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The Intersection Property in the Family of Compact Convex Sets

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In this paper we study compact convex sets having the following property: nonempty intersection of any family of translates of the set is a summand (in the sense of Minkowski) of that set. The intersection property was introduced by G. T. Sallee [13]. We call such sets Sallee sets. We prove that some sets other than polytopes and elipsoids, that is wedges, dull wedges (Theorem 2.3) and certain subsets of the Euclidean ball (Theorem 4.3), possess the intersection property. We also present the family of all three-dimensional polyhedral sets that have the intersection property (Theorem 3.2). The family coincides with the family of all three dimensional strongly monotypic polytopes [10], [1].

Keywords: Minkowski subtraction, summands of convex sets

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1. Introduction

Compact convex sets in $\mathbb{R}^2$ have a very special property: nonempty intersection of any family of translates of a given set is a summand of that set [14]. In general, convex sets in $\mathbb{R}^n$, $n \geq 3$ do not have such property; consider for example a regular octahedron and its translate in the direction of one of its edges by half of the length of it. Interestingly, by [9] this property is shared by all Euclidean balls. Sallee [13] characterized all centrally symmetric convex polytopes in $\mathbb{R}^n$ that have this property (they are direct sums of segments and centrally symmetric polygons). By the name of *Sallee sets* we will call all nonempty compact convex sets in $\mathbb{R}^n$ that have that property. Let $X = \mathbb{R}^n$, and $\mathcal{K}(X)$ be the family of all nonempty compact convex subsets of X. Let *Minkowski sum* of A, $B \in \mathcal{K}(X)$ be defined by

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

We also denote $k \cdot A = \{k \cdot a \mid a \in A\}$ for $k \in \mathbb{R}$. We say that a set $B \in \mathcal{K}(X)$ is *a summand* of $A \in \mathcal{K}(X)$ and we write $B \ll A$ if there exists $C \in \mathcal{K}(X)$ such that

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$B + C = A$. Let

$$A \dot{-} B = \{x \in X \mid x + B \subset A\}$$

be the *Minkowski subtraction* of A and B. The following equality

$$A \dot{-} B = \bigcap_{C \subset B+p} (B + p)$$

holds true [6], [8], [12]. The Minkowski subtraction is the most natural subtraction within the cone $\mathcal{K}(X)$ of nonempty compact convex sets. This subtraction is surpassed only by subtraction in Minkowski-Rådström-Hörmander space [7] after embedding the cone $\mathcal{K}(X)$ in that space. The Minkowski-Rådström-Hörmander space appears very useful in studying differences of sublinear functions [15] (DS-functions) and quasidifferential calculus [3], [4]. The set A is a Sallee set if the Minkowski difference $A \dot{-} B$ for any $B \in \mathcal{K}(X)$ is either empty or a summand of A [2]. Minkowski subtraction was studied in [8], [12], [10], [11], [6], [2].
For a subset $A \subset X$ of a vector space X we denote by

$$\operatorname{conv} A = \left\{ x = \sum_{i=1}^{k} \alpha_i a_i \;\middle|\; \alpha_i \geq 0,\; \sum_{i=1}^{k} \alpha_i = 1,\; a_i \in A,\; k \in \mathbb{N} \right\}$$

the *convex hull* of A.
In this paper we present a class of non-polyhedral sets (wedges and dull wedges) that are Sallee sets. We also show that three dimensional polytopes which are Sallee sets coincide with strongly monotypic polytopes. Finally we show that certain segments of Euclidean ball are Sallee sets.

2. Generalized wedges and dull wedges

Let $x = (x_1, x_2, x_3)$, $p_1(x) = (x_1, x_2, 0)$, $p_2(x) = (x_1, x_2, x_2)$, $p_3(x) = (x_1, 0, 0)$, $e = (0, 0, 1)$, $w \in \mathbb{R}_+$.
Let us denote $H_1 = \{x \in \mathbb{R}^3 \mid x_3 = 0\}$, $H_1^+ = \{x \in \mathbb{R}^3 \mid x_3 \geq 0\}$, $H_2 = \{x \in \mathbb{R}^3 \mid x_3 = x_2\}$, $H_2^- = \{x \in \mathbb{R}^3 \mid x_3 \leq x_2\}$, $H_2^w = \{x \in \mathbb{R}^3 \mid x_2 + w = x_3\}$, $H_2^{w-} = \{x \in \mathbb{R}^3 \mid x_3 \leq x_2 + w\}$, $H_3 = \{x \in \mathbb{R}^3 \mid x_2 = 0\}$, $H_3^+ = \{x \in \mathbb{R}^3 \mid x_2 \geq 0\}$.

