Journal of Convex Analysis
Volume 17 (2010), No. 1, 103–110

Invertibility of Order-Reversing Transforms on Convex Functions

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Received: March 31, 2008
Revised manuscript received: January 9, 2009

The invertibility of an order-reversing transform on the class of proper lower semicontinuous convex functions is completely determined by the behavior of its composition with its putative inverse (and of the reverse composition) on the subclasses of continuous affine functions over the primal and dual spaces. This strengthens a recent result of Artstein-Avidan and Milman, which characterizes order-reversing transforms of convex functions as affine adjustments of the Legendre-Fenchel transform.

Keywords: Locally convex space, Legendre-Fenchel transform, convex duality

2000 Mathematics Subject Classification: 46N10, 52A41

1. Introduction

Let $\Gamma(X)$ denote the class of proper, lower semicontinuous convex functions $f : X \to \mathbf{R} \cup \{\infty\}$, where X is a Hausdorff locally convex topological vector space. We assume that the dual Y of X is endowed with a topology compatible with the duality pairing $\langle \cdot, \cdot \rangle : Y \times X \to \mathbf{R}$. A transform $S : \Gamma(X) \to \Gamma(Y)$ is said to be *order-reversing* if $Sf_1 \geq Sf_2$ whenever $f_1 \leq f_2$. Arguably, the most important example of an order-reversing transform of convex functions is the Legendre-Fenchel transform (or *convex conjugation*),

$$f \mapsto f^* = \sup_{x \in X} \big(\langle \cdot, x \rangle - f(x) \big),$$

introduced in [2]–[4]. The present work is motivated by the following characterization.

Theorem 1.1 (Artstein-Avidan and Milman, [1]). *Consider an order-reversing transform $S : \Gamma(\mathbf{R}^n) \to \Gamma(\mathbf{R}^n)$ with an order-reversing inverse. Then there exist constant scalars $\alpha > 0$ and β, vectors $u, v \in \mathbf{R}^n$, and an invertible matrix $B \in \mathbf{R}^{n \times n}$ such that*

$$(Sf)(y) = \alpha f^*(By + v) + \langle y, u \rangle + \beta$$

for all $y \in \mathbf{R}^n$ and all $f \in \Gamma(\mathbf{R}^n)$.

Our concern in this paper is with the verification of the hypotheses of Theorem 1.1. When S and T are not presented in a form similar to the conclusion of Theorem 1.1, it may be cumbersome or difficult to verify the inverse relation for S and T over the

ISSN 0944-6532 / \$ 2.50

full class of convex functions. We demonstrate that it suffices to consider the inverse relation of S and T on the much smaller class of continuous affine functions, along with verification that S and T preserve properness and reverse order. Specifically, we prove the following result.

Theorem 1.2. *Consider order-reversing transforms $S : \Gamma(X) \to \Gamma(Y)$ and $T : \Gamma(Y) \to \Gamma(X)$. Then S and T are inverses of each other if and only if they satisfy the following conditions:*

(*a*) $ST(\langle \cdot, u\rangle - \alpha) = \langle \cdot, u\rangle - \alpha$ *for all* $(u, \alpha) \in X \times \mathbf{R}$*;*
(*b*) $TS(\langle v, \cdot\rangle - \beta) = \langle v, \cdot\rangle - \beta$ *for all* $(v, \beta) \in Y \times \mathbf{R}$*.*

Despite the intuitive appeal of this statement, its validity is not self-evident. Indeed, it seems to require an argument of similar complexity to that needed for Theorem 1.1 itself.

2. Notation and preliminaries

In this section, we introduce a bit of notation and provide a few preliminary results. For any $u \in X$, we define a function $\delta_u \in \Gamma(X)$ by

$$\delta_u(x) = \begin{cases} 0, & \text{if } x = u, \\ \infty, & \text{otherwise,} \end{cases}$$

with the same notation used for the analogous functions in $\Gamma(Y)$. We shall refer to functions of the form $x \mapsto \delta_u(x) + \alpha$ as *convex delta* functions with *base* point u. It is easily verified that the Legendre-Fenchel transform provides a natural bijection between continuous affine functions and convex delta functions, given by

$$(\langle v, \cdot\rangle - \beta)^* = \delta_v + \beta, \qquad (\langle \cdot, u\rangle - \alpha)^* = \delta_u + \alpha$$

and

$$(\delta_u + \alpha)^* = \langle \cdot, u\rangle - \alpha, \qquad (\delta_v + \beta)^* = \langle v, \cdot\rangle - \beta$$

for all $u \in X$, $v \in Y$ and $\alpha, \beta \in \mathbf{R}$. Moreover, the Legendre-Fenchel transforms on $\Gamma(X)$ and $\Gamma(Y)$ are inverses of each other (in particular, this follows from Theorem 2.2 below). These facts yield a second form of Theorem 1.2.

