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The Topological Frame of a Reflexive Result*

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A topological result of Kelley is extended, so that it contains a reflexive result of Robinson in a convex-graph-free setting.

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1. Introduction

Let X and Y be topological spaces and let $F : X \to Y$ be a multifunction. Our aim is to show that, under suitable hypotheses, openness of F at a point is equivalent to near openness of F at that point.

To begin with, we recall some basic definitions. The multifunction F is said to be *open* if for every open subset U of X the subset $F(U)$ of Y is open. This is a complex property, which can be analyzed through a simplex one. The multifunction F is said to be *open at a point* $(x, y) \in \operatorname{graph}(F)$ if for every neighborhood U of x the set $F(U)$ is a neighborhood of y. Accordingly, F is open if and only if F is open at every point $(x, y) \in \operatorname{graph}(F)$. A twin simplex property can be constructed by replacing the set $F(U)$ with its closure. In the following, $\overline{S}$ stands for the closure of any subset of a topological space. The multifunction F is said to be *nearly open at a point* $(x, y) \in \operatorname{graph}(F)$ if for every neighborhood U of x the set $\overline{F(U)}$ is a neighborhood of y. A twin complex property can be synthesized through the twin simplex one. The multifunction F is said to be *nearly open* if F is nearly open at every point $(x, y) \in \operatorname{graph}(F)$.

Obviously, if F is open at a point, then F is nearly open at that point. Therefore, if F is open, then F is nearly open. In the literature, there are many results which derive openness of F from near openness of F. In fact, most of them derive openness of F at a point from near openness of F at that point as well as at sufficiently many other points (see [13, p. 145, Theorem 4] and the references therein). However, if X and Y are quite general topological vector spaces, and F has a convex graph, then openness of F at a point is derived only from near openness of F at that point (see [12, p. 439, Lemma 3], cf. [11, p. 132, Theorem 1]). In this paper, we derive further results of

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this type (see Section 3). The forms of our results require some drastic assumptions (see [5, p. 214, Note]) on the domain of F, namely local compactness if X and Y are topological spaces and X is regular or local boundedness if X and Y are locally convex topological vector spaces and X is semi-reflexive, but do not require convexity of the graph of F in the latter setting.

Our results are corollaries of some basic lemmas (see Section 2), which have a neighborhood free substratum. To describe the matter, we rephrase openness and near openness at a point (x, y) by using no matter which bases $\mathcal{U}$ and $\mathcal{V}$ for the neighborhood systems of the points x and y respectively: F is open at (x, y) if and only if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that

$$V \subseteq F(U); \tag{1}$$

F is nearly open at (x, y) if and only if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that

$$V \subseteq \overline{F(U)}. \tag{2}$$

Obviously, near openness at a point implies openness at that point if, for example, inclusion (2) implies inclusion (1) for sufficiently many pairs (U, V) of neighborhoods. Now, consider a pair (U, V) of sets which are not necessarily neighborhoods and note that inclusion (2) does imply inclusion (1) if

$$V \cap \overline{F(U)} \subseteq F(U). \tag{3}$$

In view of the basic lemmas, inclusion (3) holds if

$$(U \times V) \cap \overline{\text{graph}(F)} \subseteq \text{graph}(F) \tag{4}$$

and if some additional assumptions are satisfied. Note parenthetically that, if F has a locally closed graph, then inclusion (4) holds for sufficiently many pairs (U, V) of neighborhoods. One of the additional assumptions mentioned above states that

$$\overline{U \cap \text{domain}(F)} \subseteq U, \tag{5}$$

but the basic lemmas do not involve inclusion (5). Note finally that the topological basic Lemma 2.1 (cf. [5, p. 203, Chapter 6, Problem A]) is the frame of the reflexive basic Lemma 2.2 (cf. [11, p. 131, Lemma 1 *b*)]).

A counterexample (see Section 4) illustrates the necessity of some assumptions of our openness results. A final counterexample shows that openness of a multifunction at a point may not imply near openness at any other point, even if that multifunction has a closed graph.

2. Basic lemmas.

Let $U \subseteq X$ and $V \subseteq Y$. Further, consider the inclusion

$$V \cap \overline{F(U)} \subseteq F\left(\overline{U \cap \text{domain}(F)}\right) \tag{6}$$

and note (6) and (5) imply (3). Finally, consider the inclusion

$$\left(\overline{U \cap \text{domain}(F)} \times V\right) \cap \overline{\text{graph}(F)} \subseteq \text{graph}(F), \tag{7}$$

and note (4) and (5) imply (7).

Lemma 2.1 (Kelley). *Let* $U \cap \mathrm{domain}(F)$ *be relatively compact. Then inclusion* (7) *implies inclusion* (6).