Definitions. We call the set $A \in \mathcal{K}(\mathbb{R}^3)$ a *wedge in proper position* if for any $x \in A$ we have

1) $0 \leq x_3 \leq x_2$,
2) $p_1(x) \in A,\ p_2(x) \in A,\ p_3(x) \in A$.

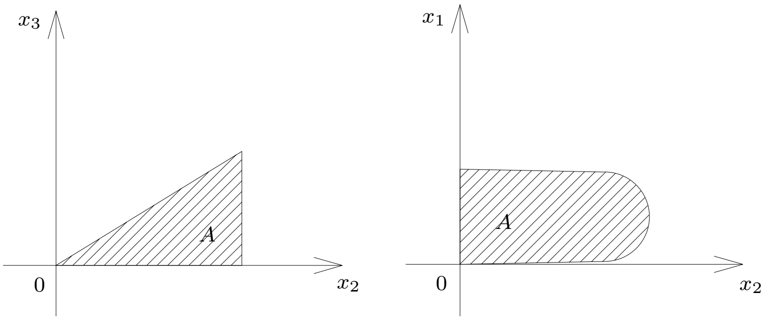
We call the set $A' \in \mathcal{K}(\mathbb{R}^3)$ a *proper base* if for any $x \in A'$ we have

1) $0 = x_3 \leq x_2$,
2) $p_3(x) \in A'$.

We call the set $A \in \mathcal{K}(\mathbb{R}^3)$ a *dull wedge in proper position* with the width $w > 0$ if for any $x \in A$

1) $0 \leq x_3 \leq x_2 + w$,
2) $p_1(x) \in A,\ p_2(x) + we \in A,\ p_3(x) \in A,\ p_3(x) + we \in A$.

Let $A_x = p_3^{-1}(p_3(x)) \cap A$ be the intersection of A and the plane containing x and perpendicular to the first axis of coordinates. Notice that if A is a wedge in proper position or the dull wedge in proper position then $p_1(A)$, $p_2(A)$ are its bases, $p_1(A)$ is a proper base and $p_3(A)$ is its edge. The set A_x is a triangle or, respectively, a trapezoid. We can describe a wedge as a dull wedge with the width $w = 0$. We call the set $A \in \mathcal{K}(\mathbb{R}^3)$ a *wedge* (a *dull wedge)* if there exists a system of coordinates in $\mathbb{R}^3$ such that A is a wedge (a dull wedge) in proper position.



Notice that a polygonal wedge is a wedge with polygonal base. Polygonal wedges and dull wedges form two out of four types of strongly monotypic polytopes [10]. In general, the side surface of a wedge (dull wedge) does not have to be a finite union of polygons. It can be a smooth union of continuum of parallel segments. The following propositions are quite obvious.

Proposition 2.1. *Any proper base A' is a base of exactly one wedge in proper position A (one dull wedge in proper position with the width w). Then $A = H_1^+ \cap H_2^- \cap p_1^{-1}(A')$, $(A = H_1^+ \cap H_2^{w-} \cap p_1^{-1}(A'))$.*

Proposition 2.2. *Let A, B be two wedges (two dull wedges with the widths w_A, w_B) in proper position. Then the following properties hold true:*

(*i*) *$A = B$ if and only if $p_1(A) = p_1(B)$ ($p_1(A) = p_1(B)$ and $w_A = w_B$);*

(*ii*) *$A + B$ is a wedge in proper position (dull wedge in the proper position with the width $w_A + w_B$) and $p_1(A + B) = p_1(A) + p_1(B)$;*

(*iii*) *B is a summand of A if and only if $p_1(B)$ is a summand of $p_1(A)$ ($p_1(B)$ is a summand of $p_1(A)$ and $w_B \leq w_A$). Moreover, $A \dot{-} B$ is a wedge in proper position (dull wedge in proper position with the width $w_A - w_B$) satisfying $p_1(A \dot{-} B) = p_1(A) \dot{-} p_1(B)$.*