Corollary 2.1. *Consider order-reversing transforms $S : \Gamma(X) \to \Gamma(Y)$ and $T : \Gamma(Y) \to \Gamma(X)$. Then S and T are inverses of each other if and only if they satisfy the following conditions:*

(*a*) $TS(\delta_u + \alpha) = \delta_u + \alpha$ *for all* $(u, \alpha) \in X \times \mathbf{R}$*;*
(*b*) $ST(\delta_v + \beta) = \delta_v + \beta$ *for all* $(v, \beta) \in Y \times \mathbf{R}$*.*

Proof. The necessity is trivial. To prove the sufficiency, assume that conditions (a) and (b) hold. Let $L : f \mapsto f^*$ for $f \in \Gamma(X)$ and $K : g \mapsto g^*$ for $g \in \Gamma(Y)$ denote the Legendre-Fenchel transforms, so that L and K are inverses of each other. Define $\hat{T} : \Gamma(Y) \to \Gamma(X)$ and $\hat{S} : \Gamma(X) \to \Gamma(Y)$ by $\hat{T} = KSK$ and $\hat{S} = LTL$, respectively.

Clearly, $\hat{T}$ and $\hat{S}$ are order-reversing. Given any $(u, \alpha) \in X \times \mathbf{R}$, we have

$$\begin{aligned}\hat{S}\hat{T}(\langle\cdot, u\rangle - \alpha) &= LTLKSK(\langle\cdot, u\rangle - \alpha) = LTSK(\langle\cdot, u\rangle - \alpha)\\ &= LTS(\delta_u + \alpha) = L(\delta_u + \alpha) = \langle\cdot, u\rangle - \alpha.\end{aligned}$$

Consequently, $\hat{S}$ and $\hat{T}$ satisfy condition (a) of Theorem 1.2; similarly, they satisfy condition (b) of that result. Therefore, Theorem 1.2 tells us that $\hat{S}$ and $\hat{T}$ are inverses of each other. Because $S = L\hat{T}L$ and $T = K\hat{S}K$, we see that $ST = L\hat{T}LK\hat{S}K = L\hat{T}\hat{S}K = LK$, and likewise $TS = KL$. Hence ST and TS are the respective identity mappings on $\Gamma(Y)$ and $\Gamma(X)$, so S and T are inverses of each other. □

Our proof of Theorem 1.2 begins with an easy characterization of the identity transform.

Theorem 2.2. *Consider an order-preserving transform $M : \Gamma(X) \to \Gamma(X)$. Then M is the identity mapping on $\Gamma(X)$ if and only if it satisfies the following conditions:*

(a) $M(\delta_u + \alpha) \le \delta_u + \alpha$ *for all* $(u, \alpha) \in X \times \mathbf{R}$*;*
(b) $M(\langle v, \cdot\rangle - \beta) \ge \langle v, \cdot\rangle - \beta$ *for all* $(v, \beta) \in Y \times \mathbf{R}$.

Theorem 2.2 is proved by combining the following two lemmas, the first of which will play a key role in the next section as well.

Lemma 2.3. *Consider an order-preserving transform $M : \Gamma(X) \to \Gamma(X)$. If $M(\langle v, \cdot\rangle - \beta) \ge \langle v, \cdot\rangle - \beta$ for all $(v, \beta) \in Y \times \mathbf{R}$, then $Mf \ge f$ for each $f \in \Gamma(X)$.*

Proof. Consider $f \in \Gamma(X)$. For any affine function $\langle v, \cdot\rangle - \beta \le f$, the hypotheses of the lemma yield $\langle v, \cdot\rangle - \beta \le M(\langle v, \cdot\rangle - \beta) \le Mf$. Hence the function $Mf \in \Gamma(X)$ dominates every affine function that f dominates. By the standard duality of convex functions, each member of $\Gamma(X)$ is the pointwise supremum of the affine functions it dominates, so we may conclude that $f \le Mf$. □

