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LMI Representations of the Convex Hulls of Quadratic Basic Semialgebraic Sets

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In this paper, we are motivated by the question of when a convex semialgebraic set in $\mathbb{R}^n$ is equal to the feasible set of a linear matrix inequality (LMI). Given a basic semialgebraic set, $\mathcal{V}$, which is defined by quadratic polynomials, we restrict our attention to closure of its convex hull, namely $\overline{\mathbf{co}(\mathcal{V})}$. Our main result is that $\overline{\mathbf{co}(\mathcal{V})}$ is equal to the intersection of a finite number of LMI sets and the halfspaces supporting $\mathcal{V}$ along a particular subset of the boundary of $\mathcal{V}$. As a corollary, we show that in $\mathbb{R}^2$, the halfspaces of concern are finite in number, so that an LMI representation for $\overline{\mathbf{co}(\mathcal{V})}$ always exists.

1. Introduction

A linear matrix inequality (LMI) is a constraint of the form

$$F(x) = F_0 + F_1 x_1 + \cdots + F_n x_n \succeq 0, \tag{1}$$

where F_i are $p \times p$ real symmetric matrices. It has recently been shown that the solvability conditions for numerous problems in systems and control theory and several other fields can be expressed as LMI conditions [1]. Since it is easily shown that they constitute convex optimization problems and a self-concordance barrier function exists, numerical solutions of LMIs can be obtained efficiently [2].

A simple fact about LMIs is that their feasible regions (*i.e.*, "LMI sets") constitute convex semialgebraic sets in $\mathbb{R}^n$. Based on this fact, one can naturally ask whether the converse is also true. That is, are all convex semialgebraic sets LMI sets? It is easily shown by the counterexample $x^4 + y^4 \leq 1$ that the answer is in the negative. However, by defining new variables, one can lift this particular problem to a higher dimensional space and obtain an LMI representation [8]. Yet, to the best our knowledge, it is currently not known whether it is possible to convert every convex semialgebraic set into an LMI set in a higher dimensional space or not.

It is then reasonable to ask "Which class of convex semialgebraic sets are LMI sets?" This question has been studied in detail by [3] and their results show that a property

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called "rigid convexity" is necessary for a semialgebraic set to have an LMI representation. In $\mathbb{R}^2$, rigid convexity is also shown to be a sufficient condition. They also conjecture that sufficiency is valid in $\mathbb{R}^n$ as well.

In this paper, we are interested in finding an LMI representation not for a semialgebraic set $\mathcal{V}$, but, for the closure of its convex hull, $\overline{\mathbf{co}(\mathcal{V})}$. We are motivated in this by the fact that minimizing a linear function over $\mathcal{V}$ yields the same result as minimizing it over $\overline{\mathbf{co}(\mathcal{V})}$ [2]. Therefore, if the LMI representation exists, a nonconvex problem can be converted into a convex LMI problem. Moreover, in many global optimization methods, a series of convex sets $\mathcal{C}_i$ satisfying $\mathcal{V} \subseteq \mathcal{C}_i$, $\mathcal{C}_{i+1} \subseteq \mathcal{C}_i$ and converging to $\overline{\mathbf{co}(\mathcal{V})}$ are constructed to obtain computationally cheap bounds for optimizing an objective function over $\mathcal{V}$ [6, 7, 4, 9, 10]. Hence, if $\mathcal{C}_i$s are chosen as LMI sets, the existence of such a representation opens up the possibility of obtaining a relaxation algorithm that converges in a finite number of steps. For a positive example, see the mixed integer programming relaxations proposed in [10], for which both $\mathcal{C}_i$ and $\overline{\mathbf{co}(\mathcal{V})}$ are polytopic regions and finite convergence is achieved.

In an attempt to give a limited answer to the problem posed above, we consider semialgebraic sets described by quadratic constraints only. Our main result is that for such a $\mathcal{V}$, $\overline{\mathbf{co}(\mathcal{V})}$ is given by the intersection of a finite number of LMI sets and the halfspaces supporting $\mathcal{V}$ along a subset of the boundary of $\mathcal{V}$. The supporting halfspaces of concern are infinitely many in $\mathbb{R}^n$, but in $\mathbb{R}^2$, their intersection is easily shown to be a polyhedral set. Therefore, for $\mathcal{V} \subseteq \mathbb{R}^2$, $\overline{\mathbf{co}(\mathcal{V})}$ is always an LMI set, regardless of the convexity of $\mathcal{V}$ itself. Note that since a semialgebraic description of the convex hull is not readily available, the application of the rigid convexity criterion presented in [3] to our problem is not possible.

The paper is organized as follows. In Section 2, we introduce the notation and some assumptions used throughout the paper. The main theorem of the paper is given in Section 3 together with a sketch of the proof. In Section 4, it is shown how a characterization in terms of infinitely many supporting halfspaces can be converted into another description involving a smaller set of supporting halfspaces and finitely many LMIs. Section 5 is devoted to the proof of the main theorem. An explanatory example regarding the LMI description in $\mathbb{R}^2$ is given in Section 6. A summary and conclusions are given in Section 7. The proofs of all intermediate results can be found in appendices.

2. Notation and Preliminaries

The set of $n \times n$-dimensional real symmetric matrices is denoted by $\mathbb{S}^n$ while $\mathbb{S}^n_+$ denotes the positive semidefinite cone. Given two matrices $A, B \in \mathbb{S}^n$, the relation $A \succeq B$ implies $A - B \in \mathbb{S}^n_+$. For $A \in \mathbb{S}^n$, the number of positive and negative eigenvalues of A are denoted by $\pi(A)$ and $\nu(A)$, respectively. We use A^+ to denote the Moore-Penrose pseudo-inverse of matrix A.

For $\mathcal{S} \subseteq \mathbb{R}^n$, $\overline{\mathcal{S}}$ and $\mathbf{int}(\mathcal{S})$ denote the closure and interior of $\mathcal{S}$, respectively. The boundary of $\mathcal{S}$ is defined as $\partial\mathcal{S} := \overline{\mathcal{S}} \setminus \mathbf{int}(\mathcal{S})$. We denote the convex hull of $\mathcal{S}$ as $\mathbf{co}(\mathcal{S})$. We define a neighborhood of $x \in \mathbb{R}^n$, as an open ball centered at x. That is,

$$N_x := \{y \in \mathbb{R}^n \mid \|y - x\| < r\} \quad \text{for some } r > 0. \tag{2}$$

Now consider the quadratic polynomial

$$p(x) = \begin{pmatrix} x \\ 1 \end{pmatrix}^T P \begin{pmatrix} x \\ 1 \end{pmatrix}, \tag{3}$$

where $P \in \mathbb{S}^{n+1}$. The feasible region of a single inequality constraint and the corresponding zero set are

$$\mathcal{Q}_p := \{x \in \mathbb{R}^n \mid p(x) \geq 0\} \quad \text{and} \quad \mathcal{Z}_p := \{x \in \mathbb{R}^n \mid p(x) = 0\}. \tag{4}$$

It is clear that $\mathcal{Z}_p = \partial \mathcal{Q}_p$ as long as P is not positive semidefinite. We also use the compact notation

$$\hat{\mathcal{Q}}_p := \{x \in \mathbb{R}^n \mid p(x) \rhd 0\}, \tag{5}$$

where $\rhd \in \{\geq, =\}$. That is, if we have an equality constraint, then "$\rhd$" stands for "$=$", and $\hat{\mathcal{Q}}_p = \mathcal{Z}_p$, while for an inequality constraint, "$\rhd$" represents "$\geq$" and $\hat{\mathcal{Q}}_p = \mathcal{Q}_p$.

Let $\mathcal{S} \subseteq \mathbb{R}^n$ be nonempty and assume $x \in \partial S$. We denote the intersection of all halfspaces supporting $\mathcal{S}$ at x as $\mathbf{C}(S, x)$. If no supporting halfspace exists, we take $\mathbf{C}(S, x) = \mathbb{R}^n$.

