Journal of Convex Analysis
Volume 17 (2010), No. 1, 349–356

A Unified Construction Yielding Precisely Hilbert and James Sequences Spaces*

Dušan Repovš
Faculty of Mathematics and Physics, and Faculty of Education,
University of Ljubljana, P. O. Box 2964, Ljubljana, Slovenia 1001
dusan.repovs@guest.arnes.si

Pavel V. Semenov
Department of Mathematics, Moscow City Pedagogical University,
2-nd Selskokhozyastvennyi pr. 4, Moscow, Russia 129226
pavels@orc.ru

Received: February 4, 2008
Revised manuscript received: February 15, 2009

Following R. C. James' approach, we shall define the Banach space $J(e)$ for each vector $e = (e_1, e_2, ..., e_d) \in \mathbb{R}^d$ with $e_1 \neq 0$. The construction immediately implies that $J(1)$ coincides with the Hilbert space l_2 and that $J(1; -1)$ coincides with the celebrated quasireflexive James space J. The results of this paper show that, up to an isomorphism, there are only these two possibilities: (i) $J(e)$ is isomorphic to l_2 if $e_1 + e_2 + ... + e_d \neq 0$, and (ii) $J(e)$ is isomorphic to J if $e_1 + e_2 + ... + e_d = 0$. Such a dichotomy also holds for every separable Orlicz sequence space l_M.

Keywords: Hilbert space, Banach space, James sequence space, quasireflexive space, invertible continuous operator, Orlicz function

2000 Mathematics Subject Classification: Primary 54C60, 54C65, 41A65; Secondary 54C55, 54C20

0. Introduction

In infinite-dimensional analysis and topology and in Banach space theory, two sequence spaces – the Hilbert space l_2 and the James space J – are certainly presented as two opposite objects. In fact, the Hilbert space is the "simplest" Banach space with nice analytical, geometrical and topological properties. On the contrary, the properties of the James space are so unusual and unexpected that J is often called a "space of counterexamples" (see [3, 5]).

Let us list some of the James space most significant properties: (a) J has a Schauder basis, but admits no isomorphic embedding into a space with an unconditional Schauder basis [1, 3, 5]; (b) J and its second conjugate J^{**} are separable, but $\dim(J^{**}/\chi(J)) = 1$, where $\chi : J \to J^{**}$ is the canonical embedding (see [1]); (c) in spite of property (b), the spaces J and J^{**} are isometric with respect to an equivalent norm (see [2]); (d) J and $J \oplus J$ are non-isomorphic and moreover, J and $B \oplus B$ are non-isomorphic

*Both authors were supported by the SRA grants J1-9643-0101, P1-0292-0101-04 and Bl-RU/05-07/7. The second author was supported by the RFBR grant No. 08-01-00663.

ISSN 0944-6532 / $ 2.50

for an arbitrary weakly complete Banach space B (see [3, 5]); (e) on J there exists a C^1-function with bounded support, but there are no C^2-functions with bounded support (see [7]); (f) there exists an infinite-dimensional manifold modelled on J which cannot be homeomorphically embedded into J (see [4, 7]); and (g) the group $GL(J)$ of all invertible continuous operators of J onto itself is homotopically non-trivial with respect to the topology generated by operator's norm (see [8]), but it is contractible in the operator pointwise convergency topology (see [3, 4, 10] for more references).

In this paper we shall define the Banach space $J(e)$ for each vector $e = (e_1, e_2, ..., e_d) \in \mathbb{R}^d$ with $e_1 \neq 0$. This construction immediately implies that $J(1) = l_2$ and $J(1; -1) = J$. Surprisingly, up to an isomorphism there are only these two possibilities. We will show that $J(e)$ is isomorphic to l_2, if $e_1 + e_2 + ... + e_d \neq 0$ (see Theorem 1.5) and $J(e)$ is isomorphic to J otherwise (see Theorem 1.6).

Such a dichotomy holds not only for the space l_2 (which is clearly define by using the numerical function $M(t) = t^2,\ t \geq 0$), but also for an arbitrary Orlicz sequence space l_M defined by an arbitrary Orlicz function $M : [0; +\infty) \to [0; +\infty)$ with the so-called Δ_2-condition. Then there are also exactly two possibilities for $J(e)$: either $J(e)$ is isomorphic to l_M, or $J(e)$ is isomorphic to the James-Orlicz space J_M (see [9]). For simplicity, we restrict ourselves below to $M(t) = t^2,\ t \geq 0$.

