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Dual Variational Formulations for a Non-Linear Model of Plates

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This article develops dual variational formulations for the two dimensional equations of the nonlinear elastic Kirchhoff-Love plate model. The first duality principle presented is the classical one and may be found in similar format in Telega, [10], Gao, [7]. It is worth noting that such results are valid only for positive definite membrane forces, however, we obtain new dual variational formulations which relax or even remove such constraints. Among our results we have a convex dual variational formulation which allows non positive definite membrane forces. In the last section, similarly to the Triality criterion introduced in Gao, [9], we obtain sufficient conditions of optimality for the present case. Finally, the results are based on fundamental tools of Convex Analysis and also relevant for the developed theory is the concept of Legendre Transform, which can easily be analytically expressed for the mentioned model.

1. Introduction

The main objective of the present article is to develop systematic approaches for obtaining dual variational formulations for systems originally modeled by non-linear differential equations.

Duality for linear systems is well established and is the main subject of classical Convex Analysis, considering that in case of linearity, both primal and dual formulations are generally convex. In case of non-linear differential equations, some complications occur and the standard models of duality for convex analysis must be modified and extended.

Particularly in the case of Kirchhoff-Love plate model, it is present a non-linearity concerning the strain tensor (that is, a geometric non-linearity). To apply the classical results of convex analysis and obtain the complementary formulation is possible only for a special class of external load (which leads to non-compressed plates, please see Telega, [10], Gao, [7] and other references therein).

Now we pass to describe the primal formulation and related duality principles. For a plate of middle surface represented by an open bounded set $S \subset \mathbb{R}^2$, whose boundary is denoted by Γ, subjected to a load to be specified, we denote by $u_\alpha : S \to \mathbb{R}$ ($\alpha = 1, 2$) the displacements on the horizontal plane and by $w : S \to \mathbb{R}$, the vertical displacement field, so that the boundary value form of the Kirchhoff-Love model can be expressed

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by the equations:

$$\begin{cases} N_{\alpha\beta,\beta}=0, \\ Q_{\alpha,\alpha}+M_{\alpha\beta,\alpha\beta}+P=0, \quad \text{a.e. in } S \end{cases} \tag{1}$$

and

$$\begin{cases} N_{\alpha\beta}.n_\beta-\bar{P}_\alpha=0, \\ (Q_\alpha+M_{\alpha\beta,\beta})n_\alpha+\dfrac{\partial(M_{\alpha\beta}t_\alpha n_\beta)}{\partial s}-\bar{P}=0, \\ M_{\alpha\beta}n_\alpha n_\beta-M_n=0, \quad \text{on } \Gamma_t, \end{cases} \tag{2}$$

where:

$$N_{\alpha\beta}=H_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu},$$
$$M_{\alpha\beta}=h_{\alpha\beta\lambda\mu}\kappa_{\lambda\mu}$$

and

$$\gamma_{\alpha\beta}(u)=\frac{1}{2}(u_{\alpha,\beta}+u_{\beta,\alpha}+w_{,\alpha}w_{,\beta})$$
$$\kappa_{\alpha\beta}(u)=-w_{,\alpha\beta},$$

with the boundary conditions

$$u_\alpha=w=\frac{\partial w}{\partial \mathbf{n}}=0, \quad \text{on } \Gamma_u.$$

Here, $\{N_{\alpha\beta}\}$ denote the membrane forces, $\{M_{\alpha\beta}\}$ denote the moments and $\{Q_\alpha\}=\{N_{\alpha\beta}w_{,\beta}\}$ stand for functions related to the rotation work of membrane forces, $P\in L^2(S)$ is a field of vertical distributed forces applied on S, $(\bar{P}_\alpha,\bar{P})\in(L^2(\Gamma_t))^3$ denote forces applied to Γ_t concerning the horizontal directions defined by $\alpha=1,2$ and vertical direction respectively, and M_n are distributed moments applied also to Γ_t, where Γ is such that $\Gamma_u\subset\Gamma$, $\Gamma=\Gamma_u\cup\Gamma_t$ and $\Gamma_u\cap\Gamma_t=\emptyset$. Finally, the matrices $\{H_{\alpha\beta\lambda\mu}\}$ and $\{h_{\alpha\beta\lambda\mu}\}$ are related to the coefficients of Hooke's Law.

The correspondening primal variational formulation to this boundary value model is represented by the functional $J:U\to\mathbb{R}$, where:

$$\begin{aligned} J(u)=&\frac{1}{2}\int_S H_{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu}dS+\frac{1}{2}\int_S h_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}\kappa_{\lambda\mu}dS \\ &-\int_S PwdS-\int_{\Gamma_t}\left(\bar{P}w+\bar{P}_\alpha u_\alpha-M_n\frac{\partial w}{\partial\mathbf{n}}\right)d\Gamma \end{aligned}$$

and

$$U=\left\{(u_\alpha,w)\in W^{1,2}(S)\times W^{1,2}(S)\times W^{2,2}(S),\ u_\alpha=w=\frac{\partial w}{\partial\mathbf{n}}=0 \ \text{ on } \Gamma_u\right\}.$$

The first duality principle presented is the Classical one (again we mention the earlier similar results in Telega, [10], Gao, [7]), and is obtained by applying a little change of Rockafellar's approach for convex analysis. We claim to have developed a slightly different proof from the one found in [10], now by using the definition of Legendre Transform and related properties. Such a result may be summarized as:

$$\inf_{u\in U}\{J(u)\}=\sup_{v^*\in\mathcal{A}^*\cap C^*}\{-G_L^*(v^*)\}. \tag{3}$$

The dual functional, denoted by $-G_L^* : \mathcal{A}^* \cap C^* \to \bar{\mathbb{R}}$ is expressed as:

$$G_L^*(v^*) = \left\{ \frac{1}{2} \int_S \bar{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} dS + \frac{1}{2} \int_S \bar{h}_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\lambda\mu} dS + \frac{1}{2} \int_S \bar{N}_{\alpha\beta} Q_\alpha Q_\beta dS \right\},$$

where C^* is defined by equations (1) and (2) and,

$$\mathcal{A}^* = \left\{ v^* \in Y^* |\ N_{11} > 0, N_{22} > 0, \text{ and } N_{11}N_{22} - N_{12}^2 > 0, \text{ a.e. in } S \right\}, \tag{4}$$

here $v^* = \{N_{\alpha\beta}, M_{\alpha\beta}, Q_\alpha\} \in Y^* = L^2(S; \mathbb{R}^{10}) \equiv \mathbf{L}^2(S)$.

Therefore, as the functional $G_L^*(v^*)$ is convex in $\mathcal{A}^*$, the duality is perfect if the optimum solution for the primal formulation satisfies the constraints indicated in (4), however it is important to emphasize that such constraints imply no compression along the plate.

For the second and third principles, we highlight that our dual formulations remove or relax the constraints concerning the external load, and are valid even for compressed plates.

Still for these two principles, we use a theorem (Toland, [12]) which does not require convexity of primal functionals. Such a result can be summarized as:

$$\inf_{u \in U} \{G(u) - F(u)\} = \inf_{u^* \in U^*} \{F^*(u^*) - G^*(u^*)\}.$$

Here $G : U \to \mathbb{R}$ and $F : U \to \mathbb{R}$ and, $F^* : U^* \to \mathbb{R}$ and $G^* : U^* \to \mathbb{R}$ denote the primal and dual functionals respectively.

Particularly for the second principle, we modify the above result by applying it to a not one to one relation between primal and dual variables, obtaining the final duality principle expressed as follows:

$$\inf_{(u,p) \in U \times Y} \{J_K(u,p)\} \leq \inf_{(\hat{u},v^*) \in U \times Y^*} \{J_K^*(\hat{u}, v^*)\}$$

where:

$$J_K(u,p) = G(\Lambda u + p) - F(u) + \frac{K}{2} \langle p, p \rangle_{\mathbf{L}^2(S)}$$

and

$$J_K^*(\hat{u}, v^*) = F^*(\Lambda^* v^*) - G_L(v^*) + K \left\| \Lambda \hat{u} - \frac{\partial g_L^*(v^*)}{\partial y^*} \right\|_{\mathbf{L}^2(S)}^2 + \frac{1}{2K} \langle v^*, v^* \rangle_{\mathbf{L}^2(S)}.$$

Here $K \in \mathbb{R}$ is a positive constant and we are particularly concerned with the fact that (even though we do not prove it in the present article, postponing a more rigorous analysis concerning the behavior of u_K below indicated as $K \to +\infty$, for a future work)

$$J_K(u_K, p_K) \to J(u_0), \quad \text{as } K \to +\infty$$

and

$$J_K^*(\hat{u}_K, v_K^*) \to J(u_0), \quad \text{as } K \to +\infty$$

where

$$J_K(u_K, p_K) = \inf_{(u,p) \in U \times Y} \{J_K(u,p)\},$$

$$J_K^*(\hat{u}_K, v_K^*) = \inf_{(\hat{u},v^*)\in U\times Y^*}\{J_K^*(\hat{u}, v^*)\}$$

and

$$J(u_0) = \inf_{u\in U}\{J(u) = G(\Lambda u) - F(u)\}.$$

Specifically for the third duality principle, the dual variables must satisfy the following constraints:

$$N_{11} + K > 0, \quad N_{22} + K > 0 \quad \text{and} \quad (N_{11} + K)(N_{22} + K) - N_{12}^2 > 0, \quad \text{a.e. in } S. \tag{5}$$

Such a principle may be summarized by the following result:

$$\inf_{u\in U}\{G(\Lambda u) - F(\Lambda_1 u) - \langle u, p\rangle_U\} \leq \inf_{z^*\in Y^*}\left\{\sup_{v^*\in B^*(z^*)}\{F^*(z^*) - G_L^*(v^*)\}\right\}$$

where:

$$B^*(z^*) = \{v^* \in Y^* \mid \Lambda^* v^* - \Lambda_1^* z^* - p = 0\}.$$

Therefore the constant $K > 0$ must be chosen so that the optimum point concerning the primal formulation satisfies the constraints indicated in (5) (because these relations also define an enlarged region in which the the analytical expression of the functional $G_L^* : Y^* \to \mathbb{R}$ is convex, so that, in this case, negative membrane forces are allowed).