In the rest of the paper, we focus on the basic semialgebraic set

$$\mathcal{V} := \{x \in \mathbb{R}^n \mid p_i(x) \geq 0, \ i = 1 : r\}, \tag{6}$$

where p_i are nonconstant quadratic polynomials. To avoid technical difficulties in the rest of the paper, we put the expression given in (6) into a more suitable form and make use of a number of assumptions. Note that these assumptions by no means put any restriction on our results as described below. They are used to simplify the presentation of the material.

If an inequality constraint and a negative multiple of it exist, we replace them with a single equality constraint. By an appropriate choice of indexing, this leads to the following formulation

$$\mathcal{V} = \{x \in \mathbb{R}^n \mid p_i(x) \geq 0, \ i = 1 : \ell, \ p_j(x) = 0, \ j = \ell + 1 : m\}. \tag{7}$$

Assumption 2.1. The matrix P_j, associated with the constraint $p_j(x) = 0$ satisfies $\nu(P_j) \geq \pi(P_j)$.

Remark 2.2. Note that this assumption clearly does not lead to any limitations. If it is not satisfied, one can simply multiply the both sides of the equality constraint by a negative number.

Assumption 2.3. For all $i = 1 : m$, P_i is indefinite.

Remark 2.4. In order to see that this assumption is not restrictive, first assume $P_i \succeq 0$. In this case, p_i cannot appear in an equality constraint because the condition $\nu(P_i) \geq \pi(P_i)$ cannot be satisfied for a nonzero polynomial. On the other hand, if it is used in an inequality constraint, the constraint is satisfied for every $x \in \mathbb{R}^n$. Therefore, it can be removed from the description. Next, assume, $P_i \preceq 0$. As can be anticipated from the spectral decomposition of P_i, for both equality and inequality constraints, the feasible set turns out to be a linear subspace of $\mathbb{R}^n$. Therefore, without altering $\mathcal{V}$, the constraint associated with p_i can be replaced by a number of equality constraints, for which the corresponding matrix has to be indefinite.

3. Main Result - Characterization of $\overline{\mathbf{co}(\mathcal{V})}$

In this section, we state the main result of this paper, which is that $\overline{\mathbf{co}(\mathcal{V})}$ can be described in terms of a combination of LMIs and supporting halfspaces.

Let $\mathcal{V} \in \mathbb{R}^n$ be a basic semialgebraic set and assume the polynomials used in its description satisfies Assumptions 2.1 and 2.3. We define the set

$$\mathcal{I} := \left(\bigcup_{i \neq j} (\mathcal{Z}_{p_i} \cap \mathcal{Z}_{p_j}) \right) \cap \mathcal{V}. \tag{8}$$

The main result of the paper is the following theorem.

Theorem 3.1. *The set $\mathcal{I}$ is a subset of $\partial\mathcal{V}$. Furthermore, $\overline{\mathbf{co}(\mathcal{V})}$ can be expressed as the intersection of a finite number of LMIs and the set*

$$\mathcal{F} := \begin{cases} \bigcap_{x \in \mathcal{I}} \mathbf{C}(\mathcal{V}, x) & \text{if } \mathcal{I} \neq \emptyset \\ \mathbb{R}^n & \text{if } \mathcal{I} = \emptyset. \end{cases} \tag{9}$$

The following is an immediate corollary of the theorem above.

Corollary 3.2. *If $\mathcal{V} \in \mathbb{R}^2$, then, $\overline{\mathbf{co}(\mathcal{V})}$ is an LMI set in $\mathbb{R}^2$.*

Proof. In $\mathbb{R}^2$, the following statements hold true. The set $\mathcal{I}$ is simply composed of a finite number of intersection points. Moreover, the intersection of all halfspaces supporting a set at a common boundary point is a polyhedral set. Therefore,

$$\bigcap_{x \in \mathcal{I}} \mathbf{C}(\mathcal{V}, x) \tag{10}$$

is an LMI set. □

3.1. Outline of the Proof of Theorem 3.1

We base proof of Theorem 3.1 on the following characterization of the convex hull.

Proposition 3.3. *Let $\mathcal{S} \subseteq \mathbb{R}^n$ be nonempty. The intersection of all supporting halfspaces of $\mathcal{S}$ is equal to the closure of convex hull of $\mathcal{S}$, that is*

$$\overline{\mathbf{co}(\mathcal{S})} = \bigcap_{x \in \partial\mathcal{S}} \mathbf{C}(\mathcal{S}, x). \tag{11}$$

This proposition is a simple consequence of well-known results from convex analysis. In general, it leads to a characterization in terms of infinitely many halfspaces, which is not practically useful. The main approach we follow is to replace these halfspaces with a finite number of LMI sets as much as possible to obtain a better description of the convex hull. This procedure is summarized below.

It is easy to see that the boundary points of $\mathcal{V}$ belong to the zero sets $\mathcal{Z}_{p_i}$. Using this fact, one can separate these points into two disjoint groups, namely, those lying at the intersections of more than one zero set, $\mathcal{I}$, and those that are elements of only one zero set, namely $\partial\mathcal{V} \setminus \mathcal{I}$. In general, $\partial\mathcal{V}$ is smooth along the latter while points where the boundary is not smooth lie in the former. To prove the theorem, we show that the halfspaces supporting $\mathcal{V}$ along $\partial\mathcal{V} \setminus \mathcal{I}$ can be replaced by a finite number of LMI sets which are, except for a few special cases, directly induced by p_is themselves. This leads to a mixed description of $\overline{\mathbf{co}(\mathcal{V})}$ in terms of LMIs and the intersection of possibly infinitely many halfspaces supporting $\mathcal{V}$ along $\mathcal{I}$.

4. An LMI Description for Supporting Halfspaces

In this section, our goal is to develop the basic results which are employed to show that some supporting halfspaces in the description of the convex hull can be replaced by LMI sets. Towards this end, we concentrate on a single quadratic constraint determined by a polynomial p and the corresponding feasible region $\hat{\mathcal{Q}}_p$, where p satisfies Assumptions 2.1 and 2.3. We begin by introducing some propositions related with the LMI representation of $\mathcal{Q}_p$.

Proposition 4.1. *If $\pi(P) = 1$, $\mathcal{Q}_p$ is either an LMI set, or the union of two LMI sets, $\mathcal{Q}_p^+$ and $\mathcal{Q}_p^-$, which have disjoint interiors and which are symmetric with respect to a point $x_c \in \mathbb{R}^n$.*

The proof of the proposition is given in Appendix A. For $\pi(P) = 1$, an explicit formulation of LMI regions of $\mathcal{Q}_p$ in terms of the matrix P can be found in the proof. Examples of $\mathcal{Q}_p$ under this condition are convex regions bounded by an ellipsoid, a paraboloid and a hyperboloid. The first two are LMI sets while the last one is union of two LMI sets each of which bounded by a sheet of the hyperboloid.

Proposition 4.2. *If $\pi(P) = 1$ and $\mathcal{Q}_p$ is composed of two LMI components, no halfspace containing $\mathcal{Q}_p$ exists.*

As an example for Proposition 4.2, consider the region bounded by a hyperboloid mentioned previously. There does not exist a halfspace containing this region even though there are hyperplanes separating its two LMI components.

Now, we are ready to focus on the main result. Consider a set $\mathcal{S} \subseteq \mathbb{R}^n$, which has a nonempty interior. Let's define

$$\mathcal{U} := \mathcal{S} \cap \hat{\mathcal{Q}}_p \tag{12}$$

and assume

$$\mathcal{B} := \mathbf{int}(\mathcal{S}) \cap \mathcal{Z}_p \neq \emptyset. \tag{13}$$

It is clear that $\mathcal{B} \subseteq \partial\mathcal{U}$. A conceptual picture illustrating these definitions in $\mathbb{R}^2$ is given in Figure 4.1 for an inequality constraint (i.e., $\hat{\mathcal{Q}}_p = \mathcal{Q}_p$).