Lemma 2.4. *Consider an order-preserving transform $M : \Gamma(X) \to \Gamma(X)$. If $M(\delta_u + \alpha) \le \delta_u + \alpha$ for all $(u, \alpha) \in X \times \mathbf{R}$, then $Mf \le f$ for each $f \in \Gamma(X)$.*

Proof. Consider any point $u \in X$ for which $f(u) = \alpha < \infty$. We clearly have $\delta_u + \alpha \ge f$, so the hypotheses of the lemma imply $\delta_u + \alpha \ge M(\delta_u + \alpha) \ge Mf$. This, in turns, says that $f(u) = \alpha \ge (Mf)(u)$. Because u is arbitrary, we conclude that $f \ge Mf$. □

Observe that, by the symmetry inherent in the duality pairing, Theorem 2.2 and the two lemmas above have completely analogous counterparts for transforms on $\Gamma(Y)$.

We close this section with an example concerning the hypothesis in Theorem 1.1 that S have an order-reversing inverse. Define $S, T : \Gamma(\mathbf{R}^n) \to \Gamma(\mathbf{R}^n)$ by

$$S = \begin{cases} f^*, & \text{if } 0 \in \operatorname{int}\operatorname{dom} f,\\ (f+1)^*, & \text{if } 0 \notin \operatorname{int}\operatorname{dom} f,\end{cases} \qquad T = \begin{cases} g^*, & \text{if } 0 \in \operatorname{int}\operatorname{dom} g^*,\\ g^* - 1, & \text{if } 0 \notin \operatorname{int}\operatorname{dom} g^*,\end{cases}$$

where we define $\operatorname{dom} f = \{x \mid f(x) < \infty\}$. It is readily verified that S is order-reversing and that S and T are inverses of each other. It is also clear that S cannot be cast in the form given by the conclusion of Theorem 1.1.

3. Bijections between affine functions and convex delta functions

This section is devoted to proving the following theorem. It implies Theorem 1.2 when combined with Theorem 2.2 (taking $M = TS$ and $M = ST$) of the preceding section.

Theorem 3.1. *Consider order-reversing transforms $S : \Gamma(X) \to \Gamma(Y)$ and $T : \Gamma(Y) \to \Gamma(X)$ satisfying the following conditions:*

(*a*) $ST(\langle \cdot, u\rangle - \alpha) = \langle \cdot, u\rangle - \alpha$ *for all* $(u, \alpha) \in X \times \mathbf{R}$;

(*b*) $TS(\langle v, \cdot\rangle - \beta) = \langle v, \cdot\rangle - \beta$ *for all* $(v, \beta) \in Y \times \mathbf{R}$.

Then $TS(\delta_u + \alpha) = \delta_u + \alpha$ *for all* $(u, \alpha) \in X \times \mathbf{R}$. *Likewise,* $ST(\delta_v + \beta) = \delta_v + \beta$ *for all* $(v, \beta) \in Y \times \mathbf{R}$.

Throughout this section, we assume S and T satisfy the hypotheses of Theorem 3.1. Our goal is to demonstrate that TS must be the identity transform on the class of convex delta functions in $\Gamma(X)$. Our proof of this fact consists of two main steps, summarized in the following propositions.

Proposition 3.2. *The transform TS is the identity on the class of all convex delta functions that lie in the image of T. Similarly, ST is the identity on the class of all convex delta functions that lie in the image of S.*

Proposition 3.3. *All convex delta functions in $\Gamma(X)$ lie in the image of T.*

3.1. Proof of Proposition 3.2

Here we establish that T is a bijection between the affine functions in $\Gamma(Y)$ and the convex delta functions in $\Gamma(X)$ that lie in the image of T. We divide the argument into lemmas, which will be needed for the subsequent proof of Proposition 3.3 as well.