Theorem 2.3. *Let A be a wedge (a dull wedge) in $\mathbb{R}^3$. Then A is a Sallee set.*

Proof. We can assume that A is a wedge in proper position. Let $B \in \mathcal{K}(\mathbb{R}^3)$ and $A \dot{-} B \neq \emptyset$. Since for any $x \in \mathbb{R}^3$ one has $A \dot{-} (B - x) = (A \dot{-} B) + x$, we may assume that $B \subset A$, $B \cap p_1(B) \neq \emptyset$, $B \cap p_2(B) \neq \emptyset$. Denote $A' = p_1(A)$. By Proposition 2.1, one has $A = H_1^+ \cap H_2^- \cap p_1^{-1}(A')$. Then

$$
\begin{aligned}
A \dot{-} B = \bigcap_{b \in B}(A - b) &= \bigcap_{b \in B}(H_1^+ - b) \cap \bigcap_{b \in B}(H_2^- - b) \cap \bigcap_{b \in B}(\mathrm{p}_1^{-1}(A') - b) \\
&= H_1^+ \cap H_2^- \cap \bigcap_{b \in B}(\mathrm{p}_1^{-1}(A' - \mathrm{p}_1(b))) \\
&= H_1^+ \cap H_2^- \cap p_1^{-1}(A' \dot{-} \mathrm{p}_1(B)).
\end{aligned}
$$

Notice that $A' \dot{-} p_1(B)$ does not have to be a proper base. However, the set $(A' \dot{-} p_1(B))$

$\cap H_3^+$ is a proper base and $H_1^+ \cap H_2^- \cap \mathrm{p}_1^{-1}(A' \dot{-} \mathrm{p}_1(B)) = H_1^+ \cap H_2^- \cap (\mathrm{p}_1^{-1}(A' \dot{-} \mathrm{p}_1(B)) \cap H_3^+)$. Hence $A \dot{-} B$ is the wedge with the base $(A' \dot{-} p_1(B)) \cap H_3^+$. By [14] this base is a summand of A' and by Proposition 2.2(*iii*) the set $A \dot{-} B$ is a summand of A.

Let A be a dull wedge with the width w in proper position and $B \in \mathcal{K}(\mathbb{R}^3)$, $B \subset A$, $B \cap H_1 \neq \emptyset$, $B \cap H_2 \neq \emptyset$. Then

$$\begin{aligned} A \dot{-} B = \bigcap_{b\in B}(A-b) &= \bigcap_{b\in B}(H_1^+ - b) \cap \bigcap_{b\in B}(H_2^{w-} - b) \cap \bigcap_{b\in B}(\mathrm{p}_1^{-1}A' - b) \\ &= H_1^+ \cap H_2^{w-} \cap \bigcap_{b\in B}(\mathrm{p}_1^{-1}(A' - \mathrm{p}_1(b))) \\ &= H_1^+ \cap H_2^{w-} \cap \mathrm{p}_1^{-1}(\mathrm{A}' \dot{-} \mathrm{p}_1(B)). \end{aligned}$$

Hence $A \dot{-} B$ is the wedge in proper position with the base $(A' \dot{-} \mathrm{p}_1(B)) \cap H_3^+$. Then $A \dot{-} B$ is a summand of A.

At last let A be a dull wedge with the width w in proper position and $B \in \mathcal{K}(\mathbb{R}^3)$, $B \subset A$, $B \cap H_1 \neq \emptyset$, $B \cap H_3 \neq \emptyset$. Then $A \dot{-} B = H_1^+ \cap H_2^{v-} \cap \mathrm{p}_1^{-1}(A' \dot{-} \mathrm{p}_1(B))$ where $v \in [0, w]$. The base $A \dot{-} B \cap H_1$ is equal to $A' \dot{-} p_1(B)$ because $A' \dot{-} p_1(B) \subset H_3^+$. Hence the set $A \dot{-} B$ is the dull wedge with the width $v \leq w$ in proper position and the base equal to $A' \dot{-} p_1(B)$. Therefore, $A \dot{-} B$ is a summand of A. □

Notice that every three-dimmensional summand of a wedge or a dull wedge is a wedge or a dull wedge. Hence such a summand is a Sallee set.

