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Normality and Quasiconvex Integrands

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Let $(T, \mathcal{A})$ be an arbitrary measurable space and f an integrand defined on $T \times \mathbb{R}^n$ such that $f(t, \cdot)$ is quasiconvex and lower semicontinuous. Here, convexity is present by the level set mapping. We show that the normality property of the integrand in the sense of Rockafellar ([10], [11]) can be characterized by the normality of the level set mapping, and that normality is preserved for quasiconvex conjugates. Finally we obtain for the integral $I_f(x(\cdot)) = \int_T f(t, x(t))\, d\mu(t)$ the equality (in appropriate topology) between the lower semicontinuous regularization and the second quasiconvex conjugate.

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1. Introduction

In many areas of applied mathematics, one has to deal with integrals of the form

$$I_f(x(\cdot)) = \int_T f(t, x(t))\, d\mu(t), \tag{1}$$

where T is an arbitrary nonempty set equipped with a $\sigma - field$ $\mathcal{A}$ and a measure μ, $f : T \times \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is a measurable integrand and $x(\cdot)$ is a measurable $\mathbb{R}^n$-valued functions on T.

Classically only finite Carathéodory-type integrands $f(\cdot, \cdot)$ were considered (i.e., $f(\cdot, x)$ measurable and $f(t, \cdot)$ continuous). However, generally in applications $f(t, \cdot)$ is discontinuous. Moreover, in convex situation ($f(t, \cdot)$ convex), Rockafellar [10], [11] initiated the notion of "normal integrand", which proved to be a very fruitful concept for the study of $I_f(\cdot)$. Since the theory of convex conjugacy plays a prominent role in this framework, we are interested in generalizing it to quasiconvex situation.

It is known [9] that one can use (as the convex analysis approach) the approximation of functionals from below by so called c-affine functions. Thus, quasiconvex regularization and the generalized biconjugate have geometrical intepretations as generalized

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Minkowski and Fenchel-Moreau theorems (Section 3). It is obvious that $I_f(\cdot)$ is convex if so is $f(t,\cdot)$, but unfortunately, even if $f(t,\cdot)$ is quasiconvex, $I_f(\cdot)$ is not necessarily quasiconvex, for example if one takes the function $f:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ given by $f(t,a)=-e^{-ta^2}$ and the function $I_f:\mathcal{C}(\mathbb{R}_+;\mathbb{R})\to\mathbb{R}$ such that $I_f(x(\cdot))=\int_{\mathbb{R}_+}f(t,x(t))\,dt$. It is obvious that for all $t\in\mathbb{R}_+$, $f(t,\cdot)$ is quasiconvex but I_f is not quasiconvex on $\mathcal{C}(\mathbb{R}_+;\mathbb{R})$, since for $x^1(\cdot)$, $x^2(\cdot)$ given by $x^1(t)=1$, $x^2(t)=-1$ for all $t\in\mathbb{R}_+$ and for $\lambda=\frac{1}{2}$ we get, $(1-\lambda)x^1(\cdot)+\lambda x^2(\cdot)\equiv 0\notin\operatorname{dom}I_f$.

However, quasiconvex conjugacy was considered by Barron and Liu [1] in the calculus of variation in L^∞ space, for functionals of the form

$$I(x(\cdot))=\operatorname{ess}\sup_{0\le t\le T}f\left(t,x(t),x'(t)\right),$$

i.e.,

$$I(x(\cdot))=\left\|f\left(\cdot,x(\cdot),x'(\cdot)\right)\right\|_{L^\infty[0,T]}.$$

They have established identity between the lower semicontinous (lsc for short) regularization and the second quasiconvex conjugate (bi-q-conjugate for short), i.e.,

$$I_q^{**}(x(\cdot))=\left\|f_q^{**}\left(\cdot,x(\cdot),x'(\cdot)\right)\right\|_{L^\infty[0,T]}$$

where I_q^{**} (resp. f_q^{**}) is the bi-q-conjugate of I_f (resp. f).

Our aim is to establish similar results for L_p variational problem. We give a short introduction to measurable closed-valued mappings and its applications to normal integrands (Section 2). Then, we consider a normal integrand and we show that the quasiconvex conjugate and bi-q-conjugate are also normal integrand (Section 4). In Section 5, we prove the identity between the lsc regularization of I_f and its bi-q-conjugacy.