In the rest of the section, we show that halfspaces supporting $\mathcal{U}$ along $\mathcal{B}$ can be replaced by an LMI set.

Lemma 4.3. *If there exists a hyperplane, $\mathcal{T}$, supporting $\mathcal{U}$ at a point $y \in \mathcal{B}$, then $\pi(P) = 1$ and $\mathcal{T} \cap \mathbf{int}(\mathcal{Q}_p) = \emptyset$.*

The main idea behind Lemma 4.3 for an inequality constraint can be interpreted as

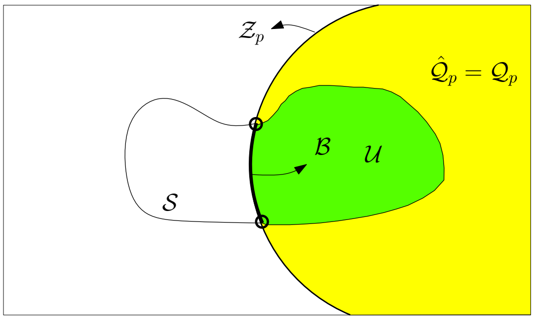

Figure 4.1: Illustration of $\mathcal{S}$, $\mathcal{Q}_p$, $\mathcal{U}$ and $\mathcal{B}$

follows. (A similar reasoning applies to an equality constraint.) If a hyperplane $\mathcal{T}$ supports $\mathcal{U}$ at a point $y \in \mathcal{B}$, clearly, $\mathcal{T} \cap \mathbf{int}(\mathcal{U}) = \emptyset$. This also implies that $\mathcal{U}$ must possess a certain convexity property around a neighborhood of y. Roughly speaking, Lemma 4.3 (in conjunction with Proposition 4.1) states that these properties also hold true globally for $\mathcal{T}$ and $\mathcal{Q}_p$.

As an example, consider Figure 4.2 a). As can be seen from the figure, there exists

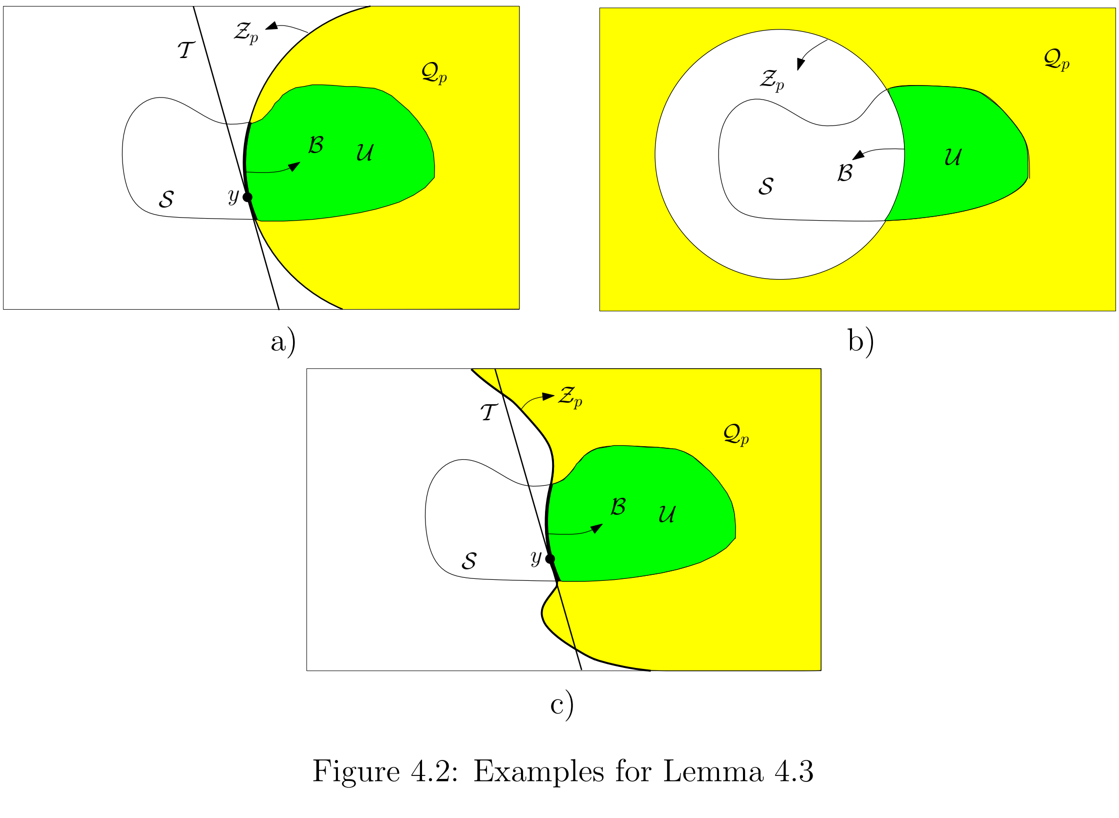

Figure 4.2: Examples for Lemma 4.3

a supporting halfspace at a point $y \in \mathcal{B}$, and hence, due to Lemma 4.3 and Proposition 4.1, $\mathcal{Q}_p$ is composed of LMI sets and $\mathcal{T}$ does not intersect its interior (for this example, $\mathcal{Q}_p$ has only one LMI component). The necessity of the condition $\pi(P) = 1$ when there is a halfspace supporting $\mathcal{U}$ along $\mathcal{B}$ is illustrated in Figure 4.2 b). Here $\mathcal{Q}_p$ is the region outside the circle for which clearly we must have $\pi(P) > 1$. As can be seen from the figure in this situation there is no hyperplane supporting $\mathcal{U}$ along $\mathcal{B}$. The lemma also guarantees that a situation like in Figure 4.2 c) (*i.e.*, $\mathcal{T}$ supports $\mathcal{U}$

but intersects the interior of $\mathcal{Q}_p$) cannot occur even in $\mathbb{R}^n$.

Lemma 4.4. *If there exists a hyperplane supporting $\mathcal{U}$ at a point $y \in \mathcal{B}$, there exists an LMI set $\mathcal{L}_p$ such that $\mathcal{U} \subseteq \mathcal{L}_p$ and any hyperplane supporting $\mathcal{U}$ at a point of $\mathcal{B}$ also supports $\mathcal{L}_p$.*

Except for some technical details, Lemma 4.4 is a simple consequence of previous results. We describe this briefly for an inequality constraint. If there exists a supporting halfspace along $\mathcal{B}$, $\mathcal{Q}_p$ is either an LMI set or a union of two LMI sets. If the former holds true, $\mathcal{Q}_p$ itself can be chosen as $\mathcal{L}_p$ and we are done (see Figure C.1 (a) below). A similar reasoning applies when $\mathcal{Q}_p$ is composed of two components and $\mathcal{U}$ is a subset of one of them as can be seen from Figure C.2. It remains to consider the case in which $\mathcal{Q}_p$ is composed of two components but $\mathcal{U}$ is not a subset of one of them as in Figure C.3 (a). However, for this situation, there does not exist a hyperplane supporting $\mathcal{U}$ along $\mathcal{B}$ if we disregard some technical details. In order to see this, assume there exists such an hyperplane, $\mathcal{T}$. Then, $\mathcal{T}$ cannot intersect $\mathbf{int}(\mathcal{Q}_p)$. Moreover, due to Proposition 4.2, a halfspace cannot contain $\mathcal{Q}_p$. Hence, $\mathcal{T}$ has to separate $\mathcal{Q}_p^+$ and $\mathcal{Q}_p^-$, which are convex sets having nonempty disjoint interiors. This leads to a contradiction.

As can be seen from the sketch given above, $\mathcal{L}_p$ is usually determined by the polynomial p itself. However, there are some exceptions as described in the proof. For these and a rigorous treatment including equality constraint, refer to Appendix B.

The following corollary of Lemma 4.4 is the main result of this section which will be utilized to obtain the characterization of $\overline{\mathbf{co}(\mathcal{V})}$.