Remarks. 1. A pair of wedges in proper position (one of them can be a dull wedge) is minimal if and only if the pair of their bases is minimal. (For the properties of pairs of compact convex sets see [11]).

2. The pair (A, B) of dull wedges with the widths w_A, w_B in proper position is equivalent to the other pair (C, D) of dull wedges in proper position if and only if the pairs (A', B'), (C', D') of their bases are equivalent and $w_A + w_D = w_B + w_C$.

3. Any two equivalent minimal pairs of wedges in proper position are translates of each other.

4. Theorem 2.3 can be generalized to higher dimensions in the following way: Let $n \geq 3$ and S be an $n-2$ dimensional simplex in $\mathbb{R}^{n-2}$. We call the set $A \in \mathcal{K}(\mathbb{R}^n)$ a wedge in proper position if for any $x = (x_1, \ldots, x_n) \in A$ and $y \in [0, x_2]$ the point $y = (x_1, y_2, y_3, \ldots, y_n)$ belongs to A if and only if $(y_3, \ldots, y_n) \in y_2 \cdot S$. Then A is a Sallee set.

3. Three-dimensional polyhedral Sallee sets.

First, we prove that skew cubes in $\mathbb{R}^3$ are Sallee sets. We denote $H_i = \{x \in \mathbb{R}^3 \mid x_i = 0\}$, $H_i^+ = \{x \in \mathbb{R}^3 \mid x_i \geq 0\}$, $H_i^w = \{x \in \mathbb{R}^3 \mid x_i = 1 + w \cdot x_{i+1}\}$, $H_i^{w-} = \{x \in \mathbb{R}^3 \mid x_i \leq 1 + w \cdot x_{i+1}\}$, $i = 1, 2, 3$.

Definition. We say that a polytope A is *a skew cube in proper position* if $A = H_1^+ \cap H_2^+ \cap H_3^+ \cap H_1^{u-} \cap H_2^{v-} \cap H_3^{w-}$ for some $u > 0$, $v > 0$, $w > 0$.

Notice that the intersection of these six halfspaces is a polytope if and only if $uvw < 1$. We also observe that a polytope is a skew cube if and only if it is a skew cube in proper position in some system of coordinates.

Theorem 3.1. *Let A be a skew cube. Then A is a Sallee set.*

Proof. Let $B \in \mathcal{K}(\mathbb{R}^3)$ and $A \dot{-} B \neq \emptyset$. Then

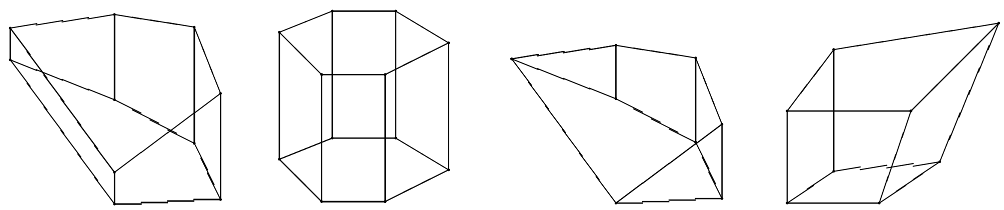
$$A \dot{-} B = (H_1^+ \dot{-} B) \cap (H_2^+ \dot{-} B) \cap (H_3^+ \dot{-} B) \\ \cap (H_1^{u-} \dot{-} B) \cap (H_2^{v-} \dot{-} B) \cap (H_3^{w-} \dot{-} B)$$

where $(H_1^+ \dot{-} B), \ldots, (H_3^{w-} \dot{-} B)$ are halfspaces determined by a planes parallel to H_1, H_2, H_3, H_1^u, H_2^v, H_3^w. Then $A \dot{-} B$ is a polytope with maximum six facets, all of them parallel to facets of A. Notice that the edges of A contained in $H_1 \cap H_2$ and $H_2 \cap H_1^u$ are parallel and the angle between the facets of A containing the first edge is the right angle and the angle between the facets of A containing the second edge is greather that the right angle. Similar facts hold true for edges of A contained in $H_2 \cap H_3$ and $H_3 \cap H_2^v$ and $H_3 \cap H_1$ and $H_1 \cap H_3^w$. Therefore, all possible edges of $A \dot{-} B$ are coparallel to the edges of A. Each facet of $A \dot{-} B$ is contained in some translate of the parallel facet of A. Consider any given edge k of $A \dot{-} B$. The edge k is contained in two facets F_1, F_2 of $A \dot{-} B$. Then k is contained in translates of two facets of A. An edge of A parallel to k dominates one of these facets. A side k dominates polygon P if and only if the sum of two angles of P adjacent to k is not greater than 180^o. Hence k cannot be longer than coparallel edge of A. Therefore by Theorem 3.2.8, p. 148 in [14] the set $A \dot{-} B$ is a summand of A. □