In Section 8, we present a convex dual variational formulation which may be expressed through the following duality principle:

$$\inf_{u\in U}\{J(u)\} = \sup_{(v^*,z^*)\in E^*\cap B^*}\{-G^*(v^*) + \langle z_\alpha^*, z_\alpha^*\rangle_{L^2(S)}/(2K)\}$$

where,

$$\begin{aligned} &G^*(v^*) = G_L^*(v^*) \\ &= \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu}\, dS + \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\lambda\mu}\, dS + \frac{1}{2}\int_S \bar{N}_{\alpha\beta}^K Q_{,\alpha} Q_{,\beta}\, dS \end{aligned}$$

if $v^* \in E^*$, where $v^* = \{\{N_{\alpha\beta}\}, \{M_{\alpha\beta}\}, \{Q_\alpha\}\} \in E^* \Leftrightarrow v^* \in L^2(S; \mathbb{R}^{10})$ and

$$N_{11} + K > 0 \quad N_{22} + K > 0 \quad \text{and} \quad (N_{11} + K)(N_{22} + K) - N_{12}^2 > 0, \quad \text{a.e. in } S,$$

where

$$\{\bar{N}_{\alpha\beta}^K\} = \left\{\begin{array}{ll} N_{11} + K & N_{12} \\ N_{12} & N_{22} + K \end{array}\right\}^{-1} \tag{6}$$

and

$$(v^*, z^*) \in B^* \Leftrightarrow \begin{cases} N_{\alpha\beta,\beta} + P_\alpha = 0, \\ Q_{\alpha,\alpha} + M_{\alpha\beta,\alpha\beta} - z_{\alpha,\alpha}^* + P = 0, \\ \bar{h}_{1212} M_{12} + z_{1,2}^*/K = 0, \\ z_{1,2}^* = z_{2,1}^*, \ \text{a.e. in } S, \ \text{and,} \ z^* = \theta \text{ on } \Gamma. \end{cases}$$

Here we are supposing the existence of $u_0 \in U$ such that $\delta J(u_0) = \theta$, and so that there exists $K > 0$ for which $N_{11}(u_0) + K > 0$, $N_{22}(u_0) + K > 0$, $(N_{11}(u_0) + K)(N_{22}(u_0) +$

$K) - N_{12}(u_0)^2 > 0$ (a.e. in S) and $h_{1212}/(2K_0) > K$ where K_0 is the constant related to *Poincaré* Inequality and,

$$N_{\alpha\beta}(u_0) = H_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(u_0).$$

Finally, in the last section, we prove a result similar to those obtained through the Triality criterion introduced in Gao, [9], and establish sufficient conditions for the existence of a minimizer for the primal formulation. Such conditions may be summarized by $\delta J(u_0) = \theta$ and

$$\frac{1}{2}\int_S N_{\alpha\beta}(u_0)w_{,\alpha}w_{,\beta}dS + \frac{1}{2}\int_S h_{\alpha\beta\lambda\mu}w_{,\alpha\beta}w_{,\lambda\mu}dS \geq 0, \quad \forall w \in W_0^{2,2}(S). \qquad \square$$

For this last result, we claim just the proof we give to be new, the statement of results themselves follows Gao, [9].

2. Preliminaries

We denote by U and Y Banach spaces which the topological dual spaces are identified with U^* and Y^* respectively. The Canonical duality pairing between U and U^* is denoted by $\langle .,. \rangle_U : U \times U^* \to \mathbb{R}$, through which the continuous linear functionals on U are represented.

Let us now recall that, given $F : U \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ its Fenchel conjugate $F^* : U^* \to \bar{\mathbb{R}}$ is defined as:

$$F^*(u^*) = \sup_{u \in U}\{\langle u, u^* \rangle_U - F(u)\}, \quad \forall u^* \in U^*.$$

Recall also that the Fenchel sub-differential $\partial F(u)$ is the subset of U^* given by

$$\partial F(u) = \{u^* \in U^*, \text{ such that } \langle v - u, u^* \rangle_U + F(u) \leq F(v), \ \forall v \in U\}.$$

The two next definitions are also relevant.

Definition 2.1 (Gâteaux Differentiability). A functional $F : U \to \bar{\mathbb{R}}$ is said to be Gâteaux differentiable at $u \in U$ if there exists $u^* \in U^*$ such that:

$$\lim_{\lambda \to 0} \frac{F(u + \lambda h) - F(u)}{\lambda} = \langle h, u^* \rangle_U, \quad \forall h \in U.$$

The vector u^* is said to be the Gâteaux derivative of $F : U \to \mathbb{R}$ at u and may be denoted as follows:

$$u^* = \frac{\partial F(u)}{\partial u} \quad \text{or } u^* = \delta F(u).$$

Definition 2.2 (Adjoint Operator). Let U and Y be Banach spaces and $\Lambda : U \to Y$ a continuous linear operator. The Adjoint Operator related to Λ, denoted by $\Lambda^* : Y^* \to U^*$ is defined through the equation:

$$\langle u, \Lambda^* v^* \rangle_U = \langle \Lambda u, v^* \rangle_Y, \quad \forall u \in U, \ v^* \in Y^*. \tag{7}$$

The next results are concerned with the representation of the polar functional. Their demonstrations can also be met in [6].

Recall that given an open subset $S \subset \mathbb{R}^n$, $g : S \times \mathbb{R}^l \to \mathbb{R}$ is a Carathéodory mapping whenever:

$$\forall \xi \in \mathbb{R}^l, \quad x \mapsto g(x,\xi) \text{ is a measurable function,}$$

and

$$\text{for almost all } x \in S, \quad \xi \mapsto g(x,\xi) \text{ is a continuous function.}$$

For the proposition below indicated we consider the particular case $U = U^* = [L^2(S)]^l$ (this is a especial situation concerning the more general hypothesis presented in [6]).

Proposition 2.3. *For $U = U^* = [L^2(S)]^l$, consider $g(x,\xi)$ a Carathéodory mapping as above indicated and the functional $G : U \to \mathbb{R}$, given by $G(u) = \int_S g(x,u(x))dS$. Thus, we may express $G^* : U^* \to \bar{\mathbb{R}}$ as:*

$$G^*(u^*) = \int_S g^*(x,u^*(x))dS,$$

where $g^*(x,y) = \sup_{\eta \in \mathbb{R}^l}(y.\eta - g(x,\eta))$, almost everywhere in S.

For non-convex functionals may be sometimes difficult to express analytically conditions for a global extremum. This fact motivates the definition of Legendre Transform, which is established through a local extremum.

Definition 2.4 (Legendre's Transform and Associated Functional). Consider a differentiable function $g : \mathbb{R}^n \to \mathbb{R}$, its Legendre Transform, denoted by $g_L^* : R_L^n \to \mathbb{R}$ is expressed as:

$$g_L^*(y^*) = x_{0^i}.y_i^* - g(x_0),$$

where x_0 is solution of the system:

$$y_i^* = \frac{\partial g(x_0)}{\partial x_i}, \tag{8}$$

and $R_L^n = \{y^* \in \mathbb{R}^n$ such that equation (8) has a unique solution$\}$.

Furthermore, considering the functional $G : Y \to \mathbb{R}$ defined as $G(v) = \int_S g(v)dS$, we define the Associated Legendre Transform Functional, denoted by $G_L^* : Y_L^* \to \mathbb{R}$ as:

$$G_L^*(v^*) = \int_S g_L^*(v^*)dS,$$

where $Y_L^* = \{v^* \in Y^* \mid v^*(x) \in R_L^n, \text{ a.e. in } S\}$.

About the Legendre Transform, we have the following well known result, connecting the local extremals of primal and dual formulations.

Theorem 2.5. *Consider the functional $J : U \to \bar{\mathbb{R}}$ defined as $J(u) = (G \circ \Lambda)(u) - \langle u, f\rangle_U$ where $\Lambda(= \{\Lambda_i\}) : U \to Y$ $(i \in \{1,...,n\})$ is a continuous linear operator and, $G : Y \to \mathbb{R}$ is a functional that can be expressed as $G(v) = \int_S g(v)dS$, $\forall v \in Y$ (here $g : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function that admits differentiable Legendre Transform denoted by $g_L^* : R_L^n \to \mathbb{R}$).*

Under these assumptions we have:

$$\delta J(u_0) = \theta \Leftrightarrow \delta(-G_L^*(v_0^*) + \langle u_0, \Lambda^* v_0^* - f\rangle_U) = \theta,$$

where $v_0^* = \frac{\partial G(\Lambda(u_0))}{\partial v}$ *is supposed to be such that:* $v_0^*(x) \in R_L^n$, *a.e. in* S *and in this case:*

$$J(u_0) = -G_L^*(v_0^*).$$

We now present the fundamental duality principle met at reference [6] and of great interest for the calculus of variations:

Theorem 2.6. *Let* $G : Y \to \bar{\mathbb{R}}$ *and* $F : U \to \mathbb{R}$ *be two convex l.s.c. (lower semi-continuous) functionals so that* $J : U \to \bar{\mathbb{R}}$ *defined as:*

$$J(u) = (G \circ \Lambda)(u) + F(u)$$

is bounded from below, where $\Lambda : U \to Y$ *is a continuous linear operator which the respective adjoint is denoted by* $\Lambda^* : Y^* \to U^*$. *Thus, if there exists* $\hat{u} \in U$ *such that* $F(\hat{u}) < +\infty$, $G(\Lambda\hat{u}) < +\infty$, *being* G *continuous at* $\Lambda\hat{u}$, *we have:*

$$\inf_{u\in U}\{J(u)\} = \sup_{v^*\in Y^*}\{-G^*(v^*) - F^*(-\Lambda^* v^*)\}$$

and there exist at least one $v_0^* \in Y^*$ *which maximizes the dual formulation. If in addition* U *is reflexive and*

$$\lim_{||u||\to+\infty} J(u) = +\infty$$

then both primal and dual formulations have global extremals so that, there exist $u_0 \in U$ *and* $v_0^* \in Y^*$ *such that*

$$J(u_0) = \min_{u\in U}\{J(u)\} = \max_{v^*\in Y^*}\{-G^*(v^*) - F^*(-\Lambda^* v^*)\} = -G^*(v_0^*) - F^*(-\Lambda^* v_0^*)$$

and also

$$G(\Lambda u_0) + G^*(v_0^*) = \langle \Lambda u_0, v_0^*\rangle_Y,$$
$$F(u_0) + F^*(-\Lambda^* v_0^*) = \langle u_0, -\Lambda^* v_0^*\rangle_U,$$

so that

$$G(\Lambda u_0) + F(u_0) = -G^*(v_0^*) - F^*(-\Lambda^* v_0^*).$$

We are now ready to enunciate the result of Toland, [12], through which will be constructed the three last duality principles.

Theorem 2.7. *Let* $J : U \longrightarrow \overline{\mathbb{R}}$ *be a functional defined as* $J(u) = G(u) - F(u)$, $\forall u \in U$, *where there exists* $u_0 \in U$ *such that* $J(u_0) = \inf_{u\in U}\{J(u)\}$ *and* $\partial F(u_0) \neq \emptyset$, *then*

$$\inf_{u\in U}\{G(u) - F(u)\} = \inf_{u^*\in U^*}\{F^*(u^*) - G^*(u^*)\}$$

and for $u_0^* \in \partial F(u_0)$ *we have,*

$$F^*(u_0^*) - G^*(u_0^*) = \inf_{u^*\in U^*}\{F^*(u^*) - G^*(u^*)\}.$$

Observe that along the proof it is clear that $u_0^* \in \partial G(u_0)$.