2. Preliminaries

Definition 2.1. 1) A set-valued mapping Γ from T to $\mathbb{R}^n$ is denoted by $\Gamma:T\rightrightarrows\mathbb{R}^n$ where

$$\operatorname{dom}\Gamma=\{t\in T:\Gamma(t)\neq\emptyset\},$$
$$\operatorname{gph}\Gamma=\{(t,x):x\in\Gamma(t)\},$$

are respectively the domain and the graph of Γ.

2) Γ is said to be measurable if for every open set $O\subset\mathbb{R}^n$ the set $\Gamma^{-1}(O)=\{t\in T:\Gamma(t)\cap O\neq\emptyset\}$ is measurable, i.e., $\Gamma^{-1}(O)\in\mathcal{A}$.

Remark 2.2. $\operatorname{dom}\Gamma$ is measurable since, $\operatorname{dom}\Gamma=\Gamma^{-1}(\mathbb{R}^n)$.

For a comprehensive treatment of measurable set-valued mappings, we refer the reader to the classical texts (Castaing and Valadier [2], Rockafellar [11]).

It is clear that the "integrand" $f:T\times\mathbb{R}^n\to\overline{\mathbb{R}}$ is a crucial element in the expression of integral functionals of the form (1) and the cornerstone will be the measurability of the set-valued mappings $E_f:T\rightrightarrows\mathbb{R}^n\times\mathbb{R}$ and $D_f:T\rightrightarrows\mathbb{R}^n$ given by

$$E_f(t)=\operatorname{epi}f(t,.)=\{(x,\alpha)\in\mathbb{R}^n\times\mathbb{R}:f(t,x)\le\alpha\}.$$

and

$$D_f(t) = \operatorname{dom} f(t,\cdot) = \{x \in \mathbb{R}^n : f(t,x) < \infty\}.$$

Let us recall that:

Definition 2.3. A function $f : T \times \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be a normal integrand if its epigraphical mapping $E_f : T \rightrightarrows \mathbb{R}^n \times \mathbb{R}$ is measurable with closed-values.

The basic consequences of normality is given by the following theorem.

Theorem 2.4 ([12], Proposition 14.28). *For any normal integrand $f : T \times \mathbb{R}^n \to \overline{\mathbb{R}}$, the following holds:*

(*1*) $D_f : T \rightrightarrows \mathbb{R}^n$ *is measurable.*
(*2*) $f(t,\cdot)$ *lsc, and for any measurable function* $x(\cdot)$, $f(\cdot, x(\cdot))$ *is measurable.*

Remark 2.5. Note that an integrand f such that $f(t,\cdot)$ is lsc and $f(\cdot, x(\cdot))$ measurable, is not necessarily normal, as is shown in ([12], page 661).

Our aim in quasiconvex setting being to retain the notions which are the closest to the corresponding notion in the convex area, it will be useful to recall some principal convex results in the following subsection.

2.1. Convexity and conjugacy

Theorem 2.6 (Castaing representation, [10]). *Let $f : T \times \mathbb{R}^n \to \overline{\mathbb{R}}$ be such that, for every $t \in T$, the function $f(t,\cdot)$ is lsc and convex. Then f is a normal integrand if and only if there is a sequence $\{x^k\}_{k\in\mathbb{N}}$ of measurable functions $x^k : T \to \mathbb{R}^n$ such that:*

(a) *for each* $k \in \mathbb{N}$, $t \mapsto f\left(t, x^k(t)\right)$ *is measurable;*
(b) *for each* $t \in T$, $\left\{x^k(t) : k \in \mathbb{N}\right\} \cap D_f(t)$ *is dense in* $D_f(t)$.

Remark 2.7. We know that if $f(t,\cdot)$ is convex then $f(t,\cdot)$ is continuous on $\operatorname{int} D_f(t)$. Then every lsc convex integrand is a normal integrand when $\operatorname{int} D_f(t) \neq \varnothing$.