1. Preliminaries

Let d be any natural number and $e = (e_1, e_2, ..., e_d) \in \mathbb{R}^d$ any d-vector with $e_1 \neq 0$. Having in our formulae many brackets, we shall choose the special notation $a * b$ for the usual scalar product of two elements $a \in \mathbb{R}^d$ and $b \in \mathbb{R}^d$. A d-subset ω of $\mathbb{N}$ is defined by setting

$$\omega = \{n(1) < n(2) < ... < n(d) < n(d+1) < ... < n(kd-1) < n(kd)\} \subset \mathbb{N}$$

for some natural k and then the subsets

$$\begin{aligned} \omega(1) &= \{n(1) < n(2) < ... < n(d)\}, \\ \omega(2) &= \{n(d+1) < n(d+2) < ... < n(2d)\}, ..., \\ \omega(k) &= \{n((k-1)d+1) < ... < n(kd)\} \end{aligned}$$

are called the d-components of the d-set ω.

For each d-set ω and each infinite sequence of reals $x = (x(m))_{m \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ we denote $x(\omega) = (x(m))_{m \in \omega}$ and $x(\omega; i) = (x(m))_{m \in \omega(i)}$.

Definition 1.1. For each $d \in \mathbb{N}, e \in \mathbb{R}^d, x \in \mathbb{R}^{\mathbb{N}}$ and d-set

$$\omega = \{n(1) < ... < n(kd)\}$$

the (e, ω)-variation of x is defined by the equality

$$(e, \omega) = \sqrt{\sum_{i=1}^{k} (e * x(\omega; i))^2}$$

$$= \sqrt{(e_1 x(n(1)) + ... + e_d x(n(d)))^2 + ... + (e_1 x(n((k-1)d+1)) + ... + e_d x(n(kd)))^2}.$$

Definition 1.2. For each $d \in \mathbb{N}$, $e \in \mathbb{R}^d$, $x \in \mathbb{R}^{\mathbb{N}}$ the e-variation of x is defined by the equality $||x||_e = sup\{e(x,\omega) : \omega \text{ is a } d\text{-subset of } \mathbb{N}\}$.

Definition 1.3. The set of all infinite sequences of reals tending to zero with finite e-variation is denoted by $J(e)$.

We omit a routine verification of the following proposition.

Proposition 1.4. *$(J(e); ||\cdot||_e)$ is a Banach space for each $e = (e_1, e_2, ..., e_d) \in \mathbb{R}^d$ with $e_1 \neq 0$.* □

Note that the restriction $e_1 \neq 0$ is purely technical. It avoids the case $e = 0 \in \mathbb{R}^d$ and guarantees that $\|(1; 0; 0; ...)\|_e > 0$. Below we fix such a hypothesis.

Theorem 1.5. *If $e_1 + e_2 + ... + e_d \neq 0$, then $J(e)$ and l_2 are isomorphic.*

Theorem 1.6. *If $e_1 + e_2 + ... + e_d = 0$, then $J(e)$ and J are isomorphic.*

Theorem 1.5 is proved in Section 2 as the corollary of Lemmata 2.1–2.4. We believe that Lemma 2.3 is of interest independently of Theorem 1.5 and its proof. Theorem 1.6 is proved in Section 3 as a corollary of Lemmata 3.1–3.3. Lemma 3.1 really stresses the importance of the equality $e_1 + e_2 + ... + e_d = 0$. Lemma 3.3 is the most difficult to prove. In the last step of its proof the special combinatorial Sublemma 3.4 is needed.

An open question concerns analogs of Theorems 1.5 and 1.6 for spaces of functions over the segment $[0; 1]$. Here the main obstruction is that the James functional space JF has a non-separable dual space [4]. Also, we believe that Theorems 1.5 and 1.6 are true for a generalizations of J, in the spirit of results of [6].