3. The Primal Variational Formulation

Let $S \subset \mathbb{R}^2$ be an open bounded set (with a boundary denoted by Γ) which represents the middle surface of a plate of thickness h. The vectorial basis related to the Cartesian system $\{x_1, x_2, x_3\}$ is denoted by $(\mathbf{a}_\alpha, \mathbf{a}_3)$, where $\alpha = 1, 2$ (in general Greek indices stand for 1 or 2), $\mathbf{a}_3$ denotes the vector normal to S and $\mathbf{t}$ is the vector tangent to Γ and $\mathbf{n}$ is the outer normal to S. The displacements will be denoted by:

$$\hat{\mathbf{u}} = \{\hat{u}_\alpha, \hat{u}_3\} = \hat{u}_\alpha \mathbf{a}_\alpha + \hat{u}_3 \mathbf{a}_3,$$

The Kirchhoff-Love relations are:

$$\hat{u}_\alpha(x_1, x_2, x_3) = u_\alpha(x_1, x_2) - x_3 w(x_1, x_2)_{,\alpha} \quad \text{and} \quad \hat{u}_3(x_1, x_2, x_3) = w(x_1, x_2),$$

where $-h/2 \leq x_3 \leq h/2$ so that we have $u = (u_\alpha, w) \in U$ where

$$U = \left\{ (u_\alpha, w) \in W^{1,2}(S) \times W^{1,2}(S) \times W^{2,2}(S),\ u_\alpha = w = \frac{\partial w}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_u \right\}.$$

Observe that $\Gamma_u \subset \Gamma$, $\Gamma = \Gamma_u \cup \Gamma_t$ and $\Gamma_u \cap \Gamma_t = \emptyset$. The strain tensors are denoted by:

$$\gamma_{\alpha\beta}(u) = \frac{1}{2}[\Lambda_{1\alpha\beta}(u) + \Lambda_{2\alpha}(u)\Lambda_{2\beta}(u)] \tag{9}$$

and

$$\kappa_{\alpha\beta}(u) = \Lambda_{3\alpha\beta}(u) \tag{10}$$

where: $\Lambda = \{\{\Lambda_{1\alpha\beta}\}, \{\Lambda_{2\alpha}\}, \{\Lambda_{3\alpha\beta}\}\} : U \to Y = Y^* = L^2(S; \mathbb{R}^{10}) \equiv \mathbf{L}^2(S)$ is defined by:

$$\Lambda_{1\alpha\beta}(u) = u_{\alpha,\beta} + u_{\beta,\alpha}, \tag{11}$$

$$\Lambda_{2\alpha}(u) = w_{,\alpha} \tag{12}$$

and

$$\Lambda_{3\alpha\beta}(u) = -w_{,\alpha\beta}. \tag{13}$$

The constitutive relations are expressed as:

$$N_{\alpha\beta} = H_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}, \tag{14}$$

$$M_{\alpha\beta} = h_{\alpha\beta\lambda\mu}\kappa_{\lambda\mu} \tag{15}$$

where: $\{H_{\alpha\beta\lambda\mu}\}$ and $\{h_{\alpha\beta\lambda\mu} = \frac{h^2}{12} H_{\alpha\beta\lambda\mu}\}$, are positive definite matrices and such that $H_{\alpha\beta\lambda\mu} = H_{\alpha\beta\mu\lambda} = H_{\beta\alpha\lambda\mu} = H_{\beta\alpha\mu\lambda}$. Furthermore $\{N_{\alpha\beta}\}$ denote the membrane forces and $\{M_{\alpha\beta}\}$ the moments. The plate stored energy, denoted by $(G \circ \Lambda) : U \to \mathbb{R}$ is expressed as:

$$(G \circ \Lambda)(u) = \frac{1}{2}\int_S N_{\alpha\beta}\gamma_{\alpha\beta}\, dS + \frac{1}{2}\int_S M_{\alpha\beta}\kappa_{\alpha\beta}\, dS \tag{16}$$

and the external work, denoted as $F : U \to \mathbb{R}$, is given by:

$$F(u) = \int_S Pw\, dS + \int_{\Gamma_t} \left(\bar{P}w + \bar{P}_\alpha u_\alpha - M_n \frac{\partial w}{\partial \mathbf{n}} \right) d\Gamma, \tag{17}$$

where P denotes a vertical distributed load applied in S and $\bar{P}, \bar{P}_\alpha$ are forces applied on $\Gamma_t \subset \Gamma$ related to directions defined by $\mathbf{a}_3$ and $\mathbf{a}_\alpha$ respectively, and, M_n denote moments also applied on Γ_t. The potential energy, denoted by $J : U \to \mathbb{R}$ is expressed as:

$$J(u) = (G \circ \Lambda)(u) - F(u).$$

It is important to emphasize that conditions for the existence of a minimizer (here denoted by u_0) related to $G(\Lambda u) - F(u)$ were presented in Ciarlet, [3]. Such $u_0 \in U$ satisfies the equation:

$$\delta(G(\Lambda u_0) - F(u_0)) = \theta$$

and we should expect at least one minimizer if $||\bar{P}_\alpha||_{L_2(\Gamma_t)}$ is small enough and $m(\Gamma_u) > 0$ (here m stands for the Lebesgue measure) and with no restrictions concerning the magnitude of $||P_\alpha||_{L_2(S)}$ if $m(\Gamma) = m(\Gamma_u)$, so that in the latter case, we consider a field of distributed forces $\{P_\alpha\}$ applied on S.

Some inequalities of Sobolev type are necessary to prove the above result, and in this work we assume some regularity hypothesis concerning S and its boundary, namely: beyond S be open and bounded, also it is supposed to be connected with a Lipschitz continuous boundary Γ, and so that S is locally on one side of Γ.

The formal proof of existence of a minimizer for $J(u) = G(\Lambda u) - F(u)$ is obtained through the Direct Method of Calculus of variations. We do not repeat this procedure here, we just refer to Ciarlet, [3] for details.

4. The Legendre Transform

In this section it will be determined the Legendre Transform related to the function $g : \mathbb{R}^{10} \to \mathbb{R}$ where:

$$g(y) = \frac{1}{2} H_{\alpha\beta\lambda\mu}[(y_{1^{\alpha\beta}} + y_{1^{\beta\alpha}} + y_{2^\alpha} y_{2^\beta})/2][(y_{1^{\lambda\mu}} + y_{1^{\mu\lambda}} + y_{2^\lambda} y_{2^\mu})/2] + \frac{1}{2} h_{\alpha\beta\lambda\mu} y_{3^{\alpha\beta}} y_{3^{\lambda\mu}} \quad (18)$$

and we recall that

$$G(\Lambda u) = \int_S g(\Lambda u) dS.$$

From Definition 2.4 we may write:

$$g_L^*(y^*) = \langle y_0, y^* \rangle_{\mathbb{R}^{10}} - g(y_0)$$

where y_0 is the unique solution of the system:

$$y_i^* = \frac{\partial g(y_0)}{\partial y_i}$$

which for the above function g, implies:

$$y^*_{1^{\alpha\beta}} = H_{\alpha\beta\lambda\mu}(y_{1^{\lambda\mu}} + y_{2^\lambda} y_{2^\mu}/2)$$

$$y^*_{2^\alpha} = H_{\alpha\beta\lambda\mu}(y_{1^{\lambda\mu}}, + y_{2^\lambda} y_{2^\mu}/2) y_{2^\beta} = y^*_{1^{\alpha\beta}} y_{2^\beta},$$

and

$$y^*_{3^{\alpha\beta}} = h_{\alpha\beta\lambda\mu} y_{3^{\lambda\mu}},$$

so that inverting this system we obtain:

$$y_{02^1} = (y^*_{1^{22}}.y^*_{2^1} - y^*_{1^{12}}.y^*_{2^2})/\Delta,$$

$$y_{02^2} = (-y^*_{1^{12}}.y^*_{2^1} + y^*_{1^{11}}.y^*_{2^2})/\Delta,$$

and

$$y_{01^{\alpha\beta}} = \bar{H}_{\alpha\beta\lambda\mu} y^*_{1^{\lambda\mu}} - y_{02^\alpha}.y_{02^\beta}/2$$

where $\Delta = y^*_{1^{11}} y^*_{1^{22}} - (y^*_{1^{12}})^2$ (we recall that $y^*_{1^{12}} = y^*_{1^{21}}$, as a result of the symmetries of $\{H_{\alpha\beta\lambda\mu}\}$), and,

$$\{\bar{H}_{\alpha\beta\lambda\mu}\} = \{H_{\alpha\beta\lambda\mu}\}^{-1}.$$

By analogy:

$$y_{03^{\alpha\beta}} = \bar{h}_{\alpha\beta\lambda\mu} v^*_{3^{\lambda\mu}}$$

where:

$$\{\bar{h}_{\alpha\beta\lambda\mu}\} = \{h_{\alpha\beta\lambda\mu}\}^{-1}.$$

Thus we can define the set R^n_L, concerning Definition 2.4 as

$$R^n_L = \{y^* \in \mathbb{R}^{10} \mid \Delta \neq 0\}. \tag{19}$$

After some simple algebraic manipulations we obtain the expression for $g^*_L : R^n_L \to \mathbb{R}$, that is:

$$g^*_L(y^*) = \frac{1}{2}\bar{H}_{\alpha\beta\lambda\mu} y^*_{1^{\alpha\beta}} y^*_{1^{\lambda\mu}} dS + \frac{1}{2}\bar{h}_{\alpha\beta\lambda\mu} y^*_{3^{\alpha\beta}} y^*_{3^{\lambda\mu}} dS + \frac{1}{2} y^*_{1^{\alpha\beta}} y_{02^\alpha} y_{02^\beta} dS \tag{20}$$

and, also from Definition 2.4, we have

$$Y^*_L = \{v^* \in Y^* = L^2(S;\mathbb{R}^{10}) \equiv \mathbf{L}^2(S) \mid v^*(x) \in R^n_L \text{ a.e. in } S\}$$

so that $G^*_L : Y^*_L \to \mathbb{R}$ may be expressed as

$$G^*_L(v^*) = \int_S g^*_L(v^*) dS$$

or, from (20):

$$G^*_L(v^*) = \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu} v^*_{1^{\alpha\beta}} v^*_{1^{\lambda\mu}} dS + \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu} v^*_{3^{\alpha\beta}} v^*_{3^{\lambda\mu}} dS + \frac{1}{2}\int_S v^*_{1^{\alpha\beta}} v_{02^\alpha} v_{02^\beta} dS. \quad \square$$

Changing the notation, as below indicated,

$$v^*_{1^{\alpha\beta}} = N_{\alpha\beta}, \qquad v^*_{2^\alpha} = Q_\alpha = v^*_{1^{\alpha\beta}} v_{02^\beta} = N_{\alpha\beta} v_{02^\beta}, \qquad v^*_{3^{\alpha\beta}} = M_{\alpha\beta}$$

we could express $G^*_L : Y^*_L \to \mathbb{R}$ as

$$G^*_L(v^*) = \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} dS + \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\lambda\mu} dS + \frac{1}{2}\int_S \bar{N}_{\alpha\beta} Q_\alpha Q_\beta dS,$$

where

$$\bar{N}_{\alpha\beta} = \{N_{\alpha\beta}\}^{-1}.$$

Remark 4.1. Also we can use the transformation

$$Q_\alpha = N_{\alpha\beta} w_{,\beta}$$

and obtain:

$$G_L^*(v^*) = \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} dS + \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\lambda\mu} dS + \frac{1}{2}\int_S N_{\alpha\beta} w_{,\alpha} w_{,\beta} dS.$$

The term denoted by $G_p : Y^* \times U \to \mathbb{R}$ and expressed as

$$G_p(v^*, w) = \frac{1}{2}\int_S N_{\alpha\beta} w_{,\alpha} w_{,\beta} dS$$

is known as the Gap Function.

5. The Classical Dual Formulation

In this section we establish the dual variational formulation in the classical sense.