Lemma 3.4. *For each $(u, \alpha) \in X \times \mathbf{R}$, $TS(\delta_u + \alpha) = \delta_u + \alpha'$ for some $\alpha' \geq \alpha$.*

Proof. Apply Lemma 2.3 to $M = TS$ and $f = \delta_u + \alpha \in \Gamma(Y)$. □

Lemma 3.5. *If $g \in \Gamma(Y)$ is affine, then Tg is a convex delta function. If $f \in \Gamma(X)$ is a convex delta function, then Sf is affine.*

Proof. First we show that $T(\langle \cdot, u\rangle - \alpha)$ cannot be dominated by two convex delta functions unless they have the same base point. Suppose $T(\langle \cdot, u\rangle - \alpha) \leq \min\{\delta_w + \epsilon,\, \delta_z + \gamma\}$, so that $\langle \cdot, u\rangle - \alpha = ST(\langle \cdot, u\rangle - \alpha) \geq \max\{S(\delta_w + \epsilon),\, S(\delta_z + \gamma)\}$. Because the only functions in $\Gamma(Y)$ dominated by $\langle \cdot, u\rangle - \alpha$ are other affine functions parallel to $\langle \cdot, u\rangle$, we must have $S(\delta_w + \epsilon) = \langle \cdot, u\rangle - \epsilon'$ and $S(\delta_z + \gamma) = \langle \cdot, u\rangle - \gamma'$ for some choice of $\epsilon', \gamma' \geq \alpha$. We may assume $\epsilon' \geq \gamma'$, so that $S(\delta_w + \epsilon) \leq S(\delta_z + \gamma)$. Hence $\delta_w + \epsilon'' = TS(\delta_w + \epsilon) \geq TS(\delta_z + \gamma) = \delta_z + \gamma''$ for some $\epsilon'' \geq \epsilon$ and $\gamma'' \geq \gamma$ (by order-reversion and Lemma 3.4). In particular, we must have $w = z$. Therefore, every delta function dominating $T(\langle \cdot, u\rangle - \alpha)$ has the same base point. This proves the first statement of the lemma.

Next we show that $S(\delta_u + \alpha)$ cannot dominate two affine functions unless they are parallel. Suppose $S(\delta_u + \alpha) \geq \max\{\langle \cdot, w\rangle - \epsilon,\, \langle \cdot, z\rangle - \gamma\}$, so that $\delta_u + \alpha' = TS(\delta_u + \alpha) \leq \min\{T(\langle \cdot, w\rangle - \epsilon),\, T(\langle \cdot, z\rangle - \gamma)\}$ for some $\alpha' \geq \alpha$ (by order-reversion and Lemma 3.4).

Because $\delta_u + \alpha'$ can only be dominated by other convex delta functions with the same base point, we must have $T(\langle\cdot, w\rangle - \epsilon) = \delta_u + \epsilon'$ and $T(\langle\cdot, z\rangle - \gamma) = \delta_u + \gamma'$ for some choice of $\epsilon', \gamma' \geq \alpha$. We may assume that $\epsilon' \geq \gamma'$, so that $T(\langle\cdot, w\rangle - \epsilon) \geq T(\langle\cdot, z\rangle - \gamma)$. Hence $\langle\cdot, w\rangle - \epsilon = ST(\langle\cdot, w\rangle - \epsilon) \leq ST(\langle\cdot, z\rangle - \gamma) = \langle\cdot, z\rangle - \gamma$. This shows that $w = z$, and so every affine function dominated by $S(\delta_u + \alpha)$ has the same coefficient vector. This proves the second statement of the lemma. □

The next result gives provides converses for Lemma 3.5.

Lemma 3.6. *If Sf is affine, then f is a convex delta function. If Tg is a convex delta function, then g is affine.*

Proof. First suppose that $Sf = \langle\cdot, u\rangle - \alpha$. By Lemma 3.5, $T(\langle\cdot, u\rangle - \alpha) = \delta_z + \gamma$ for some (z, γ). We claim that every convex delta function dominating f has z as its base point. Consider $\delta_w + \epsilon \geq f$. Then $S(\delta_w + \epsilon) \leq Sf = \langle\cdot, u\rangle - \alpha$, so $TS(\delta_w + \epsilon) \geq TSf = T(\langle\cdot, u\rangle - \alpha) = \delta_z + \gamma$. But $TS(\delta_w + \epsilon) = \delta_w + \epsilon'$ for some $\epsilon' \geq \epsilon$ (by Lemma 3.4), so we must have $w = z$, as claimed. This proves the first statement of the lemma.