2. Proof of Theorem 1.5

Lemma 2.1. *The inclusion operator $id : l_2 \to J(e)$ is well-defined and continuous.*

Proof. Let $\|\cdot\|_2$ be the standard Euclidean norm. Fix any $x = (x_1, x_2, x_3, ...) \in l_2$ and pick any d-set $\omega = \omega(1) \cup \omega(2) \cup ... \cup \omega(k)$ with d-components $\omega(1), \omega(2), ...\omega(k)$. Then $(e * x(\omega; i))^2 \leq \|e\|_2^2 \cdot \|x(\omega; i)\|_2^2$ due to the Cauchy inequality. Hence,

$$(e(x;\omega))^2 = \sum_{i=1}^{k} (e * x(\omega; i))^2 \leq \|e\|_2^2 \cdot \left(\sum_{i=1}^{k} \|x(\omega; i)\|_2^2 \right) \leq (\|e\|_2 \|x\|_2)^2$$

and therefore $\|x\|_e = \sup\{e(x;\omega) : \omega\} \leq \|e\|_2 \|x\|_2 = C\|x\|_2$. □

Lemma 2.2.

$$\det \begin{pmatrix} 0 & e_1 & e_2 \dots & e_d \\ e_1 & 0 & e_2 \dots & e_d \\ \cdot & \cdot & \cdot & \cdot \\ e_1 & e_2 \dots & e_d & 0 \end{pmatrix} = (-1)^d \left(\prod_{i=1}^{d} e_i \right) \left(\sum_{i=1}^{d} e_i \right).$$

□

Lemma 2.3. *For each $d \in \mathbb{N}$, $e \in \mathbb{R}^d$ with $e_1 \neq 0$ and $e_1 + e_2 + ... + e_d \neq 0$ there exists a constant $C = C_e > 0$ such that for every sequence of reals*

$$x(1), x(2), ..., x(d), x(d+1)$$

the inequality

$$\left| \sum_{i=1}^{d} e_i x(n(i)) \right| \geq C|x(1)|$$

holds for some d-set, $1 \leq n(1) < ... < n(d) \leq d+1$.

Proof. The assertion is obvious for $x(1) = 0$. So let $x(1) \neq 0$ and consider the case when all numbers $e_1, e_2, ..., e_d$ are non-zero. Denote by L the linear mapping of $\mathbb{R}^{d+1}$ into itself defined by the matrix from Lemma 2.2. By this lemma, $L : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ is an isomorphism. Consider $\mathbb{R}^{d+1}$ with the *max*-norm

$$\|(x(1), x(2), ..., x(d), x(d+1))\| = \max\{|x(j)| : 1 \leq j \leq d+1\},$$

i.e. as the Banach space l_∞^{d+1} of dimension $d+1$. Define the constant C as the distance between the origin and the L-image of the set of all elements with the first coordinate equal to ± 1:

$$C = \mathrm{dist}(0; \{L(y(1), y(2), ..., y(d), y(d+1) : y(1) = \pm 1\}) > 0.$$

Next, pick

$$x = (x(1), x(2), ..., x(d), x(d+1)) \in l_\infty^{d+1}$$

with $x(1) \neq 0$ and set

$$y(i) = x(i) \cdot (x(1))^{-1}, \quad i = 1, 2, ..., d, d+1.$$

Then $y(1) \neq 0$ and

$$\begin{aligned} &\|L(y(1), y(2), ..., y(d), y(d+1))\|_\infty \\ \geq\ & C, \|L(x(1), x(2), ..., x(d), x(d+1))\|_\infty \geq C|x(1)|. \end{aligned}$$

By definition of the *max* norm and by the definition of the isomorphism L we see that $|\sum_{i=1}^{d} e_i x(n(i))| \geq C|x(1)|$, for some indices $1 \leq n(1) < ... < n(d) \leq d+1$.

It is easy to check that for an arbitrary $e \in \mathbb{R}^d$ with $e_1 \neq 0$ and $e_1 + e_2 + ... + e_d \neq 0$, the constant $C_{e'}$ works properly, where the vector e' consists of all non-zero coordinates of the vector e. □

Lemma 2.4. *The inclusion operator $id : l_2 \to J(e)$ is a surjection.*

Proof. Suppose to the contrary that $\|x\|_e < \infty$ but $\|x\|_2 = \infty$, for some $x = (x(m))_{m \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$. Due to the equality

$$\sum_{m=1}^{\infty} x^2(m) = \sum_{i=1}^{d+1} \left(\sum_{k=0}^{\infty} x^2(k(d+1)+i) \right)$$

we see that for some $1 \le i \le d+1$ the series $\sum_{k=1}^{\infty} x^2(k(d+1)+i)$ is divergent.