We recall that $J : U \to \mathbb{R}$ is expressed by

$$J(u) = (G \circ \Lambda)(u) - F(u),$$

where $(G \circ \Lambda) : U \to \mathbb{R}$ and $F : U \to \mathbb{R}$ were defined by equations (16) and (17) respectively. It is known and easy to see that

$$\inf_{u \in U}\{G(\Lambda u) + F(u)\} \geq \sup_{v^* \in Y^*}\{-G^*(v^*) - F^*(-\Lambda^* v^*)\}. \tag{21}$$

Now we prove a result concerning the representation of polar functional, namely:

Proposition 5.1. *Considering the earlier definitions and assumptions on $G : Y \to \mathbb{R}$, expressed by $G(v) = \int_S g(v) dS$, where $g : \mathbb{R}^{10} \to \mathbb{R}$ is indicated in* (18)*, we have:*

$$v^* \in \mathcal{A}^* \Rightarrow G^*(v^*) = G_L^*(v^*)$$

where

$$G_L^*(v^*) = \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} dS + \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\lambda\mu} dS + \frac{1}{2}\int_S \bar{N}_{\alpha\beta} Q_\alpha Q_\beta dS \tag{22}$$

and

$$\begin{aligned}\mathcal{A}^* = \big\{v^* = \{N_{\alpha\beta}, M_{\alpha\beta}, Q_\alpha\} \in Y^* \mid N_{11} > 0,\ N_{22} > 0,& \\ \text{and } N_{11}N_{22} - N_{12}^2 > 0,\ \text{a.e. in } S\big\}&\end{aligned} \tag{23}$$

Proof. Firstly consider the quadratic inequality in x as below indicated:

$$\bar{a}x^2 + \bar{b}x + \bar{c} \leq 0, \quad \forall x \in \mathbb{R},$$

which is equivalent to

$$(\bar{a} < 0 \text{ and } \bar{b}^2 - 4\bar{a}\bar{c} \leq 0) \quad \text{or} \quad (\bar{a} = 0,\ \bar{b} = 0 \text{ and } \bar{c} \leq 0). \tag{24}$$

Consider now the inequality

$$a_1x^2 + b_1xy + c_1y^2 + d_1x + e_1y + f_1 \leq 0, \quad \forall x, y \in \mathbb{R}^2 \tag{25}$$

and the quadratic equation related to the variable x, for

$$\bar{a} = a_1, \quad \bar{b} = b_1y + d_1 \quad \text{and} \quad \bar{c} = c_1y^2 + e_1y + f_1,$$

and for $a_1 < 0$, from (24) the inequality (25) is equivalent to:

$$(b_1^2 - 4a_1c_1)y^2 + (2b_1d_1 - 4a_1e_1)y + d_1^2 - 4a_1f_1 \leq 0, \quad \forall y \in \mathbb{R}$$

and finally, for

$$\bar{a} = b_1^2 - 4a_1c_1 < 0, \quad \bar{b} = 2b_1d_1 - 4a_1e_1 \quad \text{and} \quad \bar{c} = d_1^2 - 4a_1f_1,$$

also from (24), the last inequality is equivalent to:

$$-c_1d_1^2 - a_1e_1^2 + b_1d_1e_1 - (b_1^2 - 4a_1c_1)f_1 \leq 0. \tag{26}$$

In order to represent the polar functional related to the plate stored energy, firstly we will consider the polar functional related to $g_1(y)$, where:

$$g_1(y) = \frac{1}{2}H_{\alpha\beta\lambda\mu}\left(y_{1\alpha\beta} + \frac{1}{2}y_{2\alpha}y_{2\beta}\right)\left(y_{1\lambda\mu} + \frac{1}{2}y_{2\lambda}y_{2\mu}\right),$$

$$g(y) = g_1(y) + g_2(y)$$

and

$$g_2(y) = \frac{1}{2}h_{\alpha\beta\lambda\mu}y_{3\alpha\beta}y_{3\lambda\mu}.$$

In fact it will be determined a set in which the polar functional is represented by the Legendre Transform $g_{1L}^*(y^*)$, where, from (20):

$$g_{1L}^*(y^*) = \frac{1}{2}\bar{H}_{\alpha\beta\lambda\mu}y_{1\alpha\beta}^*y_{1\lambda\mu}^* + \frac{y_{111}^*.\left(y_{22}^*\right)^2 - 2.y_{112}^*.y_{21}^*y_{22}^* + y_{122}^*.\left(y_{21}^*\right)^2}{2\left[y_{111}^*y_{122}^* - \left(y_{112}^*\right)^2\right]} \tag{27}$$

Thus, as

$$g_1^*(y^*) = \sup_{y\in\mathbb{R}^6}\{y_{1\alpha\beta}^*y_{1\alpha\beta} + y_{2\alpha}^*y_{2\alpha} - g_1(y)\}$$

we can write:

$$g_{1L}^*(y^*) = g_1^*(y^*) \Leftrightarrow g_{1L}^*(y^*) \geq y_{1\alpha\beta}^*y_{1\alpha\beta} + y_{2\alpha}^*y_{2\alpha} - g_1(y), \quad \forall y \in \mathbb{R}^6$$

or:

$$\begin{aligned} y_{1\alpha\beta}^*y_{1\alpha\beta} + y_{2\alpha}^*y_{2\alpha} - \frac{1}{2}H_{\alpha\beta\lambda\mu}\left(y_{1\alpha\beta} + \frac{1}{2}y_{2\alpha}y_{2\beta}\right)\left(y_{1\lambda\mu} + \frac{1}{2}y_{2\lambda}y_{2\mu}\right) \\ - g_{1L}^*(y^*) \leq 0, \quad \forall y \in \mathbb{R}^6 \end{aligned} \tag{28}$$

However, considering the transformation:

$$\bar{y}_{1\alpha\beta} = y_{1\alpha\beta} + \frac{1}{2} y_{2\alpha} y_{2\beta}$$

$$y_{1\alpha\beta} = \bar{y}_{1\alpha\beta} - \frac{1}{2} y_{2\alpha} y_{2\beta} \tag{29}$$

and substituting such relations into (28), we obtain

$$\begin{aligned} & g_{1L}^*(y^*) = g_1^*(y^*) \\ \Leftrightarrow\ & y_{1\alpha\beta}^* \left(\bar{y}_{1\alpha\beta} - \frac{1}{2} y_{2\alpha} y_{2\beta} \right) + y_{2\alpha}^* y_{2\alpha} - \frac{1}{2} H_{\alpha\beta\lambda\mu} \bar{y}_{1\alpha\beta} \bar{y}_{1\lambda\mu} \\ & - g_{1L}^*(y^*) \ \leq 0, \quad \forall \{\bar{y}_{1\alpha\beta}, y_{2\alpha}\} \in \mathbb{R}^6. \end{aligned} \tag{30}$$

On the other hand, since $\{H_{\alpha\beta\lambda\mu}\}$ is a positive definite matrix we have:

$$\sup_{\{\bar{y}_{1\alpha\beta}\} \in \mathbb{R}^4} \left\{ y_{1\alpha\beta}^* \bar{y}_{1\alpha\beta} - \frac{1}{2} H_{\alpha\beta\lambda\mu} \bar{y}_{1\alpha\beta} \bar{y}_{1\lambda\mu} \right\} = \frac{1}{2} \bar{H}_{\alpha\beta\lambda\mu} y_{1\alpha\beta}^* y_{1\lambda\mu}^*, \tag{31}$$

and thus considering (31) and the expression of $g_L^*(y^*)$ indicated in (27), inequality (30) is satisfied if

$$\begin{aligned} & -\frac{1}{2} y_{1\alpha\beta}^* y_{2\alpha} y_{2\beta} + y_{2\alpha}^* y_{2\alpha} \\ & \quad - \frac{y_{111}^* . \left(y_{22}^*\right)^2 - 2. y_{112}^* . y_{21}^* y_{22}^* + y_{122}^* . \left(y_{21}^*\right)^2}{2 \left[y_{111}^* y_{122}^* - \left(y_{112}^*\right)^2 \right]} \leq 0, \quad \forall \{y_{2\alpha}\} \in \mathbb{R}^2, \end{aligned} \tag{32}$$

so that, for

$$a_1 = -\frac{1}{2} y_{111}^*, \qquad b_1 = -y_{112}^*, \qquad c_1 = -\frac{1}{2} y_{122}^*, \qquad d_1 = y_{21}^*, \qquad e_1 = y_{22}^*$$

and

$$f_1 = -\frac{y_{111}^* . \left(y_{22}^*\right)^2 - 2. y_{112}^* . y_{21}^* y_{22}^* + y_{122}^* . \left(y_{21}^*\right)^2}{2 \left[y_{111}^* y_{122}^* - \left(y_{112}^*\right)^2 \right]}$$

we obtain

$$-c_1 d_1^2 - a_1 e_1^2 + b_1 d_1 e_1 - \left(b_1^2 - 4 a_1 c_1\right) f_1 = 0$$

and therefore from (26), the inequality (28) is satisfied if $a_1 < 0$ ($y_{111}^* > 0$) and $b_1^2 - 4a_1 c_1 < 0$ ($y_{111}^* y_{122}^* - (y_{112}^*)^2 > 0$ which implies $y_{122}^* > 0$).

Thus we have shown that:

$$y^* \in A^* \Rightarrow g_1^*(y^*) = g_{1L}^*(y^*), \tag{33}$$

where

$$A^* = \left\{ y^* \in \mathbb{R}^6 \mid y_{111}^* > 0,\ y_{122}^* > 0,\ y_{111}^* y_{122}^* - (y_{112}^*)^2 > 0 \right\}$$

On the other hand, by analogy to above results, it can easily be proved that

$$g_2^*(y^*) = g_{2L}^*(y^*), \quad \forall \{y_{3\alpha\beta}^*\} \in \mathbb{R}^3 \tag{34}$$

where

$$g_{2L}^*(y^*) = \frac{1}{2}\bar{h}_{\alpha\beta\lambda\mu} y_{3\alpha\beta}^* y_{3\lambda\mu}^* \tag{35}$$

and

$$g_2^*(y^*) = \sup_{y \in \mathbb{R}^3} \left\{ y_{3\alpha\beta}^* y_{3\alpha\beta} - \frac{1}{2} h_{\alpha\beta\lambda\mu} y_{3\alpha\beta} y_{3\lambda\mu} \right\}$$

From (33) and (34), we can write

$$\text{if } y^* \in A^* \text{ then } g_1^*(y^*) + g_2^*(y^*) = g_{1L}^*(y^*) + g_{2L}^*(y^*) \leq (g_1 + g_2)^*(y^*).$$

As $(g_1 + g_2)^*(y^*) \leq g_1^*(y^*) + g_2^*(y^*)$ we have

$$\text{if } y^* \in A^* \text{ then } g_L^*(y^*) = g_{1L}^*(y^*) + g_{2L}^*(y^*) = (g_1 + g_2)^*(y^*) = g^*(y^*). \tag{36}$$

However, from Proposition 2.3:

$$G^*(v^*) = \int_S g^*(v^*) dS \tag{37}$$

so that from (36) and (37) we obtain:

$$v^* \in \mathcal{A}^* \Rightarrow G^*(v^*) = \int_S g_L^*(v^*) dS = G_L^*(v^*)$$

where:

$$\mathcal{A}^* = \{v^* \in Y^* \mid v^*(x) \in A^*, \text{ a.e. in } S\}$$

or

$$\mathcal{A}^* = \left\{ v^* \in Y^* \mid v_{1_{11}}^* > 0,\ v_{1_{22}}^* > 0, \text{ and } v_{1_{11}}^* v_{1_{22}}^* - (v_{1_{12}}^*)^2 > 0, \text{ a.e. in } S \right\}, \tag{38}$$

thus, through the notation

$$v_{1_{\alpha\beta}}^* = N_{\alpha\beta}, \qquad v_{2_\alpha}^* = Q_\alpha = v_{1_{\alpha\beta}}^* v_{0_{2\beta}} = N_{\alpha\beta} v_{0_{2\beta}}, \qquad v_{3_{\alpha\beta}}^* = M_{\alpha\beta}$$

we have

$$\begin{aligned} \mathcal{A}^* = \big\{ v^* = \{N_{\alpha\beta}, M_{\alpha\beta}, Q_\alpha\} \in Y^* \mid N_{11} > 0,\ N_{22} > 0, \\ \text{and } N_{11}N_{22} - N_{12}^2 > 0, \text{ a.e. in } S \big\}. \end{aligned} \tag{39}$$