Corollary 4.5. *If the exists a hyperplane supporting $\mathcal{U}$ at a point of $\mathcal{B}$, then there exists an LMI set $\mathcal{L}_p$ such that*

$$\mathcal{U} \subseteq \mathcal{L}_p \subseteq \bigcap_{x \in \mathcal{B}} \mathbf{C}(\mathcal{V}, x). \tag{14}$$

Remark 4.6. *As one can see from the definition of $\mathcal{B}$* (13)*, the formulation given in* (14) *does not include the halfspaces supporting $\mathcal{U}$ at points of $\mathcal{Z}_p \cap \partial S$. These points are marked with small circlers in Figure* 4.1*.*

5. Proof of Theorem 3.1

We are ready to prove the main theorem employing the results developed in the previous section. Let's define

$$\mathcal{V}_{p_k} := \begin{cases} \{x \in \mathbb{R}^n \mid p_i(x) \geq 0,\ i = 1 : \ell, i \neq k; \\ \qquad p_j(x) = 0,\ j = \ell + 1 : m,\ j \neq k\} & \text{if } m > 1 \\ \mathbb{R}^n & \text{if } m = 1, \end{cases} \tag{15}$$

which is obtained by removing the constraint corresponding to polynomial p_k from the description of $\mathcal{V}$. In order to prove the theorem, for each $k \in \{1 : m\}$, we apply Corollary 4.5 to p_k and $\mathcal{V}_{p_k}$ regarding them p and $\mathcal{U}$ respectively. By this way, we can replace all supporting halfspaces used in description of $\overline{\mathbf{co}(V)}$ successively by LMI sets except the ones supporting $\mathcal{V}$ along the intersection set $\mathcal{I}$, and we are done. Note that we can not get rid of all supporting halfspaces because points lying in $\mathcal{Z}_p \cap \partial\mathcal{V}_{p_k}$ are not handled by Corollary 4.5 as stated in Remark 4.6. In the sequel, we give the formal proof of the theorem.

Proof of Theorem 3.1. We first construct a characterization of $\partial\mathcal{V}$. Because $\mathcal{V}$ is closed, $\partial\mathcal{V} \subseteq \mathcal{V}$. Moreover, as one can easily verify, an element of $\mathcal{V}$ is its boundary point if and only if it is also an element of $\mathcal{Z}_{p_i}$ for some $i = 1 : m$. Using these facts, we can express $\partial\mathcal{V}$ as the union of the two disjoint set

$$\mathcal{I} = \{x \in \mathcal{V} \mid x \in \mathcal{Z}_{p_i} \text{ for at least two different values of } i\} \tag{16}$$

and

$$\mathcal{J} := \{x \in \mathcal{V} \mid x \in \mathcal{Z}_{p_i} \text{ for exactly one } i\}. \tag{17}$$

It is clear that the former is equivalent to the definition given in (8). We define

$$\mathcal{B}_{p_i} := \mathcal{Z}_{p_i} \cap \mathbf{int}(\mathcal{V}_{p_i}), \quad i = 1 : m \tag{18}$$

and infer that

$$\mathcal{J} = \bigcup_{i=1}^{m} \mathcal{B}_{p_i}. \tag{19}$$

Without loss of generality, if there exists some nonempty elements in the collection $\{\mathcal{B}_{p_i}\}$, we can assume they are the first elements in the collection by an appropriate change of indexing. Hence, we can write

$$\mathcal{J} = \bigcup_{i=1}^{r} \mathcal{B}_{p_i}, \tag{20}$$

where r is the number of nonempty sets in $\{\mathcal{B}_{p_i}\}$. Putting all together, we can express the boundary of $\mathcal{V}$ as

$$\partial\mathcal{V} = \mathcal{I} \cup \left(\bigcup_{i=1}^{r} \mathcal{B}_{p_i} \right). \tag{21}$$

We proceed by the characterization of the closure of the convex hull of $\mathcal{V}$. Due to Proposition 3.3, we have

$$\overline{\mathbf{co}(\mathcal{V})} = \bigcap_{x \in \partial\mathcal{V}} \mathbf{C}(\mathcal{V}, x). \tag{22}$$

By substituting (21) into this expression, we can write

$$\overline{\mathbf{co}(\mathcal{V})} = \left(\bigcap_{x \in \mathcal{I}} \mathbf{C}(\mathcal{V}, x) \right) \bigcap \left(\bigcap_{i=1}^{r} \left(\bigcap_{x \in \mathcal{B}_{p_i}} \mathbf{C}(\mathcal{V}, x) \right) \right). \tag{23}$$

If all the sets in the collection $\{\mathcal{B}_{p_i}\}$ are empty, we are done. Alternatively, assume there are nonempty elements in $\{\mathcal{B}_{p_i}\}$. By definition, we can write

$$\mathcal{V} = \mathcal{V}_{p_i} \cap \mathcal{Q}_{p_i}, \quad i = 1 : r. \tag{24}$$

Therefore, for each $\mathcal{V}_{p_i}$ and $\mathcal{Q}_{p_i}$, $i \in 1:r$, we can apply Corollary 4.5 and deduce that either there exists an LMI set $\mathcal{L}_{p_i}$ such that

$$\mathcal{V} \subseteq \mathcal{L}_{p_i} \subseteq \bigcap_{x \in \mathcal{B}_{p_i}} \mathbf{C}(\mathcal{V}, x), \tag{25}$$

or there does not exist any hyperplane supporting $\mathcal{V}$ along $\mathcal{B}_{p_i}$. Using this result with (23), we come up with

$$\mathcal{V} \subseteq \left(\bigcap_{x \in \mathcal{I}} \mathbf{C}(\mathcal{V}, x) \right) \bigcap \left(\bigcap_{i=1}^{s} \mathcal{L}_{p_i} \right) \subseteq \overline{\mathbf{co}(\mathcal{V})}. \tag{26}$$

Note that we apply a change of indexing to ensure that the indices for which there exists a hyperplane supporting $\mathcal{V}$ along $\mathcal{B}_i$ are the first s ones. □

6. Example

In what follows, we illustrate our results on an example in $\mathbb{R}^2$.

Consider the set of constraints

$$\begin{aligned} p_1(x) &= 64 - x_1^2 - x_2^2 \geq 0, \\ p_2(x) &= 50 - x_1^2 - 8x_2 \geq 0, \\ p_3(x) &= x_1^2 - (x_2 - 2)^2 - 4 \geq 0, \\ p_4(x) &= (x_1 + 4)^2 + (x_2 - 4)^2 - 10 \geq 0. \end{aligned}$$

The corresponding feasible region, $\mathcal{V}$, is depicted in Figure 6.1 a). We know that $\overline{\mathbf{co}(\mathcal{V})}$

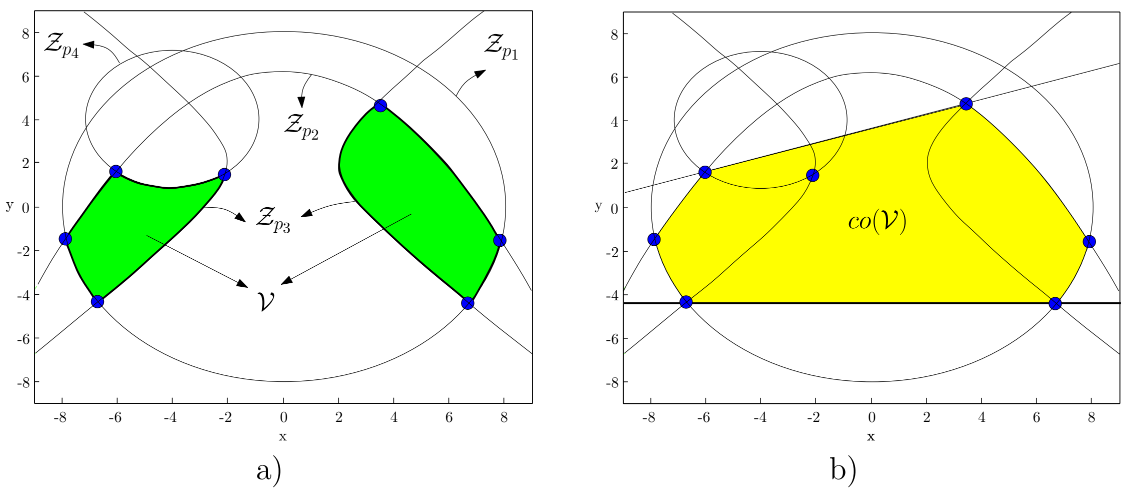

Figure 6.1: a) The region determined by quadratic inequalities; b) its convex hull.

is the intersection of its supporting halfspaces. It is possible to separate these halfspaces into two groups – those supporting $\mathcal{V}$ at the points in $\partial\mathcal{V} \setminus \mathcal{I}$ and those supporting it at the points in $\mathcal{I}$ (points of $\mathcal{I}$ are indicated by thick dots in the figure). Corollary 4.5 states that the former can always be replaced by LMIs. These LMIs can be identified as follows.