Recall that the conjugate f^* and biconjugate f^{**} of a normal integrand $f : T \times \mathbb{R}^n \to \overline{\mathbb{R}}$ are given by the Legendre-Fenchel transform of $f(t,\cdot)$ and $f^*(t,\cdot)$, i.e.,

$$f^*(t,x^*) = \sup_{x\in\mathbb{R}^n} \{\langle x, x^*\rangle - f(t,x)\}, \tag{2}$$

$$f^{**}(t,x) = \sup_{x^*\in\mathbb{R}^n} \{\langle x, x^*\rangle - f^*(t,x^*)\}. \tag{3}$$

We know by [13] that for any function $f : T \times \mathbb{R}^n \to \overline{\mathbb{R}}$, such that $f(t,\cdot)$ is proper, one has for every fixed t in T the following:

i) $f^{**}(t,\cdot) \leq f(t,\cdot)$ and
ii) $f^{**}(t,\cdot) = f(t,\cdot)$ if and only if $f(t,\cdot)$ is convex and lsc.

By $f_{\overline{c}}(t,\cdot)$ we denote the greatest convex lsc function majorized by $f(t,\cdot)$.

Proposition 2.8 ([13]). *For any function $f : T \times \mathbb{R}^n \to \overline{\mathbb{R}}$, such that $f(t,\cdot)$ is proper, one has for any $t \in T$*

i)

$$f_{\overline{c}}(t,x) = \inf\{\lambda \in \mathbb{R} : (x,\lambda) \in \overline{co} \operatorname{epi} f(t,x)\} \quad \forall x \in \mathbb{R}^n, \tag{4}$$

where $\overline{co}$ designates the convex closure operation.

ii)

$$f^{**}(t,\cdot) = f_{\overline{c}}(t,\cdot). \tag{5}$$

Theorem 2.9 ([10]). *If $f : T \times \mathbb{R}^n \to \overline{\mathbb{R}}$ is a normal integrand, then f^* and f^{**} are normal integrands.*

3. Quasiconvexity, regularization and conjugacy

3.1. Quasiconvexity

For an introduction to quasiconvex analysis we refer to [6]. Let X be a Banach space. We recall that a function $f : X \to \overline{\mathbb{R}}$ is quasiconvex if for all $\lambda \in [0,1]$ and $x_1, x_2 \in X$ one has

$$f((1-\lambda)x_1 + \lambda x_2) \leq \max(f(x_1), f(x_2)).$$

Considering the Level set mapping defined as $\operatorname{lev}_f : \mathbb{R} \rightrightarrows X$

$$\operatorname{lev}_f(\lambda) = \{x \in X : f(x) \leq \lambda\}, \tag{6}$$

it is not difficult to see that f is quasiconvex if and only if the set-valued mapping lev_f is with convex values.

Let us recall according to [4], that any function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ can be represented as follows

$$f(x) = \inf\{\lambda \in \mathbb{R} : x \in \operatorname{lev}_f(\lambda)\}. \tag{7}$$

Definition 3.1 ([4]). Let be $f : X \to \overline{\mathbb{R}}$.

1) We call the lsc hull function of f (lsc regularization) the function $\overline{f} : X \to \overline{\mathbb{R}}$, given by

$$\overline{f}(x) = \inf\left\{\lambda \in \mathbb{R} : x \in \overline{\operatorname{lev}_f(\lambda)}\right\}, \tag{8}$$

where $\overline{A}$ is the closure of the subset A.

2) We call the lsc quasiconvex hull function of f (lsc quasiconvex regularization) the function $f_{\overline{q}} : X \to \overline{\mathbb{R}}$ defined by

$$f_{\overline{q}}(x) = \inf\{\lambda \in \mathbb{R} : x \in \overline{co} \operatorname{lev}_f(\lambda)\}. \tag{9}$$

Then, the following characterization holds.

Proposition 3.2 ([4]). *For any function $f : X \to \overline{\mathbb{R}}$, the function $f_{\overline{q}}$ is the greatest lsc quasiconvex function majorized by f, and its Level set mapping satisfies the equality*

$$\operatorname{lev}_{f_{\overline{q}}}(\lambda) = \bigcap_{\beta > \lambda} \overline{co} \operatorname{lev}_f(\beta) \quad \forall \lambda \in \mathbb{R}. \tag{10}$$

3.2. Quasiconvex conjugacy

Following Moreau [8] we know that, for a Banach space Y and a function $c : X \times Y \to \overline{\mathbb{R}}$ (said coupling fuction), we can define the c-conjugate and the bi-c-conjugate of functional $f : X \to \overline{\mathbb{R}}$ as follows:

$$f^c : y \in Y \mapsto f^c(y) = \sup_{x\in X} \{c(x,y) - f(x)\} \tag{11}$$

$$f^{cc} : x \in X \longmapsto f^{cc}(x) = \sup_{y\in Y} \{c(x,y) - f^c(y)\}\,. \tag{12}$$

Since we are dealing with extended real valued functions we use the conventions $+\infty + (-\infty) = -\infty + (+\infty) = +\infty - (+\infty) = -\infty - (-\infty) = -\infty$. Several coupling functions were discussed in the literature, motivated by various aims see, e.g., [14], [7] and [9].