$\square$

5.1. The Fenchel Conjugate Functional Related to $F : U \to \bar{\mathbb{R}}$

We are concerned with the evaluation of the below indicated extremum:

$$F^*(-\Lambda^* v^*) = \sup_{u \in U}\{\langle u, -\Lambda^* v^*\rangle_U - F(u)\}$$

that is:

$$F^*(-\Lambda^* v^*) = \sup_{u \in U}\{\langle \Lambda u, -v^*\rangle_Y - F(u)\}$$

Considering that:

$$F(u) = -\left(\int_S P w dS + \int_{\Gamma_t}\left(\bar{P}w + \bar{P}_\alpha u_\alpha - M_n \frac{\partial w}{\partial n}\right) d\Gamma\right) = \langle u, f\rangle_U$$

we have:

$$F^*(-\Lambda^* v^*) = \begin{cases} 0, & \text{if } v^* \in C^*, \\ +\infty, & \text{otherwise} \end{cases} \tag{40}$$

where $v^* \in C^* \Leftrightarrow v^* \in Y^*$ and

$$\begin{cases} v^*_{1\alpha\beta,\beta} = 0, \\ v^*_{2\alpha,\alpha} + v^*_{3\alpha\beta,\alpha\beta} + P = 0, \quad \text{a.e. in } S, \end{cases} \tag{41}$$

and

$$\begin{cases} v^*_{1\alpha\beta}.n_\beta - \bar{P}_\alpha = 0, \\ \left(v^*_{2\alpha} + v^*_{3\alpha\beta,\beta}\right).n_\alpha + \dfrac{\partial\left(v^*_{3\alpha\beta} t_\alpha n_\beta\right)}{\partial s} - \bar{P} = 0, \\ v^*_{3\alpha\beta} n_\alpha n_\beta - M_n = 0, \quad \text{on } \Gamma_t \end{cases} \tag{42}$$

□

Remark 5.2. We can also denote:

$$C^* = \{v^* \in Y^* \mid \Lambda^* v^* = f\}, \tag{43}$$

where the relation $\Lambda^* v^* = f$ is defined by (41) and (42).

5.2. The Duality Principle

Considering inequality (21), the expression of $G^*(v^*)$, and the set C^* above defined, we can write:

$$\inf_{u \in U}\{(G \circ \Lambda)(u) - F(u)\} \geq \sup_{v^* \in \mathcal{A}^* \cap C^*}\{-G^*_L(v^*)\} \tag{44}$$

so that the final form of the concerned duality principle results from the following theorem:

Theorem 5.3. *Let $(G \circ \Lambda) : U \to \mathbb{R}$ and $F : U \to \mathbb{R}$ defined by (16) and (17) respectively (and here we express F as $F(u) = \langle u, f\rangle_U$). If $-G^*_L : Y^*_L \to \mathbb{R}$ attains a local extremum at $v^*_0 \in \mathcal{A}^*$ under the constraint $\Lambda^* v^* - f = 0$ then:*

$$\inf_{u \in U}\{(G \circ \Lambda)(u) + F(u)\} = \sup_{v^* \in \mathcal{A}^* \cap C^*}\{-G^*_L(v^*)\}$$

and $u_0 \in U$ *and* $v_0^* \in Y^*$ *such that:*

$$\delta\{-G_L^*(v_0^*) + \langle u_0, \Lambda^* v_0^* - f\rangle_U\} = \theta$$

are also such that:

$$J(u_0) = -G_L^*(v_0^*) \quad \text{and} \quad \delta J(u_0) = \theta.$$

The proof of above theorem is consequence of the standard necessary conditions for a local extremum for $-G_L^* : Y_L^* \to \mathbb{R}$ under the constraint $\Lambda^* v^* - f = \theta$, the inequality (44) plus an application of Theorem 2.5.

Therefore, in a more explicit format we would have:

$$\begin{aligned}
&\inf_{u\in U}\left\{\frac{1}{2}\int_S H_{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu}dS + \frac{1}{2}\int_S h_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}\kappa_{\lambda\mu}dS\right.\\
&\qquad\left.-\left(\int_S PwdS + \int_{\Gamma_t}\bar{P}wdS + \int_{\Gamma_t}\bar{P}_\alpha u_\alpha d\Gamma - \int_{\Gamma_t} M_n\frac{\partial w}{\partial \mathbf{n}}d\Gamma\right)\right\}\\
&= \sup_{v^*\in\mathcal{A}^*\cap C^*}\left\{-\frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu}N_{\alpha\beta}N_{\lambda\mu}dS - \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu}M_{\alpha\beta}M_{\lambda\mu}dS - \frac{1}{2}\int_S \bar{N}_{\alpha\beta}Q_\alpha Q_\beta dS\right\}
\end{aligned}$$

where $v^* \in C^* \Leftrightarrow v^* \in Y^*$ and,

$$\begin{cases} N_{\alpha\beta,\beta} = 0, \\ Q_{\alpha,\alpha} + M_{\alpha\beta,\alpha\beta} + P = 0, \quad \text{a.e. in } S \end{cases}$$

and

$$\begin{cases} N_{\alpha\beta}.n_\beta - \bar{P}_\alpha = 0, \\ (Q_\alpha + M_{\alpha\beta,\beta})n_\alpha + \dfrac{\partial(M_{\alpha\beta}t_\alpha n_\beta)}{\partial s} - \bar{P} = 0, \\ M_{\alpha\beta}n_\alpha n_\beta - M_n = 0, \quad \text{on } \Gamma_t, \end{cases}$$

with the set $\mathcal{A}^*$ defined by (23) and

$$\{\bar{N}_{\alpha\beta}\} = \{N_{\alpha\beta}\}^{-1}. \qquad \square$$

6. The Second Duality Principle

The next result is a extension of Theorem 2.7 and, instead of calculating the polar functional related to the main part of primal formulation, it is determined its Legendre Transform and associated functional.

Theorem 6.1. *Consider Gâteaux differentiable functionals* $G\circ\Lambda : U \to \bar{\mathbb{R}}$ *and* $F\circ\Lambda_1 : U \to \bar{\mathbb{R}}$ *where only the second one is necessarily convex, through which is defined the functional* $J_K : U \times Y \to \bar{\mathbb{R}}$ *expressed as:*

$$J_K(u,p) = G(\Lambda u + p) + K\langle p,p\rangle_{\boldsymbol{L}^2(S)} - F(\Lambda_1 u) - \frac{K\langle p,p\rangle_{\boldsymbol{L}^2(S)}}{2} - \langle u, u_0^*\rangle_U$$

so that it is supposed the existence of $(u_0, p_0) \in U \times Y$ *such that*

$$J_K(u_0,p_0) = \inf_{(u,p)\in U\times Y}\{J_K(u,p)\}$$

and $\delta J_K(u_0, p_0) = \theta$ *(here* $\Lambda = \{\Lambda_i\} : U \to Y$ *and* $\Lambda_1 : U \to Y$ *are continuous linear operators whose adjoint operators are denoted by* $\Lambda^* : Y^* \to U^*$ *and* $\Lambda_1^* : Y^* \to U^*$ *respectively).*

Furthermore it is assumed the existence of a differentiable function denoted by $g : \mathbb{R}^n \to \mathbb{R}$ *so that* $G : Y \to \bar{\mathbb{R}}$ *may be expressed as:* $G(v) = \int_\Omega g(v)dS,\ \forall v \in Y$ *where* g *admits differentiable Legendre Transform denoted by* $g_L^* : R_L^n \to \mathbb{R}$.

Under these assumptions we have:

$$\inf_{(u,p)\in U\times Y}\{J_K(u,p)\} \leq \inf_{(z^*,v^*,\hat{u})\in E^*}\{J_K^*(z^*,v^*,\hat{u})\}$$

where

$$J_K^*(z^*,v^*,\hat{u}) = F^*(z^*) + 1/(2K)\langle v^*, v^*\rangle_{\mathbf{L}^2(S)} - G_L^*(v^*) + K\sum_{i=1}^n \left(\left\|\Lambda_i\hat{u} - \frac{\partial g_L^*(v^*)}{\partial y_i^*}\right\|_{L^2(S)}^2\right)$$

and

$$E^* = \{(z^*,v^*,\hat{u}) \in Y^* \times Y_L^* \times U \mid -\Lambda_1^* z^* + \Lambda^* v^* - u_0^* = \theta\}.$$

Also, the functions z_0^*, v_0^*, *and* $\hat{u}_0$, *defined by*

$$z_0^* = \frac{\partial F(\Lambda_1 u_0)}{\partial v},$$

$$v_0^* = \frac{\partial G(\Lambda u_0 + p_0)}{\partial v}$$

and

$$\hat{u}_0 = u_0$$

are such that

$$-\Lambda_1^* z_0^* + \Lambda^* v_0^* - u_0^* = \theta$$

and thus

$$J_K(u_0,p_0) \leq \inf_{(z^*,v^*,\hat{u})\in E^*}\{J_K^*(z^*,v^*,\hat{u})\} \leq J_K(u_0,p_0) + 2K\langle p_0, p_0\rangle_{\mathbf{L}^2(S)} \tag{45}$$

where we are assuming that $v_0^* \in Y_L^*$.

Proof. Defining $\alpha = \inf_{(u,p)\in U\times Y}\{J_K(u,p)\}$, $G_1(u,p) = G(\Lambda u + p) + K\langle p, p\rangle_{\mathbf{L}^2(S)}$ and $G_2(u,p) = F(\Lambda_1 u) + (K/2)\langle p, p\rangle_{\mathbf{L}^2(S)} + \langle u, u_0^*\rangle_U$ we have:

$$G_1(u,p) \geq G_2(u,p) + \alpha, \quad \forall (u,p) \in U \times Y,$$

so that, $\forall\ v^* \in Y_L^*$, we have

$$\begin{aligned}&\sup_{(u,p)\in U\times Y}\left\{\langle v^*, \Lambda u + p\rangle_{\mathbf{L}^2(S)} - G_2(u,p)\right\}\\ &\geq \langle v^*, \Lambda u + p\rangle_{\mathbf{L}^2(S)} - G_1(u,p) + \alpha, \quad \forall (u,p) \in U \times Y,\end{aligned} \tag{46}$$

but from Theorem 2.6:

$$\begin{aligned} &\sup_{(u,p)\in U\times Y}\left\{\langle v^*,\Lambda u+p\rangle_{\mathbf{L}^2(S)}-G_2(u,p)\right\} \\ &= \inf_{z^*\in C^*(v^*)}\left\{F(z^*)+(1/2K)\langle v^*,v^*\rangle_{\mathbf{L}^2(\Omega)}\right\} \end{aligned} \tag{47}$$

where

$$C^*(v^*)=\{z^*\in Y^*\mid -\Lambda_1^*z^*+\Lambda^*v^*-u_0^*=\theta\}.$$

Furthermore

$$\langle v^*,\Lambda u+p\rangle_{\mathbf{L}^2(S)}-G_1(u,p)=\langle v^*,\Lambda u+p\rangle_{\mathbf{L}^2(S)}-G(\Lambda u+p)-K\langle p,p\rangle_{\mathbf{L}^2(S)}$$

so that choosing $u=\hat{u}$ and p satisfying the equations:

$$v_i^*=\frac{\partial G(\Lambda\hat{u}+p)}{\partial v_i}$$

which from a well known Legendre Transform property, implies that:

$$p_i=\frac{\partial G_L(v^*)}{\partial v_i^*}-\Lambda_i\hat{u}$$

we would obtain

$$\langle v^*,\Lambda u+p\rangle_{\mathbf{L}^2(S)}-G_1(u,p)=G_L^*(v^*)-K\sum_{i=1}^n\left(\left\|\Lambda_i\hat{u}-\frac{\partial g_L^*(v^*)}{\partial y_i^*}\right\|_{L^2(S)}^2\right)$$

and thus from last results and inequality (46) we have:

$$\begin{aligned} &\inf_{z^*\in C^*(v^*)}\left\{F(z^*)+1/(2K)\langle v^*,v^*\rangle_{\mathbf{L}^2(S)}\right\} \\ &\quad -G_L^*(v^*)+K\sum_{i=1}^n\left(\left\|\Lambda_i\hat{u}-\frac{\partial g_L^*(v^*)}{\partial y_i^*}\right\|_{L^2(S)}^2\right) \\ &\geq \alpha=\inf_{(u,p)\in U\times Y}\{J_K(u,p)\} \end{aligned}$$

that is,

$$\begin{aligned} &F(z^*)+1/(2K)\langle v^*,v^*\rangle_{\mathbf{L}^2(S)}-G_L^*(v^*)+K\sum_{i=1}^n\left(\left\|\Lambda_i\hat{u}-\frac{\partial g_L^*(v^*)}{\partial y_i^*}\right\|_{L^2(S)}^2\right) \\ &\geq \alpha=\inf_{(u,p)\in U\times Y}\{J_K(u,p)\},\quad \text{if } z^*\in C^*(v^*), \end{aligned}$$

or

$$\begin{aligned} &\inf_{(z^*,v^*,\hat{u})\in E^*}\Bigg\{F(z^*)+(1/2K)\langle v^*,v^*\rangle_{\mathbf{L}^2(S)} \\ &\quad -G_L^*(v^*)+K\sum_{i=1}^n\left(\left\|\Lambda_i\hat{u}-\frac{\partial g_L^*(v^*)}{\partial y_i^*}\right\|_{L^2(S)}^2\right)\Bigg\} \\ &\geq \alpha=\inf_{(u,p)\in U\times Y}\{J_K(u,p)\} \end{aligned}$$

so that:

$$\inf_{(z^*,v^*,\hat{u})\in E^*}\{J_K^*(z^*,v^*,\hat{u})\} \geq \inf_{(u,p)\in U\times Y}\{J_K(u,p)\}$$

where $E^* = C^*(v^*)\times Y_L^*\times U$, and the remaining conclusions follow from the expressions of $J_K(u_0,p_0)$ and $J_K^*(z_0^*,v_0^*,\hat{u}_0)$. □

Remark 6.2. It seems to be clear that the duality gap between the primal and dual formulations, namely $2K\langle p_0,p_0\rangle_{\mathbf{L}^2(S)}$, goes to zero as $K\to+\infty$, since $p_0\in Y$ satisfies the extremal condition:

$$\frac{1}{K}\frac{\partial G(\Lambda u_0+p_0)}{\partial v}+p_0=0,$$

and $J_K(u,p)$ is bounded from below. We do not prove it in the present work, postponing the analysis for a future work.

Specifically in the application of such result to the Kirchhoff-Love plate model, we would have: $F(\Lambda_1 u)=\theta$, and therefore the variable z^* is not present in the dual formulation. Also,

$$\langle u,u_0^*\rangle_U=\int_S PwdS+\int_{\Gamma_t}\left(\bar{P}_\alpha u_\alpha+\bar{P}w-M_n\frac{\partial w}{\partial\mathbf{n}}\right)d\Gamma \tag{48}$$

and thus the concerned duality principle could be expressed as:

$$\inf_{u\in U}\left\{G(\Lambda u+p)+K\langle p,p\rangle_{\mathbf{L}^2(S)}-\langle u,u_0^*\rangle_U-\frac{K}{2}\langle p,p\rangle_{\mathbf{L}^2(S)}\right\}$$

$$\leq \inf_{(v^*,\hat{u})\in E^*}\left\{-\frac{1}{2}\int_S\bar{H}_{\alpha\beta\lambda\mu}N_{\alpha\beta}N_{\lambda\mu}dS-\frac{1}{2}\int_S\bar{h}_{\alpha\beta\lambda\mu}M_{\alpha\beta}M_{\lambda\mu}dS\right. \tag{49}$$

$$-\frac{1}{2}\int_S\bar{N}_{\alpha\beta}Q_\alpha Q_\beta dS+\frac{1}{2K}\int_S N_{\alpha\beta}N_{\alpha\beta}dS+\frac{1}{2K}\int_S M_{\alpha\beta}M_{\alpha\beta}dS$$

$$+K\sum_{\alpha,\beta=1}^{2}\left\|\frac{1}{2}(u_{\alpha,\beta}+u_{\beta,\alpha})-\bar{H}_{\alpha\beta\lambda\mu}N_\lambda\mu+\frac{1}{2}v_{02^\alpha}v_{02^\beta}\right\|^2_{L^2(S)}$$

$$\left.+K\sum_{\alpha=1}^{2}||w_{,\alpha}-v_{02^\alpha}||^2_{L^2(S)}+K\sum_{\alpha,\beta=1}^{2}\left\|-w_{,\alpha\beta}-\bar{h}_{\alpha\beta\lambda\mu}M_{\lambda\mu}\right\|^2_{L^2(S)}\right\} \tag{50}$$

where $(v^*,\hat{u})\in E^*=C^*\times U\Leftrightarrow(v^*,\hat{u})\in Y_L^*\times U$ and,

$$\begin{cases}N_{\alpha\beta,\beta}=0,\\ Q_{\alpha,\alpha}+M_{\alpha\beta,\alpha\beta}+P=0,\quad\text{a.e. in } S\end{cases}$$

and

$$\begin{cases}N_{\alpha\beta}.n_\beta-\bar{P}_\alpha=0,\\ (Q_\alpha+M_{\alpha\beta,\beta})n_\alpha+\dfrac{\partial(M_{\alpha\beta}t_\alpha n_\beta)}{\partial s}-\bar{P}=0,\\ M_{\alpha\beta}n_\alpha n_\beta-M_n=0,\quad\text{on }\Gamma_t,\end{cases}$$

where $\{v_{02^\alpha}\}$ is defined through the equations:

$$Q_\alpha=N_{\alpha\beta}v_{02^\beta}$$

and,

$$\{\bar{N}_{\alpha\beta}\} = \{N_{\alpha\beta}\}^{-1}.$$

Finally, we recall that

$$Y_L^* = \{v^* \in Y^* \mid \Delta = N_{11}N_{22} - (N_{12})^2 \neq 0, \text{ a.e. in } S\}. \qquad \square$$

7. The Third Duality Principle

Now we establish the third result which may be summarized by the following theorem:

Theorem 7.1. *Let U be a reflexive Banach space, $(G \circ \Lambda) : U \to \bar{\mathbb{R}}$ a convex Gâteaux differentiable functional and $(F \circ \Lambda_1) : U \to \bar{\mathbb{R}}$ convex, coercive and lower semi-continuous (l.s.c.) such that the functional*

$$J(u) = (G \circ \Lambda)(u) - F(\Lambda_1 u) - \langle u, p \rangle_U$$

is bounded from below, where $\Lambda : U \to Y$ and $\Lambda_1 : U \to Y$ are continuous linear operators.

Then we may write:

$$\inf_{z^* \in Y^*} \sup_{v^* \in B^*(z^*)} \{F^*(z^*) - G^*(v^*)\} \geq \inf_{u \in U}\{J(u)\}$$

where $B^*(z^*) = \{v^* \in Y^*$ such that $\Lambda^* v^* - \Lambda_1^* z^* - p = 0\}$

Proof. By hypothesis there exists $\alpha \in \mathbb{R}$ $(\alpha = \inf_{u \in U}\{J(u)\})$ so that $J(u) \geq \alpha$, $\forall u \in U$, that is,

$$(G \circ \Lambda)(u) \geq F(\Lambda_1 u) + \langle u, p \rangle_U + \alpha, \quad \forall u \in U.$$

The above inequality clearly implies that:

$$\sup_{u \in U}\{\langle u, u^* \rangle_U - F(\Lambda_1 u) - \langle u, p \rangle_U\} \geq \sup_{u \in U}\{\langle u, u^* \rangle_U - (G \circ \Lambda)(u)\} + \alpha$$

$\forall u^* \in U^*$, and, as F is convex, coercive and l.s.c., by Theorem 2.6 we may write:

$$\sup_{u \in U}\{\langle u, u^* \rangle_U - F(\Lambda_1 u) - \langle u, p \rangle_U\} = \inf_{z^* \in A^*(u^*)}\{F^*(z^*)\},$$

where,

$$A^*(u^*) = \{z^* \in Y^* \mid \Lambda_1^* z^* + p = u^*\}$$

and, as G also satisfies the hypothesis of Theorem 2.6, we have:

$$\sup_{u \in U}\{\langle u, u^* \rangle_U - (G \circ \Lambda)(u)\} = \inf_{v^* \in D^*(u^*)}\{G^*(v^*)\},$$

where

$$D^*(u^*) = \{v^* \in Y^* \mid \Lambda^* v^* = u^*\}.$$

Therefore we may summarize the last results as below indicated:

$$F(z^*) + \sup_{v^* \in D^*(u^*)}\{-G^*(v^*)\} \geq \alpha, \quad \forall z^* \in A^*(u^*)$$

and this inequality implies that:

$$F(z^*) + \sup_{v^* \in B^*(z^*)} \{-G^*(v^*)\} \geq \alpha,$$

so that we can write:

$$\inf_{z^* \in Y^*} \sup_{v^* \in B^*(z^*)} \{F^*(z^*) - G^*(v^*)\} \geq \inf_{u \in U} \{J(u)\}$$

where $B^*(z^*) = \{v^* \in Y^* \mid \Lambda^* v^* - \Lambda_1^* z^* - p = 0\}$. $\square$

We will apply the last theorem to a changed functional concerning the primal formulation related to the Kirchhoff-Love plate model, that is we will redefine $(G \circ \Lambda) : U \to \bar{\mathbb{R}}$ and $(F \circ \Lambda_1) : U \to \bar{\mathbb{R}}$ as below indicated:

$$\begin{aligned}(G \circ \Lambda)(u) = \frac{1}{2} \int_S & H_{\alpha\beta\lambda\mu} \gamma_{\alpha\beta}(u) \gamma_{\lambda\mu}(u) dS \\ & + \frac{1}{2} \int_S h_{\alpha\beta\lambda\mu} \kappa_{\alpha\beta}(u) \kappa_{\lambda\mu}(u) dS + \frac{1}{2} K \int_S w,_\alpha w,_\alpha dS\end{aligned}$$

if $N_{11}(u) + K > 0$, $N_{22}(u) + K > 0$ and $(N_{11}(u) + K)(N_{22}(u) + K) - N_{12}(u)^2 > 0$ and, $+\infty$ otherwise.

Remark 7.2. Notice that $(G \circ \Lambda) : U \to \bar{\mathbb{R}}$ is convex and *Gâteaux* differentiable on its effective domain, which is sufficient for our purposes, since the concerned Fenchel conjugate may be easily expressed through the region of interest.

Also, we define:

$$F(\Lambda_1 u) = \frac{1}{2} K \int_S w,_\alpha w,_\alpha dS$$

$$\langle u, p \rangle_U = \int_S P w dS + \int_S P_\alpha u_\alpha dS$$

where

$$u = (u_\alpha, w) \in U = W_0^{1,2}(S) \times W_0^{1,2}(S) \times W_0^{2,2}(S).$$

Observe that these boundary conditions refer to a clamped plate. Furthermore:

$$\Lambda_1(u) = \{w,_1, w,_2\}$$

and

$$\Lambda = \{\Lambda_{1\alpha\beta}, \Lambda_{2\alpha}, \Lambda_{3\alpha\beta}\}$$

as indicated in (11), (12) and (13).