The halfspaces supporting $\mathcal{V}$ along $\mathcal{Z}_{p_1}$ and $\mathcal{Z}_{p_2}$ can be replaced by $\mathcal{Q}_{p_1}$ and $\mathcal{Q}_{p_2}$. Because these sets are determined by convex quadratic constraints, they admit the LMI representations

$$\mathcal{L}_{p_1}:\begin{bmatrix}1 & 0 & x_1\\ 0 & 1 & x_2\\ x_1 & x_2 & 64\end{bmatrix}\succeq 0;\qquad \mathcal{L}_{p_2}:\begin{bmatrix}1 & x_1\\ x_1 & 50-8x_2\end{bmatrix}\succeq 0 \tag{27}$$

which can be derived from (32) and (37) respectively. On the other hand, there does not exist any halfspace supporting $\mathcal{V}$ along the points of $\partial\mathcal{V}\setminus\mathcal{I}$ lying on $\mathcal{Z}_{p_3}$ or $\mathcal{Z}_{p_4}$. This fact was also mentioned just after Lemma 4.3.

Now it remains to consider the halfspaces supporting $\mathcal{V}$ at points of $\mathcal{I}$. As shown in the proof of Corollary 3.2, using only finitely many of them is enough. In this example, we have only two

$$\mathcal{L}_{I_1}: 0.3322x_1 - x_2 + 3.6438 \geq 0;\qquad \mathcal{L}_{I_2}: x_2 + 4.385 \geq 0 \tag{28}$$

which are depicted in Figure 6.1 b). As a result, the following LMI representation is obtained

$$\overline{\mathbf{co}(\mathcal{V})} = \mathcal{L}_{p_1}\cap\mathcal{L}_{p_2}\cap\mathcal{L}_{I_1}\cap\mathcal{L}_{I_2}. \tag{29}$$

Notice that, Corollary 3.2 only shows that the convex hull is an LMI set and it is not constructive. The main difficulty in obtaining an LMI representation arises in finding the halfspaces supporting $\mathcal{V}$ at the points of $\mathcal{I}$. In the example given above, we could compute the convex hull by inspection.

7. Summary and Conclusions

In this paper, we have focused on the LMI representation of the convex hull of a basic semialgebraic set $\mathcal{V}\in\mathbb{R}^n$. Our interest in the problem stems from the fact that if such a representation exists a nonconvex problem can be converted into a convex LMI problem. Moreover, it would be possible to develop an LMI relaxation algorithm having finite convergence property. Assuming that the defining polynomials are quadratic, we have derived a characterization of $\overline{\mathbf{co}(\mathcal{V})}$ in terms of a combination of LMIs and a class of supporting halfspaces. Then, using this characterization, we have proved that $\overline{\mathbf{co}(\mathcal{V})}$ is an LMI set when the problem is restricted to $\mathbb{R}^2$. A future research direction is to develop an algorithm which constructs the LMI representation of $\overline{\mathbf{co}(\mathcal{V})}$. We are currently investigating the conditions under which one can find an exclusive LMI representation for $\overline{\mathbf{co}(\mathcal{V})}$ in $\mathbb{R}^n$.

A. Proofs of Propositions 4.1 and 4.2

Proof of Proposition 4.1. By partitioning the matrix P, we can express the inequality constraint as

$$p(x) = \begin{pmatrix}x\\1\end{pmatrix}^T\begin{bmatrix}A & b\\ b^T & c\end{bmatrix}\begin{pmatrix}x\\1\end{pmatrix}\geq 0 \tag{30}$$

Let's apply the linear transformation $x = \bar{x} - A^{+}b$. The constraint in the new space can now be written as

$$\bar{p}(\bar{x}) = \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix}^{T} \bar{P} \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} \geq 0, \qquad \bar{P} := \begin{bmatrix} A & \bar{b} \\ \bar{b}^{T} & \bar{c} \end{bmatrix}, \tag{31}$$

where $\bar{b} := (I - AA^{+})b$, $\bar{c} := c - b^{T}A^{+}b$.

We proceed by investigating a number of scenarios. First, assume $b \in \mathrm{Im}(A)$. In that case, $\bar{b} = 0$ and (31) simplifies to $\bar{c} + \bar{x}^{T}A\bar{x} \geq 0$. If $\bar{c} > 0$, A has to be negative semidefinite because $\pi(P) = 1$. Therefore, A can be decomposed as $A = -VV^{T}$, where V is the matrix of eigenvectors corresponding to the negative eigenvalues of A. Using the Schur complement formula, we obtain

$$\begin{bmatrix} I & V^{T}\bar{x} \\ \bar{x}^{T}V & \bar{c} \end{bmatrix} = \begin{bmatrix} I & V^{T}(x + A^{+}b) \\ (x + A^{+}b)^{T}V & c - b^{T}A^{+}b \end{bmatrix} \succeq 0. \tag{32}$$

Now, assume $\bar{c} \leq 0$. This implies $\pi(A) = 1$. Therefore, A can be expressed as $A = uu^{T} - VV^{T}$, where u is the eigenvector corresponding to the positive eigenvalue and V is the matrix of eigenvectors corresponding to the negative eigenvalues of A. With this decomposition, (31) becomes

$$\bar{x}^{T}uu^{T}\bar{x} - \bar{x}^{T}VV^{T}\bar{x} - d^{2} \geq 0 \tag{33}$$

where $-d^{2} := \bar{c}$. The set of points satisfying the last inequality are the solutions of one of the two constraints

$$\begin{bmatrix} \pm u^{T}\bar{x}I & \begin{pmatrix} V^{T}\bar{x} \\ d \end{pmatrix} \\ \begin{pmatrix} \bar{x}^{T}V & d \end{pmatrix} & \pm u^{T}\bar{x} \end{bmatrix} = \begin{bmatrix} \pm u^{T}(x + A^{+}b)I & \begin{pmatrix} V^{T}(x + A^{+}b) \\ d \end{pmatrix} \\ \begin{pmatrix} (x + A^{+}b)^{T}V & d \end{pmatrix} & \pm u^{T}(x + A^{+}b) \end{bmatrix} \succeq 0 \tag{34}$$

and we denote their feasible sets as $\mathcal{Q}_p^{+}$ and $\mathcal{Q}_p^{-}$. It can be easily inferred that $\mathcal{Q}_p^{+}$ and $\mathcal{Q}_p^{-}$ are symmetric with respect to the point $x_c := -A^{+}b$. Note that we obtain (34) using the Schur complement formula for positive semidefinite inequalities. See [2] for details.

