^{-1}(u)}{M^{-1}(2u)} = 2^{(ps-1)/2p(1-s)} > 1.$$

In other words, there are $u_1 > 0$ and $u_2 > 0$ such that $M^{-1}(u) > M^{-1}(2u)$ if $0 < u < u_1$ or $u > u_2$. A contradiction because the inverse function of any N-function is strictly increasing on $[0, \infty)$.

Acknowledgements. The author would like to thank the referee, who has informed him that first estimates for the Kottman constants of Orlicz spaces were given by H. Hudzik and T. Landes (Packing constant in Orlicz spaces equipped with the Luxemburg norm, Boll. Unione Mat. Ital., VII. Ser., A 9 (1995) 225–237), and that their result was improved by Rao and Ren [8, Ch. 5.3]. The present author considered this problem in 1985 (cf. **MR** 87j: 46061; Rao and Ren [7, p. 256]).

References

[1] S. T. Chen: Geometry of Orlicz Spaces, Dissertationes Math. 356, Polish Academy of Sciences, Warsaw (1996).

[2] C. E. Cleaver: Packing spheres in Orlicz spaces, Pacific J. Math. 65 (1976) 325–335.

[3] Ju. I. Gribanov: Zur Theorie der Orliczschen Koordinatenräume und der unendlichen Matrizenabbildungen, Wiss. Z. Humboldt-Univ. Berlin, Math.-Naturw. Reihe 8(3) (1958/59) 339–349.

[4] J. Han, X. L. Li: Exact value of packing spheres constant in class of Orlicz function spaces (in Chinese), J. Tongji Univ. 30(7) (2002) 895–899.

[5] H. Hudzik: Every nonreflexive Banach lattice has the packing constant equal to 1/2, Collect. Math. 44 (1993) 129–134.

[6] C. A. Kottman: Packing and reflexivity in Banach spaces, Trans. Amer. Math. Soc. 150 (1970) 565–576.

[7] M. M. Rao, Z. D. Ren: Theory of Orlicz Spaces, Marcel Dekker, New York (1991).

[8] M. M. Rao, Z. D. Ren: Applications of Orlicz Spaces, Marcel Dekker, New York (2002).

[9] Z. D. Ren: Nonsquare constants of Orlicz spaces, in: Stochastic Processes and Functional Analysis (Riverside / CA, 1994), J. A. Goldstein et al. (ed.), Lect. Notes Pure Appl. Math. 186, Marcel Dekker, New York (1997) 179–197.

[10] D. V. Salehov: On a property of N-functions, Math. Notes 4 (1968) 662–667.

[11] Y. Q. Yan: On the exact value of packing spheres in a class of Orlicz function spaces, J. Convex Analysis 11(2) (2004) 391–400.

[12] Y. N. Ye: Packing spheres in Orlicz sequence spaces (in Chinese), Chin. Ann. Math., Ser. A 4 (1983) 487–493.