Notice that a summand of a skew cube is a skew cube or a quadrilateral wedge or a simplex or a singleton, hence it is a Sallee set.

Theorem 3.2. *A three-dimensional polytope A is a Sallee set if and only if A is one of four types of polytopes:*

(i) *polygonal dull wedge*
(ii) *polygonal prism*
(iii) *polygonal wedge*
(iv) *skew cube.*



(i) (ii) (iii) (iv)

Figure 3.1: Four types of Sallee polytopes in $\mathbb{R}^3$

Proof. By [10] we know that all three-dimensional polyhedral Sallee sets are strongly monotypic and the family of all strongly monotypic polytopes in $\mathbb{R}^3$ consists of four types of polytopes [10], [1]; namely, polygonal prisms, polygonal wedges, polygonal dull wedges, skew cubes. Then it is enough to show that any set belonging to any one of these four types is a Sallee set.

First, polygonal prism is a direct sum of two Sallee sets and by [2] it is a Sallee set itself. Polygonal wedges and dull wedges are Sallee sets due to Theorem 2.3. Skew cubes are Sallee sets by Theorem 3.1. □

4. Parts of the ball and other non-polyhedral sets

Theorem 2.3 gives us a broad class of non-polyhedral Sallee sets in $\mathbb{R}^3$. In this section we show some other non-polyhedral Sallee sets being parts of the ball. We also give some examples of non-polyhedral sets not being Sallee sets in $\mathbb{R}^3$. By $A \vee B$ we denote the convex hull of $A \cup B$.

Lemma 4.1. *Let H be the hyperplane $H = \{x \in \mathbb{R}^n \mid x_n = 0\}$, H^+ be the halfspace $\{x \in \mathbb{R}^n,\, x_n \geq 0\}$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ be the isometry $T(x) = (x_1, x_2, \ldots, x_{n-1}, -x_n\}$. If $A,\, B \in \mathcal{K}(\mathbb{R}^n)$, $T(A) = A$, $T(B) = B$, then*

$$A \cap H^+ + B \cap H^+ = (A + B) \cap H^+.$$

Proof. Let $a \in A$ i $b \in B$, $a_n + b_n \geq 0$. If $a \notin H^+$ then denote $a^{'} = (a_1, \ldots, a_{n-1}, 0)$, $b^{'} = (b_1, \ldots, b_{n-1}, a_n + b_n)$. Both $a^{'}$ and b' belong to H^+. Notice that $a^{'} \in a \vee T(a) \subset A$, $b^{'} \in b \vee T(b) \subset B$. Then $a + b = a' + b' \in A \cap H^+ + B \cap H^+$. The case of $b \notin H^+$ is similar. Therefore $(A + B) \cap H^+ \subset A \cap H^+ + B \cap H^+$. The reverse inclusion is obvious. □

Lemma 4.2. *Let $A \in \mathcal{K}(\mathbb{R}^n)$ and $T(A) = A$. If $B \in \mathcal{K}(\mathbb{R}^n)$, $B \subset A \cap H^+$ and $B \cap H \neq \emptyset$, then*

$$A \cap H^+ \dot{-} B = (A \dot{-} (B \vee TB)) \cap H^+.$$

Proof. Let $x + B \subset A \cap H^+$. Since $x + B \subset H^+$ then $x_n \geq 0$. We have $Tx + TB \subset A$. Since $x + Tb \subset (x + b) \vee (Tx + Tb)$ for all $b \in B$ then $x + TB \subset (x + B) \vee (Tx + TB)$. Therefore, $x + (B \vee TB) \subset A$. Hence $x \in (A \dot{-} (B \vee TB)) \cap H^+$. On the other hand, if $x + (B \vee TB) \subset A$ and $x \in H^+$, then $x + B \subset A$ and by the fact that $B \subset H^+$ also $x + B \subset H^+$. Therefore, $x + B \subset A \cap H^+$. □