Next, suppose that $Tg = \delta_u + \alpha$. By Lemma 3.5, $S(\delta_u + \alpha) = \langle\cdot, z\rangle - \gamma$ for some (z, γ). We claim that every affine function dominated by g has z as its coefficient vector. Consider $\langle\cdot, w\rangle - \epsilon \leq g$. Then $T(\langle\cdot, w\rangle - \epsilon) \geq Tg = \delta_u + \alpha$, so $\langle\cdot, w\rangle - \epsilon = ST(\langle\cdot, w\rangle - \epsilon) \leq STg = S(\delta_u + \alpha) = \langle\cdot, z\rangle - \gamma$. This implies that $w = z$, as claimed. This proves the second statement of the lemma. □

An immediate consequence of Lemma 3.6 is that $TS(\delta_u + \alpha) = \delta_u + \alpha$ whenever $\delta_u + \alpha = Tg$ for some $g \in \Gamma(Y)$, which proves the first statement in Proposition 3.2. Here is the analogue of Lemmas 3.5 and 3.6 needed for the second statement of Proposition 3.2.

Lemma 3.7. *A function $f \in \Gamma(X)$ is affine if and only if Sf is a convex delta function. A function $g \in \Gamma(Y)$ is a convex delta function if and only if Tg is affine.*

3.2. Proof of Proposition 3.3

Now we demonstrate that, under the hypotheses of Theorem 3.1, all convex delta functions in $\Gamma(X)$ lie in the image of T. To do so, we restate our question on transforming functions as one about mapping vectors, where the latter are the coefficients uniquely parameterizing the former. Specifically, we note that Lemma 3.5 above defines mappings T^{aff} and S^{del} of $X \times \mathbf{R}$ to itself, according to:

- $T^{\text{aff}}(u, \alpha) = (z, \gamma)$, whenever $T(\langle\cdot, u\rangle - \alpha) = \delta_z + \gamma$;
- $S^{\text{del}}(z, \gamma) = (u, \alpha)$, whenever $S(\delta_z + \gamma) = \langle\cdot, u\rangle - \alpha$.

Observe that $S^{\text{del}} \circ T^{\text{aff}}$ is the identity on $X \times \mathbf{R}$, because ST is the identity on the class of affine functions in $\Gamma(Y)$. With this notation, Proposition 3.3 is equivalent to the following.

Proposition 3.8. *The mapping T^{aff} is surjective.*

This change in perspective helps identify a course of action, summarized intuitively as follows. Denote the vector coordinates of the image of $T^{\mathrm{aff}} : (u, \alpha) \mapsto (z, \gamma)$ by

$$(u, \alpha) \mapsto z^{\mathrm{aff}}(u, \alpha) \in X, \qquad (u, \alpha) \mapsto \gamma^{\mathrm{aff}}(u, \alpha) \in \mathbf{R}.$$

Our proof of Proposition 3.8 consists of demonstrating three properties:

(P1) for each $\alpha \in \mathbf{R}$, the mapping $z^{\mathrm{aff}}(\cdot, \alpha) : X \to X$ is surjective;
(P2) for each $u \in X$, the function $\gamma^{\mathrm{aff}}(u, \cdot) : \mathbf{R} \to \mathbf{R}$ is surjective;
(P3) for each $u \in X$, the mapping $z^{\mathrm{aff}}(u, \cdot) : \mathbf{R} \to X$ is constant.

To see why these suffice to prove the surjectivity of T^{aff}, consider $(\bar{z}, \bar{\gamma}) \in X \times \mathbf{R}$. First, we use (P1) with $\alpha = 0$ to obtain $\bar{u} \in X$ so that $z^{\mathrm{aff}}(\bar{u}, 0) = \bar{z}$. Next, (P2) gives us $\bar{\alpha} \in \mathbf{R}$ with $\gamma^{\mathrm{aff}}(\bar{u}, \bar{\alpha}) = \bar{\gamma}$. Finally, (P3) tells us that $z^{\mathrm{aff}}(\bar{u}, \bar{\alpha}) = z^{\mathrm{aff}}(\bar{u}, 0) = \bar{z}$. Hence $T^{\mathrm{aff}}(\bar{u}, \bar{\alpha}) = \big(z^{\mathrm{aff}}(\bar{u}, \bar{\alpha}), \gamma^{\mathrm{aff}}(\bar{u}, \bar{\alpha})\big) = (\bar{z}, \bar{\gamma})$, and so T^{aff} is surjective.

Verifying the above properties (P1)–(P3) requires proving similar results about S^{del} along the way, so we introduce the notation $u^{\mathrm{del}}(z, \gamma)$ and $\alpha^{\mathrm{del}}(z, \gamma)$ to represent the first and second coordinates of $S^{\mathrm{del}}(z, \gamma) = (u, \alpha)$. We begin by proving property (P3).