So let C be the constant from Lemma 2.3. Applying this lemma for each natural k to the reals

$$x(k(d+1)+i), x(k(d+1)+i+1), x(k(d+1)+i+2), ..., x(k(d+1)+i+d)$$

we find some d-set, say $\omega(k)$, with $\omega(k) \subset [k(d+1)+i, k(d+1)+i+d]$ such that $|e * x(\omega(k))| \ge C|x(k(d+1)+i)|$. Hence,

$$\sum_{k=1}^{\infty}(e * x(\omega(k)))^2 \ge C \sum_{k=1}^{\infty} x^2(k(d+1)+i) = \infty.$$

Thus $\|x\|_e = \infty$ and this is a contradiction. $\square$

Note that Theorem 1.5 implies that for $e_1 + e_2 + ... + e_d \neq 0$ it suffices to define $J(e)$ as the set of all sequences with a finite e-variation. In this situation the convergence of coordinates to zero is a corollary of finiteness of the e-variation.

3. Proof of Theorem 1.6

As it was mentioned above, we first explain the reason for the appearance of the restriction $e_1 + e_2 + ... + e_d = 0$.

Lemma 3.1. *The inclusion operator $id : J(1; -1) \to J(e)$ is well-defined and continuous.*

Proof. For arbitrary reals $t_1, t_2, ..., t_d$ we see that

$$\begin{aligned} & |e_1 t_1 + e_2 t_2 + ... + e_d t_d| \\ = & |e_1(t_1 - t_2) + (e_1 + e_2)t_2 + ... + e_d t_d| \\ = & |e_1(t_1 - t_2) + (e_1 + e_2)(t_2 - t_3) + (e_1 + e_2 + e_3)t_3 + ... + e_d t_d| \\ = & \left| \sum_{i=1}^{d-1}(e_1 + e_2 + ... + e_i)(t_i - t_{i+1}) \right| \le C \sum_{i=1}^{d-1} |t_i - t_{i+1}| \end{aligned}$$

and

$$(e_1 t_1 + e_2 t_2 + ... + e_d t_d)^2 \le \left(C \sum_{i=1}^{d-1} |t_i - t_{i+1}| \right)^2 \le C^2(d-1) \sum_{i=1}^{d-1}(t_i - t_{i+1})^2,$$

where $C = \max\{|e_1 + e_2 + ... + e_i| : 1 \le i \le d-1\}$.

Now pick any d-set $\omega = \omega(1) \cup \omega(2) \cup ... \cup \omega(k)$ with d-components $\omega(1), \omega(2), ..., \omega(k)$. Making the estimates above we see that

$$\begin{aligned} (e(x;\omega))^2 = & \sum_{j=1}^{k}(e_1 x(n((j-1)d+1)) + e_2 x(n((j-1)d+2)) + ... + e_d x(n(jd)))^2 \\ \le & C^2(d-1) \sum_{j=1}^{k} \sum_{j=1}^{d-1}(x(n((j-1)d+i)) - x(n((j-1)d+i+1)))^2 \\ \le & C^2(d-1) \|x\|^2_{J(1;-1)}, \end{aligned}$$

where $\|x\|_{J(1;-1)}$ denotes one of the equivalent norms in the James space $J = J(1;-1)$ (see [1, 3]). Hence$\|x\|_{J(e)} \leq C\sqrt{d-1}\|x\|_{J(1;-1)}$. $\square$

The following lemma gives a chance to pass from an arbitrary vector $e = (e_1, e_2, ..., e_d) \in \mathbb{R}^d$ to the special$(d+1)$-vector $u_d = (1, -1, 0, 0, ...0) \in \mathbb{R}^{d+1}$.

Lemma 3.2. *The inclusion operator $id : J(e) \to J(u_d)$ is well-defined and continuous.*

Proof. Fix $x \in J(e)$ and pick any $(d+1)$-set $\omega = \omega(1) \cup \omega(2) \cup ... \cup \omega(k)$ with $(d+1)$-components $\omega(1), \omega(2), ..., \omega(k)$. For each component

$$\omega(j) = \{n((j-1)(d+1)+1) < n((j-1(d+1)+2) < ... < n(j(d+1))\}$$

let $\omega\text{'}(j) = \omega(j) \setminus \{n((j-1)(d+1)+1)\}$ and $\omega\text{''}(j) = \omega(j) \setminus \{n((j-1)(d+1)+2)\}$.