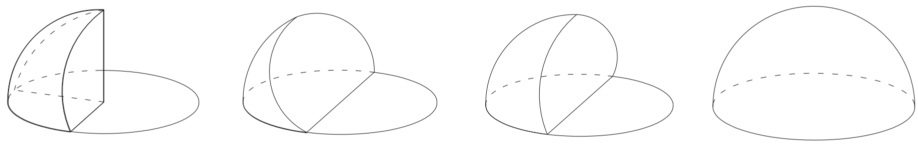
Theorem 4.3. *Let A be a Sallee set in $\mathbb{R}^n$ which is symmetric with respect to hyperplane H. Then half of the set A, namely $A \cap H^+$, cut by H, is a Sallee set.*

Proof. Let T be the isometry in $\mathbb{R}^n$ such that $Tx = x$ if and only if $x \in H$. Let $B \in \mathcal{K}(\mathbb{R}^n)$ and $A \cap H^+ \dot{-} B \neq \emptyset$. We can assume that $B \subset A \cap H^+$. Since $T(A) = A$ we have $TB \subset A \cap H^-$ and for some $x \in \mathbb{R}^n$, $x + B \subset A \cap H^+$ and $(x + B) \cap H \neq \emptyset$. Hence replacing $x + B$ with B we can assume that the assumptions of Lemma 4.2 are fulfilled. Then $A \cap H^+ \dot{-} B = (A \dot{-} (B \vee TB)) \cap H^+$. Since A is a Sallee set there exists $C \in \mathcal{K}(\mathbb{R}^n)$ such that $(A \dot{-} (B \vee TB)) + C = A$. Since $T(A) = A$ and $T(B \vee TB) = B \vee TB$ then $T(A \dot{-} (B \vee TB)) = A \dot{-} (B \vee TB)$ and $T(C) = C$. By Lemma 4.1 we conclude that $(A \cap H^+ \dot{-} B) + C \cap H^+ = (A \dot{-} (B \vee TB)) \cap H^+ + C \cap H^+ = A \cap H^+$. □

Theorem 4.3 applied to Sallee polytopes will not give us additional types of polytopes. Half of dull wedge is a dull wedge. The same is true of prisms and wedges. And skew cubes are symmetric with respect to no plane. However, we can apply Theorem 4.3 to the Euclidean ball.

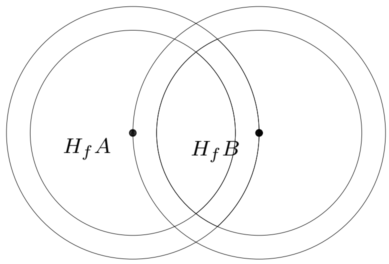
Corollary. *Let $\mathbb{B}^n$ be the unit ball in $\mathbb{R}^n$. Let H_i^+ be the halfspace $\{x \in \mathbb{R}^n,\ x_i \geq 0\}$, $i = 1, 2, \ldots, n$ and $\hat{H}_k^+$ be the halfspace $\{x \in \mathbb{R}^n \mid x_n \leq x_1 \cdot tg(2^{-k} \cdot \pi)\}$ where k is an integer and $k \geq 2$. Then the sets $A_m = \mathbb{B}^n \cap \bigcap_{i \in I} H_i^+$, $m = 1, \ldots, n$ and $A_{m,k} = \mathbb{B}^n \cap \hat{H}_k^+ \cap \bigcap_{i \in I} H_i^+$, $m = 1, \ldots, n-1$ are Sallee sets.*

Proof. Applying Theorem 4.3 to the ball we obtain that half of the ball is a Sallee set in $\mathbb{R}^n$. Applying Theorem 4.3 repeatedly, we obtain that a quarter of the ball, one eighth of the ball, etc. are Sallee sets. □



In the following example we give non polyhedral sets that are not Sallee sets.