Lemma 3.9. *For any $u \in X$, the value of $z^{\mathrm{aff}}(u, \alpha)$ is independent of α. Similarly, for any $z \in X$, the value of $u^{\mathrm{del}}(z, \gamma)$ is independent of γ.*

Proof. Consider $\gamma \le \gamma'$ and let $(u, \alpha) = S^{\mathrm{del}}(z, \gamma)$ and $(w, \epsilon) = S^{\mathrm{del}}(z, \gamma')$. Then $\delta_z + \gamma \le \delta_z + \gamma'$, so $\langle \cdot, u\rangle - \alpha = S(\delta_z + \gamma) \ge S(\delta_z + \gamma) = \langle \cdot, w\rangle - \epsilon$. This implies $u = w$, proving that $u^{\mathrm{del}}(z, \gamma)$ is independent of γ. The proof for z^{aff} and T^{aff} is similar. $\square$

Lemma 3.9 allows us to write $z^{\mathrm{aff}}(u)$ and $u^{\mathrm{del}}(z)$. Next, we obtain property (P3).

Lemma 3.10. *The mappings z^{aff} and u^{del} on X are inverses of each other.*

Proof. Because $S^{\mathrm{del}} \circ T^{\mathrm{aff}}$ is the identity on $X \times \mathbf{R}$, we have

$$\Big(u^{\mathrm{del}}\big(z^{\mathrm{aff}}(\bar{u})\big),\ \alpha^{\mathrm{del}}\big(z^{\mathrm{aff}}(\bar{u}), \gamma^{\mathrm{aff}}(\bar{u}, \bar{\alpha})\big)\Big) = (\bar{u}, \bar{\alpha}) \tag{1}$$

for all $(\bar{u}, \bar{\alpha})$. Equality of the first coordinates in (1) shows that $u^{\mathrm{del}} \circ z^{\mathrm{aff}}$ is the identity on X. Consequently, it suffices to prove that z^{aff} is both injective and surjective. For the injectivity, suppose $z^{\mathrm{aff}}(u) = z = z^{\mathrm{aff}}(w)$. Then $T(\langle u, \cdot\rangle - \alpha) = \delta_z + \alpha'$ and $T(\langle w, \cdot\rangle - \epsilon) = \delta_z + \epsilon'$ for some $\alpha, \alpha', \epsilon, \epsilon' \in \mathbf{R}$. We may assume $\alpha' \ge \epsilon'$, so that $S(\delta_z + \alpha') \le S(\delta_z + \epsilon')$. Hence $\langle u, \cdot\rangle - \alpha \le \langle w, \cdot\rangle - \epsilon$ and we see that $u = w$. Therefore, z^{aff} is injective. For the surjectivity, consider $z \in X$. Lemma 3.5 tells us that $S\delta_z = \langle u, \cdot\rangle - \alpha$ for some (u, α), whereas Lemma 3.4 gives $TS\delta_z = \delta_z + \gamma$ or some $\gamma \ge 0$. Combining these yields $T(\langle u, \cdot\rangle - \alpha) = \delta_z + \gamma$, so that $(z, \gamma) = T^{\mathrm{aff}}(u, \alpha) = \big(z^{\mathrm{aff}}(u), \gamma^{\mathrm{aff}}(u, \alpha)\big)$. In particular, the first coordinate shows that $z = z^{\mathrm{aff}}(u)$. Therefore, z^{aff} is surjective. $\square$

Our proof of the remaining property, (P2), relies on the following technical observation.

Lemma 3.11. *For each $z \in X$, the function $\alpha^{\mathrm{del}}(z, \cdot) : \mathbf{R} \to \mathbf{R}$ is nondecreasing and surjective. Moreover, $\alpha^{\mathrm{del}}(z, \cdot) \circ \gamma^{\mathrm{aff}}\big(u^{\mathrm{del}}(z), \cdot\big)$ is the identity on $\mathbf{R}$.*

Proof. The claim that $\alpha^{\text{del}}(z,\cdot)$ is nondecreasing follows immediately from the order-reversion property of S and T. Next, we note that equality of the second coordinates in (1) shows that $\alpha^{\text{del}}\big(z^{\text{aff}}(\bar u),\, \gamma^{\text{aff}}(\bar u,\bar\alpha)\big) = \bar\alpha$ for all $\bar u$ and $\bar\alpha$. By Lemma 3.10, choosing $\bar u = u^{\text{del}}(z)$ gives $z = z^{\text{aff}}(\bar u)$, so the equation in the preceding sentence becomes $\alpha^{\text{del}}\big(z, \gamma^{\text{aff}}(u^{\text{del}}(z),\bar\alpha)\big) = \bar\alpha$ for all $\bar\alpha$, as claimed. □

We are now prepared to prove property (P2).