Proof. Let the inclusion (7) hold, let $v \in V \cap \overline{F(U)}$, and let $A = \overline{U \cap \mathrm{domain}(F)}$. We have to show that $v \in F(A)$. Since $F(U) = F(U \cap \mathrm{domain}(F)) \subseteq F(A)$, it follows $v \in \overline{F(A)}$. Consider the family $\mathcal{Q}$ of neighborhoods Q of the point v and note the family of sets $\{A \cap F^{-1}(Q); Q \in \mathcal{Q}\}$ is a filter base in the compact set A, hence there exits a point $u \in A$ such that

$$u \in \bigcap_{Q \in \mathcal{Q}} \overline{A \cap F^{-1}(Q)}.$$

Now, consider the family $\mathcal{P}$ of neighborhoods P of the point u and note that for every $P \in \mathcal{P}$ as well as for every $Q \in \mathcal{Q}$ the set $P \cap F^{-1}(Q)$ is nonempty, that is, the set $(P \times Q) \cap \mathrm{graph}(F)$ is nonempty, hence $(u, v) \in \overline{\mathrm{graph}(F)}$. Since $(u, v) \in A \times V$, it follows from inclusion (7) that $(u, v) \in \mathrm{graph}(F)$, hence $v \in F(A)$. □

In view of Lemma 2.1, if F has a closed graph and U is compact, then $F(U)$ is closed (see [5, p. 203, Chapter 6, Problem A]).

The "topological space" result above is the frame of the "locally convex topological vector space" result below.

In the following, F^{-1} stands for the inverse of F, that is, $x \in F^{-1}(y)$ if and only if $y \in F(x)$. Moreover, one hypotheses below states that F^{-1} maps the convex subsets of the vector space Y to convex subsets of the vector space X. Obviously, if F has a convex graph, then both F maps the convex subsets of X to convex subsets of Y and F^{-1} maps the convex subsets of Y to convex subsets of X, but the converse implication may fail. A counterexample is provided by the multifunction $F : R \to R$ given through $F(x) = \{x^3\}$.

Recall that a locally convex topological vector space X is semi-reflexive (see [1, p. 87, Définition 3] and [2, IV, p. 16, Définition 2]; cf. [4, p. 508], [6, p. 189], [7, p. 298], and [10, p. 72]) if and only if each bounded subset of X is weakly relatively compact (see [1, p. 88, Théorème 1] and [2, IV, p. 16, Théorème 1]; cf. [4, p. 508, 8.4.2 Theorem], [6, p. 190, 20.1 Criterion for Semi-Reflexiveness], [7, p. 299, (1)], and [10, p. 72, Proposition 4]).

Lemma 2.2 (Robinson). *Let* X *and* Y *be separated, locally convex topological vector spaces, let* X *be semi-reflexive, and let* F^{-1} *map the convex subsets of* Y *to convex subsets of* X*. Let* U *be convex and let* $U \cap \mathrm{domain}(F)$ *be bounded. Then inclusion* (7) *implies inclusion* (6)

The proof of Lemma 2.2 depends on Lemma 2.1 above and on Proposition 2.3 below. Denote by X^w the vector space X endowed with the weak topology, denote by $\overline{S}^{X^w \times Y}$ the $(X^w \times Y)$-closure of any subset S of the vector space $X \times Y$, and note the obvious inclusion $\overline{S} \subseteq \overline{S}^{X^w \times Y}$ can be improved to the equality $\overline{S} = \overline{S}^{X^w \times Y}$ if S is convex. The equality still holds if $S = \mathrm{graph}(F)$ and F^{-1} maps the convex subsets of Y to convex subsets of X.

Proposition 2.3. *Let X and Y be locally convex topological vector spaces, and let F^{-1} map the convex subsets of Y to convex subsets of X. Then* $\overline{\text{graph}(F)} = \overline{\text{graph}(F)}^{X^w \times Y}$.

Proof. We have to show that, if $(p,q) \in \overline{\text{graph}(F)}^{X^w \times Y}$, then $(p,q) \in \overline{\text{graph}(F)}$. Assume, by contradiction, that $(p,q) \notin \overline{\text{graph}(F)}$. Then there exist an X-neighborhood P of p and an Y-neighborhood Q of q such that $(P \times Q) \cap \text{graph}(F) = \emptyset$, that is, $P \cap F^{-1}(Q) = \emptyset$. We can suppose, taking smaller P and Q if necessary, that P is X-open, and both P and Q are convex. Since the set Q is convex, so is the set $F^{-1}(Q)$, hence there exist a linear X-continuous function $\xi : X \to R$ and a real number ρ such that $\xi(\alpha) < \rho \leq \xi(\beta)$ whenever $\alpha \in P$ and $\beta \in F^{-1}(Q)$ (see [3, p. 642]). Denote by Π the set of all points $\alpha \in X$ such that $\xi(\alpha) < \rho$. Clearly, $(\Pi \times Q) \cap \text{graph}(F) = \emptyset$, that is, $\Pi \cap F^{-1}(Q) = \emptyset$. Since ξ is also X^w-continuous, it follows Π is an X^w-neighborhood of the point p, hence $(p,q) \notin \overline{\text{graph}(F)}^{X^w \times Y}$, a contradiction. □