Examples. 1. Let $A = \{x \in \mathbb{B}^n \mid x_1 \geq \alpha\}$, $n \geq 3$, where $-1 < \alpha < 0$. Let $u = \{-\alpha, 1, 0, \ldots, 0\}$. Denote $B = A \cap (A + u)$. Let $f(x) = -x_1$. Then $H_f A = \{x \in \mathbb{R}^n \mid x_1 = \alpha,\ x_2^2 + \ldots + x_n^2 \leq \sqrt{1-\alpha^2}\}$, $H_f B = \{x \in \mathbb{R}^n \mid x_1 = 0,\ x_2^2 + \ldots + x_n^2 \leq 1,\ (x_2 - 1)^2 + x_3^2 + \ldots + x_n^2 \leq \sqrt{1-\alpha^2}\}$. Notice that $H_f B$ is not a summand of $H_f A$. Therefore B is not a summand of A and A is not a Sallee set. Hence north part of the ball cut by the plane containing any southern parallel is not a Sallee set.

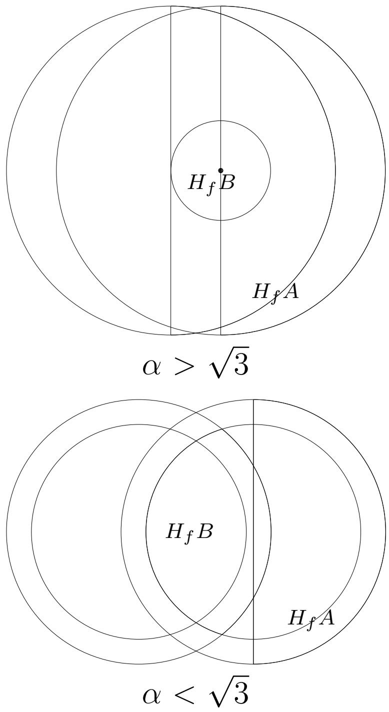


2. Let $A = \{x \in \mathbb{B}^n \mid x_1 \geq 0,\ x_2 \geq -\alpha \cdot x_1\}$, $n \geq 3$, $\alpha > 0$. Let $u = \left(\frac{1}{\sqrt{1+\alpha^2}}, -\frac{\alpha}{\sqrt{1+\alpha^2}}, 0, \ldots, 0\right) \in \mathbb{R}^n$. Denote $B = A \cap (A + u)$. Then

$$H_f A = \{x \in \mathbb{R}^n \mid x_1 = 0,\ x_2 \geq 0,\ x_2^2 + \ldots + x_n^2 \leq 1\}$$

and $H_f B = \left\{x \in \mathbb{R}^n \mid x_1 = \frac{1}{\sqrt{1+\alpha^2}},\ x_2^2 + \ldots + x_n^2 \leq \frac{\alpha^2}{1+\alpha^2},\ \left(x_2 - \frac{\alpha}{\sqrt{1+\alpha^2}}\right)^2 + x_3^2 + \ldots + x_n^2 \leq 1\right\}$.

If $\alpha \leq \sqrt{3}$ then $H_f B$ is an $(n-1)$-dimensional ball, if $\alpha > \sqrt{3}$ then $H_f B$ is an intersection of two balls. In either case $H_f B$ is not a summand of $H_f A$ which is a halfball. Therefore, A is not a Sallee set. Consider convex part of the ball cut by two halfplanes containing different meridians. If the angle between the two hyperplanes is greater than 90 degrees and less than 180 degrees then the part of ball is not a Sallee set.



3. The union of cylinder and two halfballs (or Minkowski sum of a ball and a segment) is not a Sallee set.

4. Let T be a Reuleaux triangle (see [5], p. 122, [16]) with the width equal to 2 lying on the plane $y = 0$ in $\mathbb{R}^3$. Let 0 be the center of this triangle and let one of its verticles lie on the positive halfaxis OZ. Let A be the set created by rotating the triangle T around the axis OZ. Then A is a set of constant width. The set A is not a Sallee set.

5. Let $A \subset \mathbb{R}^3$ be the set of constant width $2\sqrt{2}$ containing the verticles $a = (1, 1, 1)$, $b = (1, -1, -1)$, $c = (-1, 1, -1)$, $d = (-1, -1, 1)$ described in [16], p. 88 (three dimensional generalization of the Reuleaux triangle). The set A is not a Sallee set.

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