Lemma 3.12. *For each $u \in X$, the function $\gamma^{\text{aff}}(u,\cdot) : \mathbf{R} \to \mathbf{R}$ is surjective.*

Proof. By Lemma 3.7, the transforms S and T define mappings S^{aff} and T^{del} on $Y \times \mathbf{R}$ by

- $S^{\text{aff}}(v,\beta) = \big(w^{\text{aff}}(v), \epsilon^{\text{aff}}(v,\beta)\big) = (w,\epsilon)$, whenever $S(\langle v,\cdot\rangle - \beta) = \delta_w + \epsilon$;
- $T^{\text{del}}(w,\epsilon) = \big(v^{\text{del}}(w), \beta^{\text{del}}(w,\epsilon)\big) = (v,\beta)$, whenever $T(\delta_w + \epsilon) = \langle v,\cdot\rangle - \beta$.

As in Lemma 3.10, the mappings w^{aff} and v^{del} are inverses of each other. In analogy with Lemma 3.11, we also have the following two properties:

(a) the function $\beta^{\text{del}}(w,\cdot) : \mathbf{R} \to \mathbf{R}$ is nondecreasing and surjective;
(b) the composition $\beta^{\text{del}}(w,\cdot) \circ \epsilon^{\text{aff}}\big(v^{\text{del}}(w),\cdot\big)$ is the identity on $\mathbf{R}$.

Consider any $\alpha \in \mathbf{R}$ and fix an element $w \in Y$, then define $z = z^{\text{aff}}(u)$ and $v = v^{\text{del}}(w)$. By the trivial inequality $\langle\cdot,u\rangle - \alpha \le \delta_w + \langle w,u\rangle - \alpha$ and the order-reversion property of T, we have

$$\delta_z + \gamma^{\text{aff}}(u,\alpha) = T\big(\langle\cdot,u\rangle - \alpha\big) \ge T\big(\delta_w + \langle w,u\rangle - \alpha\big) = \langle v,\cdot\rangle - \beta^{\text{del}}(w, \langle w,u\rangle - \alpha),$$

which implies

$$\gamma^{\text{aff}}(u,\alpha) \ge \langle v,z\rangle - \beta^{\text{del}}(w,\langle w,u\rangle - \alpha). \tag{2}$$

By similar reasoning, we also have $\epsilon^{\text{aff}}(v,\beta) \ge \langle w,u\rangle - \epsilon^{\text{del}}(z,\langle v,z\rangle - \beta)$ for any $\beta \in \mathbf{R}$. Taking $\beta = \langle v,z\rangle - \gamma^{\text{aff}}(u,\alpha)$, we obtain

$$\epsilon^{\text{aff}}\big(v, \langle v,z\rangle - \gamma^{\text{aff}}(u,\alpha)\big) \ge \langle w,u\rangle - \epsilon^{\text{del}}(z,\gamma^{\text{aff}}(u,\alpha)) = \langle w,u\rangle - \alpha,$$

where we use the second statement of Lemma 3.11 to simplify the right-hand side. Next, we use property (a) above: applying the function $\beta^{\text{del}}(w,\cdot)$ to the left- and right-hand sides preserves the inequality and yields

$$\beta^{\text{del}}\Big(w, \epsilon^{\text{aff}}\big(v, \langle v,z\rangle - \gamma^{\text{aff}}(u,\alpha)\big)\Big) \ge \beta^{\text{del}}(w,\langle w,u\rangle - \alpha).$$

Property (b) simplifies this to $\langle v,z\rangle - \gamma^{\text{aff}}(u,\alpha) \ge \beta^{\text{del}}(w,\langle w,u\rangle - \alpha)$, which is the reverse of inequality (2). Therefore, (2) holds as equality for every α and the claimed surjectivity of $\gamma^{\text{aff}}(u,\cdot)$ follows from that of $\beta^{\text{del}}(w,\cdot)$. □

This completes the proof of Proposition 3.8 and, therefore, of Theorems 1.2 and 3.1.

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