Lastly, assume $b \notin \mathrm{Im}(A)$. This implies $\bar{b} \neq 0$, and $A^{T}\bar{b} = 0$. As a result, it can be verified that the vectors

$$\begin{bmatrix} (\bar{c}/2 \pm \Delta)\bar{b} \\ -\bar{b}^{T}\bar{b} \end{bmatrix}, \quad \text{where } \Delta := \sqrt{\bar{c}^{2}/4 + \bar{b}^{T}\bar{b}}, \tag{35}$$

are eigenvectors of $\bar{P}$, the corresponding eigenvalues of which are $-\frac{\bar{b}^{T}\bar{b}}{\bar{c}/2 \pm \Delta}$. It is clear that one of these eigenvalues has to be negative while the other is positive. Therefore, $\bar{P}$ cannot have another positive eigenvalue because $\pi(P) = 1$ and inertia is preserved under congruence. Based on this fact, it can be shown that $A \preceq 0$. In order to see this, assume A has a positive eigenvalue α with the associated eigenvector u. Because $u \in \mathrm{Im}(A)$, the condition $\bar{b}^{T}u = 0$ must hold true. This leads to the fact that the vector $(u^{T}\ 0)^{T}$ must be another eigenvector of $\bar{P}$ with the associated positive eigenvalue α, which is a contradiction.

Now, because A is negative semidefinite, we can rewrite (31) as

$$\bar{c} + 2\bar{b}^T\bar{x} - \bar{x}^T VV^T\bar{x} \geq 0 \tag{36}$$

using the decomposition $A = -VV^T$. This leads to the LMI representation

$$\begin{bmatrix} I & V^T\bar{x} \\ \bar{x}^T V & \bar{c} + 2\bar{b}^T\bar{x} \end{bmatrix} = \begin{bmatrix} I & V^T(x + A^+b) \\ (x + A^+b)^T V & \bar{c} + 2\bar{b}^T(x + A^+b) \end{bmatrix} \succeq 0. \tag{37}$$

□

Proof of Proposition 4.2. We use the linear transformation introduced in the proof of Proposition 4.1. Since $\mathcal{Q}_p$ is composed of two LMI components, we have $\bar{c} \leq 0$, $\pi(A) = 1$ and the polynomial constraint takes the form

$$p(\bar{x}) = \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix}^T \begin{bmatrix} A & 0 \\ 0 & \bar{c} \end{bmatrix} \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} \geq 0. \tag{38}$$

Consider an arbitrary halfspace defined by the inequality

$$q(\bar{x}) = \alpha^T\bar{x} + \beta = \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix}^T \begin{bmatrix} 0 & \alpha/2 \\ \alpha^T/2 & \beta \end{bmatrix} \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} \geq 0. \tag{39}$$

Now, assume this halfspace contains $\mathcal{Q}_p$. This means $q(\bar{x}) \geq 0$ for every $\bar{x}$ satisfying $p(\bar{x}) \geq 0$. Because $q(\bar{x})$ can be made strictly positive, the S-procedure [2] can be applied and one can infer that there exists a $\lambda \geq 0$ such that

$$\begin{bmatrix} 0 & \alpha/2 \\ \alpha^T/2 & \beta \end{bmatrix} - \lambda \begin{bmatrix} A & 0 \\ 0 & \bar{c} \end{bmatrix} \succeq 0. \tag{40}$$

However, because $\pi(A) = 1$, the above given inequality cannot be satisfied for any $\lambda \geq 0$, which is a contradiction. Hence, there does not exist a halfspace containing $\mathcal{Q}_p$. □

B. Proof of Lemma 4.3

Proof of Lemma 4.3. Consider parametrization of the hyperplane $\mathcal{T}$ given by

$$x = Uw + y, \tag{41}$$

where $w \in \mathbb{R}^{n-1}$ and $U \in \mathbb{R}^{n\times(n-1)}$ has full-column rank. By substituting this parametrization into polynomial p, one obtains

$$\begin{gathered} p(Uw + y) = \begin{pmatrix} w \\ 1 \end{pmatrix}^T \tilde{P} \begin{pmatrix} w \\ 1 \end{pmatrix}, \\ \tilde{P} := \begin{bmatrix} U & y \\ 0 & 1 \end{bmatrix}^T P \begin{bmatrix} U & y \\ 0 & 1 \end{bmatrix}, \qquad P = \begin{bmatrix} A & b \\ b^T & c \end{bmatrix}. \end{gathered} \tag{42}$$

Because $y \in \mathcal{B}$, it satisfies $p(y) = 0$ and $\tilde{P} \in \mathbb{R}^n$ simplifies to

$$\tilde{P} = \begin{bmatrix} U^T A U & U^T(Ay+b) \\ (Ay+b)^T U & 0 \end{bmatrix}. \tag{43}$$

We now show that $\xi := U^T(Ay+b) = 0$. To that end, assume $\xi \neq 0$. This means $\xi_i \neq 0$ for an $i \in 1 : n-1$. Let's substitute the vector $w := \alpha e_i$ into (42), where e_i denotes the i^{th} standard basis vector. We obtain

$$p(\alpha U e_i + y) = \bar{a}_{ii}\alpha^2 + 2\xi_i \alpha, \tag{44}$$

where $\bar{a}_{ii} := (U^T A U)_{ii}$. This expression can be made both positive and negative for values of α having arbitrarily small absolute value. This means that the hyperplane $\mathcal{T}$ does not stay on the same side of the surface $\mathcal{Z}_p$ within the set $\mathcal{S}$. Hence, it cannot be a supporting hyperplane of $\mathcal{U}$, which is a contradiction. As a result, we must have $U^T(Ay+b) = 0$.

(From a geometric point of view, this conditions means the hyperplane $\mathcal{T}$ is either tangent to the surface $\mathcal{Z}_p$ or the tangent of $\mathcal{Z}_p$ at y is not defined.)

For later use, we note that employing the Poincaré separation theorem [5] and the fact that $U^T(Ay+b) = 0$, one can obtain the inertia inequalities

$$\pi(P) \geq \pi(\tilde{P}) = \pi(U^T A U) \geq \pi(P) - 1 \tag{45a}$$

$$\nu(P) \geq \nu(\tilde{P}) = \nu(U^T A U) \geq \nu(P) - 1. \tag{45b}$$

We now consider inequality and equality constraints separately and begin with inequality constraints first. Consider a neighborhood of the origin $N_0 \subseteq \mathbb{R}^{n-1}$ such that $Uw + y \in \mathbf{int}(\mathcal{S})$ for every $w \in N_0$. Note that such a neighborhood exists because $y \in \mathbf{int}(\mathcal{S})$. Since $\mathcal{T}$ is a supporting hyperplane, $p(Uw+y) \leq 0$ for all $w \in N_0$. Otherwise, $\mathcal{T} \bigcap \mathbf{int}(\mathcal{U})$ would be non-empty and $\mathcal{T}$ would not be a supporting hyperplane. Since $U^T(Ay+b) = 0$, we infer $p(Uw+y) = w^T U^T A U w$. Hence, we obtain $w^T U^T A U w \leq 0$ for every $w \in N_0$. This can be satisfied if and only if $U^T A U \preceq 0$, which means $p(Uw+y) \leq 0$ for every $w \in \mathbb{R}^{n-1}$. Therefore, $\mathcal{T} \cap \mathbf{int}(\mathcal{Q}_p) = \emptyset$. Since $U^T A U \preceq 0$, we conclude that $\pi(P) \leq 1$ by (45a). Due to Assumption 2.3, $\pi(P) = 1$.

Now consider the equality constraint. There exists a neighborhood $N_0 \subseteq \mathbb{R}^{n-1}$ such that the polynomial $p(Uw+y)$ is sign-definite over N_0. Otherwise, there would be points of $\mathcal{B}$, which is a subset of $\mathcal{U}$, on both sides of $\mathcal{T}$, and hence, it would not be a supporting hyperplane. Consequently, $U^T A U$ must be sign-definite. In order to show that it is in fact negative semi-definite, note the following. Assumptions 2.1 and 2.3 imply $\nu(P) \geq \pi(P) \geq 1$. Together with sign-definiteness of $U^T A U$ and (45), this implies that $\pi(P) = 1$. Moreover, $U^T A U \preceq 0$ if $\nu(P) > 1$. Hence, it only remains to show negative semi-definiteness when $\pi(P) = \nu(P) = 1$. For this case, if $U^T A U \succeq 0$, we can multiply both sides of the corresponding constraint by a negative number without loss of generality. This is because p appears in an equality constraint and Assumption 2.1 is not violated. Hence it can be ensured that $U^T A U \preceq 0$. Consequently, $\mathcal{T} \cap \mathbf{int}(\mathcal{Q}_p) = \emptyset$. □

C. Proof of Lemma 4.4

Before proving Lemma 4.4, we need the following proposition.