For our part and following [6], we shall be concerned with the coupling function given by

$$c(x,(x^*,r)) = c_r(\langle x, x^*\rangle)\,,$$

where $\langle , \rangle$ is the duality pairing between X and X^* and

$$c_r(t) = \begin{cases} r & \text{if } t > r \\ -\infty & \text{otherwise.} \end{cases} \tag{13}$$

Formulas (11) and (12) become:

$$f^c(x^*,r) = \sup_{x\in X} \{c_r(\langle x, x^*\rangle) - f(x)\} \tag{14}$$

and

$$f^{cc}(x) = \sup_{(x^*,r)\in X^*\times\mathbb{R}} \{c_r(\langle x, x^*\rangle) - f^c(x^*,r)\}\,. \tag{15}$$

As a consequence of Corollary 4.2, [7] and Corollary 1.4, [5] the following proposition establishes, for bi-quasiconvex operation, a result in the line of Proposition 2.8.

Proposition 3.3 ([7]). *For any function $f : X \to \overline{\mathbb{R}}$, we have*

$$f^{cc} = f_{\overline{q}}.$$

4. Level sets, integrands and normality

The main properties of convex normal integrand f are consequences of the closure, measurability and convexity of the epigraphical mapping $E_f : T \rightrightarrows \mathbb{R}^n \times \mathbb{R}$. Let us recall some essential results about the mapping E_f.

Theorem 4.1 ([12], Theorem 14.8). *Let $f : T \times \mathbb{R}^n \rightrightarrows \mathbb{R}$*

1) *The implication $(a) \Rightarrow (b)$ holds for the following properties:*
   (a) *$E_f : T \rightrightarrows \mathbb{R}^n \times \mathbb{R}$ is with closed-valued and measurable i.e., f is normal.*
   (b) $\operatorname{gph} E$ *is an $\mathcal{A} \otimes \mathcal{B}(X)$-measurable subset of $T \times X$.*
2) *When the $\sigma-$field $\mathcal{A}$ is $\mu-$complete, these two properties are equivalent.*

This section is devoted to integrands functions $f(\cdot,\cdot)$ such that $f(t,\cdot)$ is quasi-convex, then the level-set mapping will be crucial in the analysis.

Theorem 4.2 ([12], Proposition 14.32). *For a normal integrand $f : T \times \mathbb{R}^n \to \overline{\mathbb{R}}$ and any $\alpha \in \mathbb{R}$, the level-set mapping*

$$t \mapsto \operatorname{lev}_{f(t,\cdot)}(\alpha) = \{x \in \mathbb{R}^n : f(t,x) \leq \alpha\}$$

is closed-valued and measurable.

Actually the converse holds whenever the σ-field is μ-complete.

Proposition 4.3. *Assume that the the $\sigma-$field $\mathcal{A}$ is $\mu-$complete. Then a lsc integrand $f : T \times X \to \overline{\mathbb{R}}$ is normal if and only if for each $\alpha \in \mathbb{R}$ the level set mapping $t \mapsto \operatorname{lev}_{f(t,.)}(\alpha)$ is measurable.*

Proof. It is sufficient to show the measurability of E_f. From Theorem 4.1, the sets

$$\begin{aligned}\operatorname{gphlev}_{f(.,.)}(\alpha) &= \left\{(t,x) \in T \times \mathbb{R}^n : x \in \operatorname{lev}_{f(.,.)}(\alpha)\right\} \\ &= \{(t,x) \in T \times \mathbb{R}^n : f(t,x) \leq \alpha\}\end{aligned}$$

are $\mathcal{A} \otimes \mathcal{B}(X)$-measurable for all $\alpha \in \mathbb{R}$. This implies that f is $\mathcal{A} \otimes \mathcal{B}(X)$-measurable and so is $\widehat{f}$ where

$$\widehat{f}(t,x,\alpha) = f(t,x) - \alpha$$

which leads to the measurability of

$$\operatorname{gph} E_f = \left\{(t,x,\alpha) : \widehat{f}(t,x,\alpha) \leq 0\right\}.$$