Calculating $G^* : Y^* \to \bar{\mathbb{R}}$ and $F^* : Y^* \to \bar{\mathbb{R}}$ we would obtain:

$$\begin{aligned}G^*(v^*) = \frac{1}{2} \int_S & \bar{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} dS + \frac{1}{2} \int_S \bar{h}_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\lambda\mu} dS \\ & + \frac{1}{2} \int_S N_{\alpha\beta} w,_\alpha w,_\beta dS + \frac{1}{2} K \int_S w,_\alpha w,_\alpha dS \qquad (51)\end{aligned}$$

if $v^* \in E^*$, where:

$$v^* = \{N_{\alpha\beta}, M_{\alpha\beta}, w_{,\alpha}\},$$

$$E^* = \big\{v^* \in Y^* \mid N_{11} + K > 0,\ N_{22} + K > 0 \\ \text{and } (N_{11} + K)(N_{22} + K) - N_{12}^2 > 0, \text{ a.e. in } S\big\}$$

and,

$$F^*(z^*) = \frac{1}{2K}\int_S (z_1^*)^2 dS + \frac{1}{2K}\int_S (z_2^*)^2 dS.$$

Furthermore, $v^* \in B^*(z^*) \Leftrightarrow v^* \in Y^*$ and,

$$\begin{cases} N_{\alpha\beta,\beta} + P_\alpha = 0, \\ -(z_\alpha^*)_{,\alpha} + (N_{\alpha\beta}w_{,\beta})_{,\alpha} + M_{\alpha\beta,\alpha\beta} + Kw_{,\alpha\alpha} + P = 0, \quad \text{a.e. in } S. \end{cases}$$

Finally, we can express the application of last theorem as:

$$\begin{aligned} &\inf_{z^* \in Y^*} \sup_{v^* \in B^*(z^*) \bigcap E^*} \Big\{ \frac{1}{2K}\int_S (z_1^*)^2 dS + \frac{1}{2K}\int_S (z_2^*)^2 dS - \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} dS \\ &\quad - \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\lambda\mu} dS - \frac{1}{2}\int_S N_{\alpha\beta} w_{,\alpha}\, w_{,\beta}\, dS - \frac{1}{2}K\int_S w_{,\alpha}\, w_{,\alpha}\, dS \Big\} \\ &\geq \inf_{u \in U}\{J(u)\} \end{aligned} \tag{52}$$

□

The above inequality can in fact represents an equality if the positive real constant K is chosen so that the point of local extremum $v_0^* = \frac{\partial G(\Lambda u_0)}{\partial v} \in E^*$ (which means $N_{11}(u_0) + K > 0$, $N_{22}(u_0) + K > 0$, and $(N_{11}(u_0) + K)(N_{22}(u_0) + K) - N_{12}(u_0)^2 > 0$). The mentioned equality is a result of a little change concerning Theorem 2.5.

Remark 7.3. For the determination of $G^*(v^*)$ in (51) we have used the transformation

$$Q_\alpha = N_{\alpha\beta}w_{,\beta} + Kw_{,\alpha},$$

similarly as indicated at Remark 4.1.

8. A Convex Dual Formulation

Remark 8.1. In this section we assume:

$$H_{\alpha\beta\lambda\mu} = h\left\{\frac{4\lambda_0\mu_0}{\lambda_0 + 2\mu_0}\delta_{\alpha\beta}\delta_{\lambda\mu} + 2\mu_0(\delta_{\alpha\lambda}\delta_{\beta\mu} + \delta_{\alpha\mu}\delta_{\beta\lambda})\right\},$$

and

$$h_{\alpha\beta\lambda\mu} = \frac{h^2 H_{\alpha\beta\lambda\mu}}{12},$$

where $\delta_{\alpha\beta}$ denotes the Kronecker delta and λ_0, μ_0 are appropriate constants.

The next result may be summarized by the following theorem, which also establishes sufficient conditions for optimality.

Theorem 8.2. *Consider the functionals* $(G \circ \Lambda) : U \to \bar{\mathbb{R}}$, $(F \circ \Lambda_1) : U \to \bar{\mathbb{R}}$ *and* $\langle u, p \rangle_U$ *defined as*

$$(G \circ \Lambda)(u) = \frac{1}{2}\int_S H_{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(u)dS + \frac{1}{2}\int_S h_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}(u)\kappa_{\lambda\mu}(u)dS + \frac{1}{2}K\int_S w_{,\alpha}\, w_{,\alpha}\, dS,$$

$$F(\Lambda_1 u) = \frac{1}{2}K\int_S w_{,\alpha}\, w_{,\alpha}\, dS$$

and

$$\langle u, p \rangle_U = \int_S P w dS + \int_S P_\alpha u_\alpha dS$$

where

$$u = (u_\alpha, w) \in U = W_0^{1,2}(S) \times W_0^{1,2}(S) \times W_0^{2,2}(S),$$

Observe that these boundary conditions refer to a clamped plate. The operators $\{\gamma_{\alpha\beta}\}$ *and* $\{\kappa_{\alpha\beta}\}$ *are defined in* (9) *and* (10)*, respectively. Furthermore, we define* $J(u) = (G \circ \Lambda)(u) - F(\Lambda_1 u) - \langle u, p \rangle_U$*, and*

$$\Lambda_1(u) = \{w_{,1}\,, w_{,2}\,\}.$$

Thus, supposing the existence of $u_0 \in U$ *such that* $\delta J(u_0) = 0$*, and so that there exists* $K > 0$ *for which* $N_{11}(u_0) + K > 0$, $N_{22}(u_0) + K > 0$, $(N_{11}(u_0) + K)(N_{22}(u_0) + K) - N_{12}(u_0)^2 > 0$ *(a.e. in S) and* $h_{1212}/(2K_0) > K$ *where* K_0 *is the constant related to Poincaré Inequality where also,*

$$N_{\alpha\beta}(u_0) = H_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(u_0),$$

we have that:

$$\begin{aligned} J(u_0) =& \min_{u \in U}\{J(u)\} = \max_{(v^*, z^*) \in E^* \cap B^*}\{-G^*(v^*) + \langle z^*_\alpha, z^*_\alpha \rangle_{L^2(S)}/(2K)\} \\ =& -G^*(v^*_0) + \langle z^*_{0\alpha}, z^*_{0\alpha} \rangle_{L^2(S)}/(2K) \end{aligned} \tag{53}$$

where,

$$v^*_0 = \frac{\partial G(\Lambda u_0)}{\partial v} \quad \text{and} \quad z^*_{0\alpha} = K w_{0,\alpha},$$

$$G^*(v^*) = G^*_L(v^*) = \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu}N_{\alpha\beta}N_{\lambda\mu}dS + \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu}M_{\alpha\beta}M_{\lambda\mu}dS + \frac{1}{2}\int_S \bar{N}^K_{\alpha\beta}Q_{,\alpha}Q_{,\beta}dS$$

if $v^* \in E^*$*, where* $v^* = \{\{N_{\alpha\beta}\}, \{M_{\alpha\beta}\}, \{Q_\alpha\}\} \in E^* \Leftrightarrow v^* \in L^2(S, \mathbb{R}^{10})$ *and*

$$N_{11} + K > 0 \quad N_{22} + K > 0 \quad \text{and} \quad (N_{11} + K)(N_{22} + K) - N_{12}^2 > 0, \quad \text{a.e. in } S$$

and,

$$(v^*, z^*) \in B^* \Leftrightarrow \begin{cases} N_{\alpha\beta,\beta} + P_\alpha = 0, \\ Q_{\alpha,\alpha} + M_{\alpha\beta,\alpha\beta} - z^*_{\alpha,\alpha} + P = 0, \\ \bar{h}_{1212}M_{12} + z^*_{1,2}/K = 0, \\ z^*_{1,2} = z^*_{2,1}, \ \text{a.e. in } S, \ \text{and, } z^* = \theta \text{ on } \Gamma, \end{cases}$$

being $\{\bar{N}^K_{\alpha\beta}\}$ *as indicated in* (6)*.*

Proof. Similarly to Proposition 5.1, we may obtain: if $v^* \in E^*$ then:

$$G_L^*(v^*) = G^*(v^*) \geq \langle v^*, \Lambda u\rangle_Y - G(\Lambda u), \quad \forall u \in U,$$

so that

$$G_L^*(v^*) - \frac{1}{2K}\langle z_\alpha^*, z_\alpha^*\rangle_{L^2(S)} \geq \langle v^*, \Lambda u\rangle_Y - \frac{1}{2K}\langle z_\alpha^*, z_\alpha^*\rangle_{L^2(S)} - G(\Lambda u), \quad \forall u \in U,$$

and thus, as $\Lambda^* v^* - \Lambda_1^* z^* - p = 0$ (see the definition of B^*) we obtain:

$$Q_{\alpha,\alpha} + M_{\alpha\beta,\alpha\beta} - z^*_{\alpha,\alpha} + P = 0 \text{ a.e. in } S,$$

and, through this equation we may symbolically write

$$M_{12} = \Lambda_{3^{12}}^{-1}\{(-Q_{\alpha,\alpha} + z^*_{\alpha,\alpha} - \bar{M}_{\alpha\beta,\alpha\beta} - P)/2\}, \tag{54}$$

where $\bar{M}_{\alpha\beta,\alpha\beta}$ denotes $M_{11,11} + M_{22,22}$, in S, so that substituting such a relation in the last inequality we have:

$$\begin{aligned} &\frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} dS + \frac{1}{2}\int_S \bar{h}_{1111} M_{11}^2 dS + \int_S \bar{h}_{1122} M_{11} M_{22} dS + \frac{1}{2}\int_S \bar{h}_{2222} M_{22}^2 dS \\ &\quad + 2\int_S \bar{h}_{1212}(\Lambda_{3^{12}}^{-1}(v^*, z^*))^2 dS + \frac{1}{2}\int_S \bar{N}^K_{\alpha\beta} Q_{,\alpha} Q_{,\beta} dS - \frac{1}{2K}\langle z_\alpha^*, z_\alpha^*\rangle_{L^2(S)} \\ &\geq \langle \Lambda_1 u, z^*\rangle_{L^2(S;\mathbb{R}^2)} - \frac{1}{2K}\langle z_\alpha^*, z_\alpha^*\rangle_{L^2(S)} - G(\Lambda u) + \langle u, p\rangle_U, \quad \forall u \in U, \end{aligned} \tag{55}$$

where M_{12} is made explicit through equation (54), and, this equation makes z^* an independent variable, so that evaluating the supremum concerning z^*, particularly for the left side of above inequality, the global extremum is achieved through the equation:

$$-\left(\left[\Lambda_{3^{12}}^{-1}\right]^* \left[\bar{h}_{1212}\Lambda_{3^{12}}^{-1}(v^*, z^*)\right]\right)_{,\alpha} - z_\alpha^*/K = 0, \quad \text{a.e. in } S$$

which means:

$$-\bar{h}_{1212}\Lambda_{3^{12}}^{-1}(v^*, z^*) - z^*_{\alpha,\beta}/K = 0, \quad \text{a.e. in } S \text{ and } z_1^* = z_2^* = 0 \text{ on } \Gamma$$

or

$$\bar{h}_{1212} M_{12} + z^*_{\alpha,\beta}/K = 0, \quad \text{a.e. in } S \text{ and } z_1^* = z_2^* = 0 \text{ on } \Gamma$$

for $(\alpha, \beta) = (1, 2)$ and $(2, 1)$. Therefore, after evaluating the suprema in both sides of (55), we may write:

$$\begin{aligned} &G_L^*(v^*) - \frac{1}{2K}\langle z_\alpha^*, z_\alpha^*\rangle_{L^2(S)} \\ &\geq F(\Lambda_1 u) - G(\Lambda u) + \langle u, p\rangle_U, \quad \forall u \in U, \text{ and } (v^*, z^*) \in B^* \cap E^* \end{aligned}$$

and it seems to be clear that the condition $h_{1212}/(2K_0) > K$ guarantees coercivity for the expression of left side in the last inequality, so that the unique local extremum concerning z^* is also a global extremum. The equality and remaining conclusions results from the Gâteaux differentiability of primal and dual formulations and an application (with little changes) of Theorem 2.5. □