Then $\omega\text{'} = \omega\text{'}(1) \cup \omega\text{'}(2) \cup ... \cup \omega\text{'}(k)$ and $\omega\text{''} = \omega\text{''}(1) \cup \omega\text{''}(2) \cup ... \cup \omega\text{''}(k)$ are two d-sets with d-components $\omega\text{'}(1), ..., \omega\text{'}(k)$ and with d-components $\omega\text{''}(1), \omega\text{''}(2), ..., \omega\text{''}(k)$.

Consider for simplicity the case $j = 1$. Then

$$\begin{aligned} & e_1(x(n(2))) - x(n(1))) \\ = {} & (e_1 x(n(2))) + e_2 x(n(3))) + ... + e_d x(d+1)))) \\ & \quad - (e_1 x(n(1))) + e_2 x(n(3))) + ... + e_d x(d+1)))) \\ = {} & e * x(\omega\text{'}; 1) - e * x(\omega\text{''}; 1) \end{aligned}$$

and

$$(x(n(2))) - x(n(1)))^2 \leq \frac{2}{e_1^2}((e * x(\omega\text{'}; 1))^2 + (e * x(\omega\text{''}; 1))^2).$$

Having such an estimate for each $j = 2, 3, ..., k$ and summing up all inequalities, we see that

$$(u_d(x;\omega))^2 \leq \frac{2}{e_1^2}(e(x;\omega\text{'}))^2 + (e(x;\omega\text{''}))^2) \leq \frac{4}{e_1^2}\|x\|_e^2 = C^2\|x\|_e^2.$$

Passing to the supremum over all $(d+1)$-sets, we finally obtain $\|x\|_{u_d} \leq C\|x\|_e$. $\square$

So our final lemma shows that dependence on $d \in \mathbb{N}$ can in fact be eliminated, and we can return to the original vector $(1;-1) = u_1$. Together with Lemmata 3.1 and 3.2 this completes the proof of the theorem.

Lemma 3.3. *The inclusion operator $id : J(u_d) \to J(u_1)$ is well-defined and continuous.*

Proof. First, we need the following purely combinatorial sublemma. We will temporarily say that a 2-set

$$\Delta = \{p(1) < p(2) < \ldots < p(2s-1) < p(2s)\} \subset \mathbb{N}$$

is $d-$ *dispersed* if $s = 1$, or if $s > 1$ and $p(2j+1) \geq p(2j) + d$ for all $j = 1, 2, \ldots, s-1$.

Sublemma 3.4. *Every 2-set can be decomposed into a union of at most $[\frac{d}{2}]+1$ pairwise disjoint, d-dispersed 2-subsets.*

Proof of Sublemma 3.4. For an arbitrary 2-set $\omega = \{n(1) < n(2) < ... < n(2k-1) < n(2k)\}$ let $N = [\frac{d}{2}]+1$ and $k = qN+r$ for some integer q and r with $0 \le r \le N-1$. If $s = 0$, then $r > 0$. So, $\omega = \omega_1 \cup ... \cup \omega_r$, where $\omega_m = \{n(2m-1); n(2m)\}$, $m = 1, 2..., r$ are pairwise disjoint d-dispersed 2-sets. If $s > 0$ then for every $r < m \le N$ we define the 2-subset ω_m of ω by setting

$$\omega_m = \bigcup_{i=0}^{q-1} \{n(2(iN+m)-1);\, n(2(iN+m))\}$$

and for every $1 \le m \le r$ by setting

$$\omega_m = \bigcup_{i=0}^{q} \{n(2(iN+m)-1);\, n(2(iN+m))\}.$$

Clearly $\omega_1, ..., \omega_N$ are pairwise disjoint 2-sets and $\omega = \omega_1 \cup ... \cup \omega_N$. Moreover,

$$\begin{aligned} &n(2((i+1)N+m)-1)) - n(2(iN+m)) \\ &\ge 2((i+1)N+m)-1) - 2(iN+m) = 2N-1 \ge d. \end{aligned}$$