Theorem 4.1 works and the measurability of E_f follows. The proof is then complete. $\square$

Remark 4.4. Let $x(\cdot)$ be a measurable function then the integrand $g : (t,x^*) \in T \times X^* \longmapsto g(t,x^*) = \langle x(t), x^* \rangle$ is of Carathéodory but the integrand $h : (t,x^*) \in T \times X^* \longmapsto h(t,x^*) = c_r(\langle x(t), x^* \rangle)$ is not. This is the crucial difference between convex and quasiconvex cases.

The following results will be useful to generalize Theorem 2.9 to quasiconvex setting.

Proposition 4.5 ([11]). *Let h be an integrand on $T \times \mathbb{R}^n$ of the form*

$$h(t,x) = \phi(t, g(t,x)) \tag{16}$$

such that:

1) *g: $T \times \mathbb{R}^n \to \mathbb{R}$ is a normal integrand.*
2) *ϕ is a normal integrand on $T \times \mathbb{R}$ with $\phi(t,\cdot)$ for every $t \in T$ nondecreasing.*

Then f is normal.

Proposition 4.6. *Let $f : T \times \mathbb{R}^n \to \overline{\mathbb{R}}$ be a normal integrand. Then $f^c(.,.,r)$ is normal for every fixed r.*

Proof. $\operatorname{dom} E_f = \{t \in T : f(t, \cdot) \neq +\infty\}$ is measurable. Then for $(x_n, \alpha_n)_{n\in\mathbb{N}}$ a Castaing representation of E_f, we have

$$E_f(t) = \overline{\cup_{n\in\mathbb{N}} \{(x_n(t), \alpha_n(t))\}} \quad t \in \operatorname{dom} E_f \text{ a.e.}$$

For every $n \in \mathbb{N}$ and $r \in \mathbb{R}$ we define $g_r^n : T \times X^* \to \overline{\mathbb{R}}$ such that

$$g_r^n(t, x^*) = \begin{cases} c_r(\langle x^n(t), x^*\rangle) - \alpha_n(t) & t \in \operatorname{dom} E_f \\ -\infty & t \in T \setminus \operatorname{dom} E_f. \end{cases}$$

By formula (14) and the density of Castaing representation we have

$$f^c(t, x^*, r) = \sup_{n\in\mathbb{N}} g_r^n(t, x^*, r) \tag{17}$$

for every fixed r, so $f^c(\cdot, \cdot, r)$ will be normal if each g_r^n is normal on $\operatorname{dom} E_f$ i.e., $(t, x^*) \mapsto c_r(\langle x^n(t), x^*\rangle)$ is normal. We can see that the function c_r defined by relation (13) is lsc and nondecreasing, hence for $h(t, x^*) = c_r(\langle x^n(t), x^*\rangle)$ Proposition 4.5 works with $\phi(t, \alpha) = c_r(\alpha)$ and $g(t, x^*) = \langle x^n(t), x^*\rangle$. The normality of g_r^n follows from above and that of $f^c((\cdot, \cdot), r)$ from ([12], Proposition 14.11). For $t \in T \setminus \operatorname{dom} E_f$, we have $f(t, x) = +\infty$ for all x and consequently $f^c((t, x^*), r) = -\infty$ for any x^*, i.e., that $\operatorname{epi} f^c((t, \cdot), r) \equiv \mathbb{R}^n \times \mathbb{R}$, thus $\operatorname{epi} f^c((\cdot, \cdot), r)$ is measurable with respect to $T \setminus \operatorname{dom} E_f$. Hence $\operatorname{epi} f^c((\cdot, \cdot), r)$ is measurable on T and the proof is complete. □

Proposition 4.7. *Assume that the σ-field $\mathcal{A}$ is μ-complete. If $f : T \times \mathbb{R}^n \to \overline{\mathbb{R}}$ is a normal integrand, then so is f^{cc}.*