Proposition C.1. *Consider two distinct hyperplanes*

$$\mathcal{T}_i = \{x \in \mathbb{R}^n \mid \alpha_i^T x + \beta_i = 0\}, \quad i = 1, 2. \tag{46}$$

Assume $\pi(P) = 1$ and $\mathcal{Q}_p$ is union of two LMI sets, $\mathcal{Q}_p^+$ and $\mathcal{Q}_p^-$. If the conditions

$$\alpha_i^T x + \beta_i \geq 0, \quad \forall x \in \mathcal{Q}_p^+, \; i = 1, 2 \tag{47a}$$

$$\alpha_i^T x + \beta_i \leq 0, \quad \forall x \in \mathcal{Q}_p^-, \; i = 1, 2 \tag{47b}$$

are satisfied, then

$$\{x \in \mathcal{Q}_p \mid \alpha_i^T x + \beta_i \geq 0, \; i = 1, 2\} = \mathcal{Q}_p^+. \tag{48}$$

Proof. Suppose that (48) does not hold true. Then, there exists a point $y \in \mathcal{Q}_p^- \setminus \mathcal{Q}_p^+$ which is an element of the set appearing on the left-hand side of (48). Hence, taking into account (47b), one can infer that $\alpha_i^T y + \beta_i = 0$, $i = 1, 2$. This implies $y \notin \mathbf{int}(\mathcal{Q}_p)$. Otherwise, at least one of (47a) and (47b) is violated. Hence, one can see that $y \in \mathcal{Z}_p$. Without loss of generality, assume that the coordinate system is translated so that $p(x) = c + x^T A x \geq 0$ as in the proof of Proposition 4.1. The normal of the surface $\mathcal{Z}_p$ at y can be found as Ay. When $Ay \neq 0$, the normal is well-defined and $\mathcal{T}_1$ and $\mathcal{T}_2$ have to be tangents of $\mathcal{Z}_p$ at y. This is because they become supporting hyperplane of $\mathcal{Q}_p^+$ or $\mathcal{Q}_p^-$ at y. However, the tangent is unique, which is a contradiction. Hence, we must have $Ay = 0$.

Because y is in the null space of A, it has to be orthogonal to eigenvectors of A. This means $u^T y = 0$, where u is defined as in (33). Hence, from (33) and (34), $y \in \mathcal{Q}_p^+ \bigcap \mathcal{Q}_p^-$. This contradicts the fact that $y \in \mathcal{Q}_p^- \setminus \mathcal{Q}_p^+$, which completes the proof. □

Proof of Lemma 4.4. Because there exists a hyperplane supporting $\mathcal{U}$ at y, Lemma 4.3 implies that $\pi(P) = 1$. Due to Proposition 4.1, this means $\mathcal{Q}_p$ is either an LMI set or a union of two LMI sets having disjoint interiors. If the former holds true, we simply choose $\mathcal{L}_p$ as $\mathcal{Q}_p$, which ensures $\mathcal{U} \subseteq \hat{\mathcal{Q}}_p \subseteq \mathcal{Q}_p = \mathcal{L}_p$.

Now, consider an arbitrary hyperplane supporting $\mathcal{U}$ at a point of $\mathcal{B}$. Using Lemma 4.3 again, we can infer that it does not intersect $\mathbf{int}(\mathcal{Q}_p)$. Because $\mathcal{Q}_p$ is a convex set having a nonempty interior, this leads to the fact that the hyperplane supports $\mathcal{L}_p$. (See Figure C.1.)

Assume $\mathcal{Q}_p$ is the union of two LMI sets $\mathcal{Q}_p^+$ and $\mathcal{Q}_p^-$. If $\mathcal{U}$ is a subset of only one of these components, the one containing $\mathcal{U}$ can be chosen as $\mathcal{L}_p$. The preceding argument can be directly utilized to show that hyperplanes supporting $\mathcal{U}$ along $\mathcal{B}$ also supports $\mathcal{L}_p$. (See Figure C.2.)

In the rest of the proof, we validate our claim when $\mathcal{Q}_p$ is composed of two components under the condition $\mathcal{U} \nsubseteq \mathcal{Q}_p^+$ and $\mathcal{U} \nsubseteq \mathcal{Q}_p^-$. First, we show that if $\nu(P) > 1$ or we have an inequality constraint, there exists at most one hyperplane supporting $\mathcal{U}$ at points of $\mathcal{B}$. To this end, assume there are two such distinct hyperplanes given by

$$\mathcal{T}_i = \{x \in \mathbb{R}^n \mid \alpha_i^T x + \beta_i = 0\}, \quad i = 1, 2. \tag{49}$$

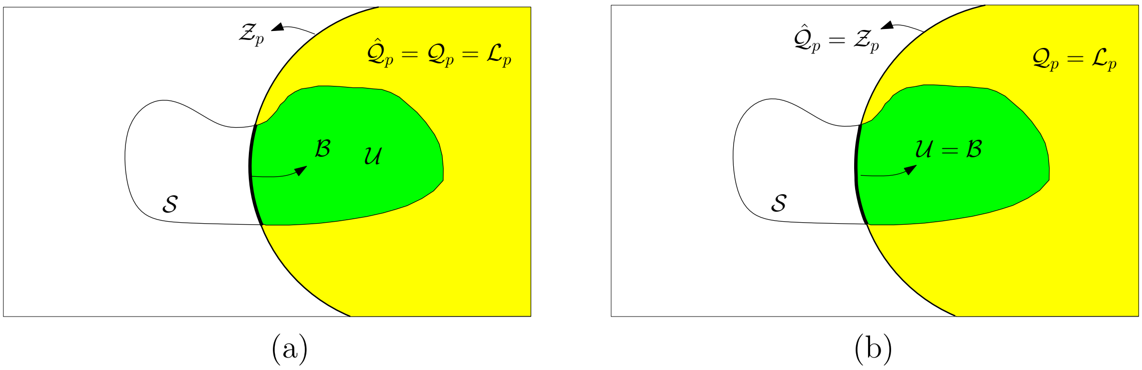

Figure C.1: Example for Lemma 4.4 with $\pi(P) = 1$ and $\mathcal{Q}_p$ has a single component. (a) Inequality constraint, (b) equality constraint.