Hence each ω_m is d-dispersed. □

Let us return to the proof of Lemma 3.3. The main advantage of a d-dispersed 2-set Δ is that one can "extend" it to a $(d+1)$- set ∇ by adding the $(d-1)$ natural numbers which immediately follow $p(2j)$, to each 2-component $\{p(2j-1); p(2j)\}$ of Δ. Namely,

$$\Delta(1) = \{p(1); p(2)\} \Rightarrow \nabla(1) = \{p(1); p(2); p(2)+1; p(2)+2; ...; p(2)+d-1\},$$

$$\Delta(2) = \{p(3); p(4)\} \Rightarrow \nabla(2) = \{p(3); p(4); p(4)+1; p(4)+2; ...; p(4)+d-1\},$$

$$\Delta(s) = \{p(2s-1); p(2s)\} \Rightarrow \nabla(s) = \{p(2s-1); p(2s); p(2s)+1; ...; p(2s)+d-1\}.$$

Clearly,

$$\max\nabla(1) < \min\nabla(2) < \max\nabla(2) < \min\nabla(3) < ... \max\nabla(s-1) < \min\nabla(s)$$

and this is why the sets $\nabla(1), \nabla(2), ..., \nabla(s)$ really are $(d+1)-$ components of their union ∇. So for each $x = (x(m))_{m\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ we have

$$\begin{aligned} &(u_1(x,\nabla))^2 \\ &= \sum_{j=1}^{s} (x(p(2j)) - x(p(2j-1)))^2 \\ &= \sum_{j=1}^{s} (x(p(2j)) - x(p(2j-1) + 0\cdot x(p(2j)+1) + ... + 0\cdot x(p(2j)+d-1))^2 \\ &= (u_d(x,\nabla))^2 \end{aligned}$$

and finally, for an arbitrary 2-set $\omega = \{n(1) < n(2) < ... < n(2k-1) < n(2k)\}$ we obtain

$$\begin{aligned}(u_1(x,\omega))^2 &= \sum_{i=1}^{k}(x(n(2i)) - x(n(2i-1)))^2 \\ &= \sum_{j=1}^{N}\left(\sum_{i\in\Delta_j}(x(n(2i)) - x(n(2i-1)))^2\right) \\ &= \sum_{j=1}^{N}(u_1(x,\Delta_j))^2 = \sum_{j=1}^{N}(u_d(x,\nabla_j))^2 \le N \cdot \|x\|^2_{J(u_d)}.\end{aligned}$$

Hence the inclusion operator $id : J(u_d) \to J(u_1)$ is a well-defined mapping and its norm does not exceed $\sqrt{N} = \sqrt{[\frac{d}{2}] + 1}$. $\square$

Acknowledgements. We acknowledge the comments and suggestions from the referees.

References

[1] R. C. James: Bases and reflexivity of Banach spaces, Ann. Math. (2) 52 (1950) 518–527.

[2] R. C. James: A non-reflexive Banach space isometric with its second conjugate space, Proc. Natl. Acad. Sci. USA 37 (1951) 174–177.

[3] H. Fetter, B. G. de Buen: The James Forest, London Math. Soc. Lecture Note Series 236, Cambridge University Press, Cambridge (1997).

[4] J. Lindenstrauss, C. Stegall: Examples of separable spaces which do not contain l_1 and whose duals are not separable, Stud. Math. 54 (1975) 81–105.

[5] J. Lindenstrauss, L. Tzafriri: Classical Banach Spaces Vols 1 and 2, Springer, Berlin (1996).

[6] R. H. Lohman, P. G. Casazza: A general construction of spaces of the type of R. C. James, Can. J. Math. 27 (1975) 1263–1270.

[7] V. Meshkov: On smooth functions in the James space, Vestn. Mosk. Univ., Ser. Math 4 (1974) 9–13 (in Russian).

[8] B. S. Mityagin, I. S. Edelshtein: Homotopy type of linear groups of two classes of Banach spaces, Funct. Anal. Appl. 4(3) (1970) 61–72.

[9] P. V. Semenov: James-Orlicz spaces, Russ. Math. Surv. 34(4) (1979) 185–186.

[10] P. V. Semenov: A counterexample to a Geoghegan-West problem, Proc. Amer. Math. Soc. 124 (1996) 939–943.