Proof. If f is lsc and quasiconvex we have $f^{cc} = f$ (Proposition 3.3) and there is nothing to prove. If f is any normal integrand then, $\forall\alpha \in \mathbb{R}, \operatorname{lev}_{f(t,\cdot)}(\alpha)$ is measurable, therefore by ([12], Proposition 14.12) $\overline{co} \operatorname{lev}_{f(t,\cdot)}(\alpha)$ is measurable, and by relation (10) $\operatorname{lev}_{f_{\overline{q}}(t,\cdot)}(\alpha) = \bigcap_{\beta>\alpha} \overline{co} \operatorname{lev}_{f(t,\cdot)}(\beta) = \bigcap_{\substack{\beta>\alpha \\ \beta\in\mathbb{Q}}} \overline{co} \operatorname{lev}_{f(t,\cdot)}(\beta)$ is measurable. Proposition 4.4 leads to the normality of f^{cc}. □

5. Quasiconvex-regularization as bi-q-conjugacy

We are going to prove that quasiconvex regularization procedure and bi-q-conjugacy are the same when the integrand are positive functional defined on $\Omega \times \mathbb{R}^n$, where Ω is an open bounded set of $\mathbb{R}^m$. We consider the space $L_n^p(\Omega)$, $1 \leq p < \infty$, that we provide with the weak topology $\sigma(L^p, L^q)$ where $\frac{1}{p} + \frac{1}{q} = 1$. The functional I_f is defined now from $L_n^p(\Omega)$ to $\overline{\mathbb{R}}_+$. We shall use the approximation procedure described in [5]. We recall that:

1) let's consider a family $(\alpha_k)_{1\leq k\leq\tau}$, $\tau \in \mathbb{N}$, $\alpha_k \geq 0$ $\forall k$, and $\sum_{k=1}^{\tau} \alpha_k = 1$.

2) For $i \in \mathbb{N}$, we denote $\mathcal{K}_i$ the set of the pavements of Ω whose edges are parallel to the coordinates axis and length 2^{-i}, and whose vertices have coordinates multiples of 2^{-i}.

In other words, $K \in \mathcal{K}_i$ if:

$$K = \Pi_{j=1}^m [m_j 2^{-i}, (m_j + 1) 2^{-i}] \subset \Omega, \quad \text{where } m_j \in \mathbb{Z}.$$

3) For $B_i = \bigcup_{K \in \mathcal{K}_i} K$, we have:

$$B_i \subset B_{i+1} \subset \Omega,\, \Omega = \bigcup_{i=1}^{\infty} B_i \quad \text{and} \quad \mu(\Omega) = \lim_{i \to \infty} \mu(B_i).$$

4) For i fixed, we will divide B_i into τ subsets B_i^k, $1 \leq k \leq \tau$, corresponding to the τ numbers α_k as follows: we cut out each pavement $K \in \mathcal{K}_i$ in τ sections K^k, perpendicularly to the first coordinates axis, the thickness of the Kth section being $\alpha_k \cdot 2^{-i}$.

In other words,

$$K^k = \left[\left(m_1 + \sum_{l=1}^{k-1} \alpha_l \right) 2^{-i}, \left(m_1 + \sum_{l=1}^{k} \alpha_l \right) 2^{-i} \right] \times \Pi_{j=2}^{m} [m_j 2^{-i}, (m_j + 1) 2^{-i}]$$

and

$$K = \bigcup_{k=1}^{\tau} K^k \quad \text{also} \quad \alpha_k \mu(K) = \mu\left(K^k\right).$$

5) Let us denote $B_i^k = \bigcup_{K \in \mathcal{K}_i} K^k$, for $i \in \mathbb{N}$ and $k \in \{1, ..., m\}$ fixed. Since the pavements K^k, for $K \in \mathcal{K}_i$ and $1 \leq k \leq m$, are not necessarily disjoint, we will thus call N_i the reunion of the borders of these pavements, which is thus a set of null measure, and we put $N = \cup_{i \in \mathbb{N}} N_i$, which is also of a null measure. Then, for all fixed $i \in N$, the $B_i^k \cap \complement N$, $1 \leq k \leq m$, are disjointed.