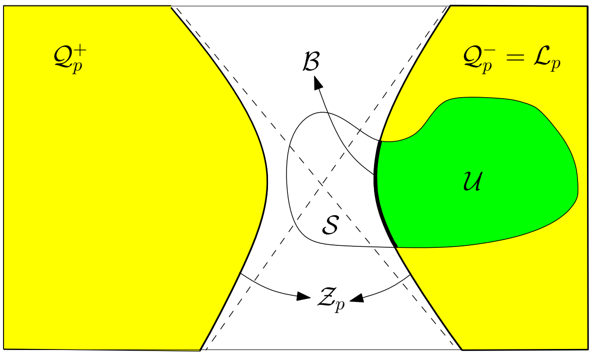

Figure C.2: Example for an inequality constraint in Lemma 4.4, for which $\pi(P) = 1$ and $\mathcal{U}$ is a subset of a convex component of $\mathcal{Q}_p$

We know from Lemma 4.3 that they cannot intersect the nonempty disjoint sets $\mathbf{int}(\mathcal{Q}_p^+)$ and $\mathbf{int}(\mathcal{Q}_p^-)$ (nonemptiness of these sets is due to Assumption 2.3). Moreover, due to Proposition 4.2, the halfspaces induced by $\mathcal{T}_1$ and $\mathcal{T}_2$ cannot contain $\mathcal{Q}_p$. Consequently, one can infer that these hyperplanes separate $\mathcal{Q}_p^+$ and $\mathcal{Q}_p^-$. Therefore, multiplying the equations used in the definition (49) by a negative number if necessary, one can always guarantee

$$\begin{array}{l} \alpha_i^T x + \beta_i \geq 0, \quad \forall x \in \mathcal{Q}_p^+, \ i = 1, 2 \\ \alpha_i^T x + \beta_i \leq 0, \quad \forall x \in \mathcal{Q}_p^-, \ i = 1, 2. \end{array} \tag{50}$$

Based on this fact, in what follows, we show that elements of $\mathcal{U}$ must lie on the set defined by the constraint

$$(\alpha_1^T x + \beta_1)(\alpha_2^T x + \beta_2) = 0. \tag{51}$$

In order to see this, assume the contrary. Then, there exists a point $y \in \mathcal{U}$ for which $\alpha_i^T y + \beta_i \neq 0$, $i = 1, 2$. Let's investigate different possibilities.

- If $\alpha_i^T y + \beta_i > 0$, $i = 1, 2$, we have

$$\alpha_i^T x + \beta_i \geq 0, \quad \forall x \in \mathcal{U}, \ i = 1, 2. \tag{52}$$

Otherwise, there would exist points of $\mathcal{U}$ on both sides of one of its supporting hyperplanes, which is a contradiction. However, due to Proposition C.1, (52)

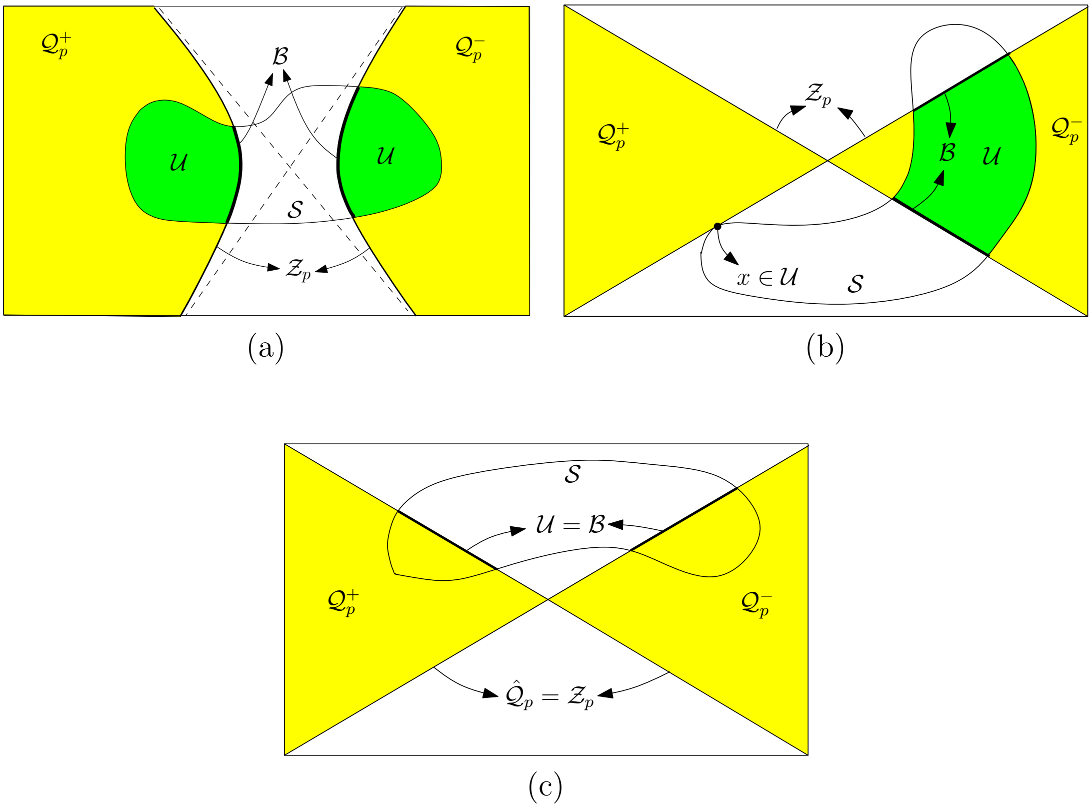

Figure C.3: An illustration of Lemma 4.4 for $\pi(P) = 1$ and two components under the condition $\mathcal{U} \nsubseteq \mathcal{Q}_p^+$, $\mathcal{U} \nsubseteq \mathcal{Q}_p^-$: (a) for an inequality constraint, $\mathcal{U}$ does not have a supporting halfspace at a point of $\mathcal{B}$, (b) for an inequality constraint, it has a single supporting halfspace; (c) for an equality constraint, it has two supporting halfspaces

implies $\mathcal{U} \subseteq \mathcal{Q}_p^+$, which clearly contradicts our hypothesis that $\mathcal{U} \nsubseteq \mathcal{Q}_p^+$. Hence, the condition $\alpha_i^T y + \beta_i > 0$, $i = 1, 2$ cannot be satisfied. Similarly, it can also be shown that these linear functions cannot be made strictly negative by y.

- If $\alpha_1^T y + \beta_1 > 0$ and $\alpha_2^T y + \beta_2 < 0$, then y cannot be an element of $\mathcal{Q}_p$ due to (50), which means it is not also an element $\mathcal{U}$. Therefore, these inequalities cannot be satisfied. Similarly, y cannot satisfy $\alpha_1^T y + \beta_1 < 0$ and $\alpha_2^T y + \beta_2 > 0$. As a result, we have shown that (51) must be satisfied for every element of $\mathcal{U}$.

Now, consider a point $x \in \mathcal{B}$. If $\nu(P) > 1$, $\mathcal{Z}_p$ exhibits a nonplanar characteristics around x (*i.e.*, it does not lie on a hyperplane) as opposed to the set defined by (51). Therefore, this condition cannot be satisfied. If $\mathcal{U}$ is defined by an inequality constraint, its interior is nonempty. Hence, $\mathcal{U}$ cannot be a subset of the set induced by (51). Consequently, we have shown that when $\nu(P) > 1$ or we have an inequality constraint, there do not exist two distinct supporting halfspaces $\mathcal{T}_1$ and $\mathcal{T}_2$, which means there exists at most one hyperplane supporting $\mathcal{U}$ along $\mathcal{B}$. This also shows that there exists one such halfspace because $\mathcal{U}$ does not lie on a hyperplane.

Lastly, let's consider the case in which $\pi(P) = \nu(P) = 1$ and $\mathcal{U}$ is defined by an equality constraint. For this case, the constraint defining $\mathcal{Z}_p$ takes the form given in (51). Hence, $\mathcal{Z}_p$ becomes the union of two hyperplanes. Because $\mathcal{B}$ is smooth, any hyperplane supporting $\mathcal{U}$ at a point of $\mathcal{B}$ has to be tangent to $\mathcal{Z}_p$. This means there exists at most two distinct supporting hyperplanes (hyperplanes of $\mathcal{Z}_p$ themselves). This also implies that there exists at most two supporting halfspaces.

When $\mathcal{U} \nsubseteq \mathcal{Q}_p^+$ and $\mathcal{U} \nsubseteq \mathcal{Q}_p^-$, we have shown that there exists finitely many halfspaces (at most two) supporting $\mathcal{U}$ along $\mathcal{B}$. We choose the intersection of these halfspaces as $\mathcal{L}_p$ when they exist. Conceptual pictures of this case are given in Figure C.3.

For the example depicted in Figure C.3 (a), there does not exist any supporting halfspace. In Figure C.3 (b), an example with one supporting halfspace is given. The last figure shows a case with two supporting halfspaces. $\square$

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