Remark 8.3. Observe that the dual functional could be expressed as

$$\begin{aligned} & G_L^*(v^*) - \frac{1}{2K}\langle z_\alpha^*, z_\alpha^*\rangle_{L^2(S)} \\ = & \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} dS + \frac{1}{2}\int_S \bar{h}_{1111} M_{11}^2 dS + \int_S \bar{h}_{1122} M_{11} M_{22} dS \\ & + \frac{1}{2}\int_S \bar{h}_{2222} M_{22}^2 dS + \frac{1}{2}\int_S \bar{N}_{\alpha\beta}^K Q_{,\alpha} Q_{,\beta} dS + \int_S h_{1212}(z_{1,2}^*)^2/K^2 dS \\ & + \int_S h_{1212}(z_{2,1}^*)^2/K^2 dS - \frac{1}{2K}\langle z_\alpha^*, z_\alpha^*\rangle_{L^2(S)} \end{aligned}$$

and thus, through the relation $h_{1212}/(2K_0) > K$ (where K_0 is the constant related to *Poincaré* inequality), it is now clear that the dual formulation is convex on $E^* \cap B^*$.

9. A Final Result, Other Sufficient Conditions for Optimality

This final result is developed similarly to the Triality criterion introduced in Gao, [9], which describes in some situations, sufficient conditions for optimality.

We prove the following result:

Theorem 9.1. *Consider $J : U \to \mathbb{R}$ where $J(u) = G(\Lambda u) + F(u)$,*

$$G(\Lambda u) = \frac{1}{2}\int_S H_{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(u) dS + \frac{1}{2}\int_S h_{\alpha\beta\lambda\mu} w_{,\alpha\beta} w_{,\lambda\mu} dS,$$

here the operators $\gamma_{\alpha\beta}$ must be considered as defined in (9)*, and*

$$F(u) = -\int_S Pw dS \equiv -\langle u, f\rangle_U,$$

and also

$$U = W_0^{1,2}(S) \times W_0^{1,2}(S) \times W_0^{2,2}(S).$$

Thus, if $u_0 \in U$ is such that $\delta J(u_0) = \theta$ and

$$\frac{1}{2}\int_S N_{\alpha\beta}(u_0) w_{,\alpha} w_{,\beta} dS + \frac{1}{2}\int_S h_{\alpha\beta\lambda\mu} w_{,\alpha\beta} w_{,\lambda\mu} dS \geq 0, \quad \forall w \in W_0^{2,2}(S), \tag{56}$$

then

$$J(u_0) = \min_{u \in U}\{J(u)\}.$$

Proof. It is clear that

$$G(\Lambda u) + F(u) \geq -(G \circ \Lambda)^*(u^*) - F^*(-u^*), \quad \forall u \in U,\ u^* \in U^*,$$

so that

$$G(\Lambda u) + F(u) \geq -(G \circ \Lambda)^*(\Lambda^* v^*) - F^*(-\Lambda^* v^*), \quad \forall u \in U,\ v^* \in Y^*. \tag{57}$$

Consider u_0 for which $\delta J(u_0) = \theta$ and such that (56) is satisfied.

Defining

$$v_0^* = \frac{\partial G(\Lambda u_0)}{\partial v},$$

from Theorem 2.5 we have that

$$\delta(-G_L^*(v_0^*) + \langle u_0, \Lambda v_0^* - f\rangle_U) = \theta,$$

$$J(u_0) = -G_L(v_0^*),$$

and

$$\Lambda^* v_0^* = f,$$

which means

$$F^*(-\Lambda^* v_0^*) = 0.$$

On the other hand

$$(G \circ \Lambda)^*(\Lambda^* v_0^*) = \sup_{u \in U}\{\langle \Lambda u, v_0^*\rangle_Y - G(\Lambda u)\},$$

or

$$\begin{aligned}
&(G \circ \Lambda)^*(\Lambda v_0^*) \\
&= \sup_{u \in U}\Bigg\{ \left\langle \frac{u_{\alpha,\beta} + u_{\beta,\alpha}}{2}, N_{\alpha\beta}(u_0)\right\rangle_{L^2(S)} + \langle -w_{,\alpha\beta}, M_{\alpha\beta}(u_0)\rangle_{L^2(S)} \\
&\quad + \langle w_{,\alpha} Q_\alpha(u_0)\rangle_{L^2(S)} - \frac{1}{2}\int_S H_{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(u)dS - \frac{1}{2}\int_S h_{\alpha\beta\lambda\mu} w_{,\alpha\beta} w_{,\lambda\mu} dS\Bigg\}.
\end{aligned}$$

Since

$$\gamma_{\alpha\beta}(u) = \frac{u_{\alpha,\beta} + u_{\beta,\alpha}}{2} + \frac{1}{2} w_{,\alpha} w_{,\beta},$$

from last equality, we may write

$$\begin{aligned}
&(G \circ \Lambda)^*(\Lambda^* v_0^*) \\
&= \sup_{u \in U}\Bigg\{ \langle \gamma_{\alpha\beta}(u), N_{\alpha\beta}(u_0)\rangle_{L^2(S)} - \left\langle \frac{w_{,\alpha} w_{,\beta}}{2}, N_{\alpha\beta}(u_0)\right\rangle_{L^2(S)} + \langle -w_{,\alpha\beta}, M_{\alpha\beta}(u_0)\rangle_{L^2(S)} \\
&\quad + \langle w_{,\alpha}, Q_\alpha(u_0)\rangle_{L^2(S)} - \frac{1}{2}\int_S H_{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(u)dS - \frac{1}{2}\int_S h_{\alpha\beta\lambda\mu} w_{,\alpha\beta} w_{,\lambda\mu} dS\Bigg\}.
\end{aligned}$$

As $(Q_\alpha(u_0))_{,\alpha} + (M_{\alpha\beta}(u_0))_{,\alpha\beta} + P = 0$, we obtain

$$\begin{aligned}
&(G \circ \Lambda)^*(\Lambda^* v_0^*) \\
&\le \sup_{u \in U}\left\{ -\left\langle \frac{w_{,\alpha} w_{,\beta}}{2}, N_{\alpha\beta}(u_0)\right\rangle_{L^2(S)} - \frac{1}{2}\int_S h_{\alpha\beta\lambda\mu} w_{,\alpha\beta} w_{,\lambda\mu} dS + \int_S Pw dS\right\} \\
&\quad + \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta}(u_0) N_{\lambda\mu}(u_0) dS. \qquad (58)
\end{aligned}$$

Therefore, from hypothesis (56) the extremum indicated in (58) is attained for functions satisfying

$$(N_{\alpha\beta}(u_0)\hat{w}_\beta)_{,\alpha} - (h_{\alpha\beta\lambda\mu}\hat{w}_{\lambda\mu})_{,\alpha\beta} + P = 0, \qquad (59)$$

which, from $\delta J(u_0)=\theta$ and boundary conditions implies that

$$\hat{w}=w_0,$$

so that

$$(G\circ\Lambda)^*(\Lambda^* v_0^*)\leq \left\langle \frac{w_{0,\alpha}w_{0,\beta}}{2}, N_{\alpha\beta}(u_0)\right\rangle_{L^2(S)}+\frac{1}{2}\int_S h_{\alpha\beta\lambda\mu}w_{0,\alpha\beta}w_{0,\lambda\mu}dS$$
$$+\frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu}N_{\alpha\beta}(u_0)N_{\lambda\mu}(u_0)dS. \qquad (60)$$

However, since

$$Q_\alpha(u_0)=N_{\alpha\beta}(u_0)w_{0,\beta},$$

and

$$M_{\alpha\beta}(u_0)=-h_{\alpha\beta\lambda\mu}w_{0,\lambda\mu}$$

from (60) we obtain

$$(G\circ\Lambda)^*(\Lambda^* v_0^*)\leq \frac{1}{2}\int \bar{N}_{\alpha\beta}(u_0)Q_\alpha(u_0)Q_\beta(u_0)dS+\frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu}M_{\alpha\beta}(u_0)M_{\lambda\mu}(u_0)dS$$
$$+\frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu}N_{\alpha\beta}(u_0)N_{\lambda\mu}(u_0)dS$$

and hence

$$(G\circ\Lambda)^*(\Lambda^* v_0^*)\leq G_L(v_0^*)=-J(u_0),$$

and thus as $F^*(-\Lambda^* v_0^*)=0$, we have that

$$J(u_0)\leq -(G\circ\Lambda)^*(\Lambda^* v_0^*)-F^*(-\Lambda^* v_0^*),$$

which, from (57) completes the proof. □

10. Final Remarks

In this paper we present four different dual variational formulations for the Kirchhoff-Love plate model. Earlier results (see references [10], [7]) present a constraint concerning the gap functional to establish the complementary energy (dual formulation). In the present work the dual formulations are established on the hypothesis of existence of a global extremum for the primal functional and the results are applicable even for compressed plates. Particularly the second duality principle is obtained through an extension of a theorem met in [12], and in this case we are concerned with the solution behavior as $K\to+\infty$, even though a rigorous and complete analysis of such behavior has been postponed for a future work. However, what seems to be interesting is that the dual formulation as indicated in (50) is represented by a natural extension of the results found in [12] (particularly Theorem 2.7), plus a kind of penalization concerning the inversion of constitutive equations.

It is worth noting that the third dual formulation was also established based on the commented theorem, despite the fact such a result had not been directly used, we followed a similar idea to prove the mentioned duality principle. For this last result, the membrane forces are allowed to be negative since it is observed the restriction

$N_{11}+K>0$, $N_{22}+K>0$ and $(N_{11}+K)(N_{22}+K)-N_{12}^2>0$, a.e. in S, where $K\in\mathbb{R}$ is a positive suitable constant.

In Section 8, we obtained a convex dual variational formulation for the concerned plate model, which allows non positive definite membrane force matrices. In this formulation, the *Poincaré inequality* plays a fundamental role.

Finally, in the last section, we developed a result similarly as in Gao's Triality criterion presented in [9], now for the present case, obtaining sufficient conditions for optimality. We present a proof, which seems to be new, for sufficient conditions of existence of a global extremum for the primal problem. As earlier mentioned, such conditions may be summarized by $\delta J(u_0)=\theta$ and

$$\frac{1}{2}\int_S N_{\alpha\beta}(u_0)w_{,\alpha}w_{,\beta}dS+\frac{1}{2}\int_S h_{\alpha\beta\lambda\mu}w_{,\alpha\beta}w_{,\lambda\mu}dS\geq 0,\quad \forall w\in W_0^{2,2}(S).\qquad\square$$

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