Definition 5.1. For $\tau \in \mathbb{N}$, $u = (u_1, ..., u_\tau) \in [L^p(\Omega)]^\tau$ and for each $i \in \mathbb{N}$, one defines a measurable mapping $T_i u$ from Ω to $\mathbb{R}$ by:

$$T_i u(t) = \begin{cases} u_k(t) & \text{if } t \in B_i^k \cap \complement N, \\ u_1(t) & \text{if } t \in N \cup \left(\Omega - \cup_{k=1}^{n} B_i^k\right). \end{cases}$$

Theorem 5.2 ([5], Corollary IX.1.1 and IX.1.2).

1) *The mapping T_i is linear continuous from $[L^p(\Omega)]^\tau$ to $L^p(\Omega)$, and we have:*

$$\lim_{i \to \infty} T_i u = \sum_{k=1}^{\tau} \alpha_k u_k \quad \textit{with respect } \sigma(L^p, L^q).$$

2) *Let f be a function from $\Omega \times \mathbb{R}^n$ with values in $\mathbb{R}$ such that the functions $t \mapsto f(t, u_k(t))$ belongs to $L^1(\Omega)$ for all k.*

Then,

$$\lim_{i \to \infty} \int_\Omega f(t, T_i u(t))\, dt = \sum_{k=1}^{\tau} \alpha_k \int_\Omega f(t, u_k(t))\, dt.$$

We are now in position to prove the following main result.

Theorem 5.3. *If $f : \Omega \times \mathbb{R}^n \to \overline{\mathbb{R}}$ is a positive normal integrand and if $L_n^p(\Omega)$ is provided with the weak topology $\sigma(L^p, L^q)$, then the bi-q-conjugate of I_f and its lsc regularization coincide, i.e.,*

$$I_f^{cc} = \overline{I_f}.$$

Proof. From relations (8), (9) and Proposition 3.3 we deduce that:

$$I_f^{cc} \le \overline{I_f}.$$

The opposite inequality is equivalent to prove that $\text{lev}_{I_f^{cc}}(\lambda) \subset \text{lev}_{\overline{I}_f}(\lambda)$ for every $\lambda \in \mathbb{R}$. Let be $\overline{u} \in \text{lev}_{I_f^{cc}}(\lambda)$. From Proposition 3.3 and relation (10) we have $\overline{u} \in \bigcap_{\beta>\lambda} \overline{co}\, \text{lev}_{I_f}(\beta)$. Fix any real number $\beta > \lambda$. Hence, for convex neighborhood $\mathcal{V}$ of the origin in $\sigma(L^p, L^q)$, there exist a family $(u_k)_{1\le k\le\tau}$ in L_n^p and τ positive numbers α_k with $\sum_{k=1}^{\tau} \alpha_k = 1$ such that:

$$\forall k,\ u_k \in \text{lev}_{I_f}(\beta) \tag{18}$$

and

$$\overline{u} - \sum_{k=1}^{\tau} \alpha_k u_k \in \mathcal{V}. \tag{19}$$

Since relation (18) implies $I_f(u_k) < \infty$ for all k, then, the function $f(\cdot, u_k(\cdot))$ belongs to L^1 for all k. By Theorem 5.1, with $u = (u_1, ...u_\tau) \in [L^p(\Omega)]^\tau$ we are able for each $\varepsilon > 0$ to take i_0 large enough so that $T_i u$ verifies for all $i \ge i_0$:

$$T_i u - \sum_{k=1}^{\tau} \alpha_k u_k \in \mathcal{V} \tag{20}$$

and

$$\left| I_f(T_i u) - \sum_{k=1}^{\tau} \alpha_k I_f(u_k) \right| \le \varepsilon. \tag{21}$$

Combining (19) and (20) we obtain

$$T_i u - \overline{u} \in 2\mathcal{V}$$

and by (18) and (21) we see that

$$I_f(T_i u) \le \beta + \varepsilon.$$

Hence,

$$T_i u \in \text{lev}_{I_f}(\beta) \quad \text{for all } i \ge i_0 \text{ and hence } \overline{u} \in \overline{\text{lev}}_{I_f}(\beta + \varepsilon).$$

Therefore

$$\overline{u} \in \bigcap_{\beta>\lambda, \varepsilon>0} \overline{\text{lev}}_{I_f}(\beta + \varepsilon) = \bigcap_{\beta>\lambda} \overline{\text{lev}}_{I_f}(\beta) = \text{lev}_{\overline{I}_f}(\lambda),$$

i.e., $I_f^{cc} \ge \overline{I_f}$ and the proof is complete. □

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