Proof of Lemma 2.2. Let the inclusion (7) hold. Denote by $\overline{S}^{X^w}$ the X^w-closure of any subset S of the vector space X. Since $\text{domain}(F) = F^{-1}(Y)$, it follows $U \cap \text{domain}(F)$ is convex, hence

$$\overline{U \cap \text{domain}(F)} = \overline{U \cap \text{domain}(F)}^{X^w},$$

and the bounded, X^w-closed set $\overline{U \cap \text{domain}(F)}^{X^w}$ is X^w-compact In view of Proposition 2.3,

$$\left(\overline{U \cap \text{domain}(F)}^{X^w} \times V\right) \cap \overline{\text{graph}(F)}^{X^w \times Y} \subseteq \text{graph}(F).$$

In view of Lemma 2.1,

$$V \cap \overline{F(U)} \subseteq F\left(\overline{U \cap \text{domain}(F)}^{X^w}\right),$$

and the inclusion (6) holds. □

In view of Lemma 2.2, if F has a closed graph, if F^{-1} maps convex subsets of Y to convex subsets of X, and U is convex, bounded, and closed, then $F(U)$ is closed. Accordingly, if F has a closed graph and a bounded domain, and if F^{-1} maps convex subsets of Y to convex subsets of X, then range(F) is closed (cf. [11, p. 131, Lemma 1 *b*)], where F has a convex graph).

A counterexample shows that, even if F has a convex graph, closeness of $F(U)$ may fail if X is not reflexive. Define $F : l^1(N) \to R$ through

$$\text{graph}(F) = \left\{ \left(x, \sum_{i=1}^{\infty} \frac{i}{i+1}\, x(i) \right); x \in l^1(N) \right\},$$

let U be the closed unit ball in $l^1(N)$, and note $F(U) = (-1, +1)$.

Another counterexample shows that closeness of range(F) may fail if domain(F) is not bounded. Define $F : R \to R$ through $\text{graph}(F) = \{(x,y); x > 0, xy \geq 1\}$, and note $\text{domain}(F) = \text{range}(F) = (0, +\infty)$.

3. Locally closed graph results.

First, recall that *local relative compactness* of the set domain(F) means that for every point $x \in$ domain(F) there exists a neighborhood U of x such that the set $U \cap$ domain(F) is relatively compact. Further, recall that *local closeness* of the set graph(F) means that for every point $(x, y) \in$ graph(F) there exists a neighborhood W of (x, y) such that the set $W \cap$ graph(F) is closed. In the following $\overset{\circ}{W}$ stands for the interior of W. Since

$$\overset{\circ}{W} \cap \overline{\text{graph}(F)} \subseteq \overline{W \cap \text{graph}(F)}$$

whenever $W \subseteq X \times Y$, it follows local closeness of the graph of F implies that for every point $(x, y) \in$ graph(F) there exists a neighborhood W of (x, y) such that

$$W \cap \overline{\text{graph}(F)} \subseteq \text{graph}(F).$$

The converse implication holds too in case $X \times Y$ is a regular topological space, that is, both X and Y are regular topological spaces.

Theorem 3.1. *Let X and Y be topological spaces, let X be regular, and let the multifunction F have a locally closed graph and a locally relatively compact domain. If F is nearly open at a point, then F is also open at that point.*

Proof. Let F be nearly open at the point $(x, y) \in$ graph(F). Since F has a locally closed graph, it follows there exist a neighborhood U^* of x and a neighborhood V^* of y such that

$$(U^* \times V^*) \cap \overline{\text{graph}(F)} \subseteq \text{graph}(F).$$

Since domain(F) is locally relatively compact, we can suppose (taking a smaller U^* if necessary) that the set domain$(F) \cap U^*$ is relatively compact. Further, denote by $\mathcal{U}$ the family of all closed neighborhoods U of x such that $U \subseteq U^*$, denote by $\mathcal{V}$ the family of all neighborhoods V of y such that $V \subseteq V^*$, and note $\mathcal{U}$ and $\mathcal{V}$ are bases for the neighborhood systems of the points x (recall X is regular) and y respectively. Now, it is easy to prove openness of F at (x, y). Let $U \in \mathcal{U}$. Since F is nearly open at (x, y), it follows there exists $V \in \mathcal{V}$ such that the inclusion (2) holds. In view of Lemma 2.1, the inclusion (1) holds too, and openness of F at (x, y) follows. □

The next result concerns the locally convex topological vector spaces X and Y. Recall that *local boundedness* of the set domain(F) means that for every point $x \in$ domain(F) there exists a neighborhood U of x such that the set $U \cap$ domain(F) is bounded. Obviously, F does have a locally bounded domain if X is a normed space.

Theorem 3.2. *Let X and Y be separated, locally convex topological vector spaces, let X be semi-reflexive, let F^{-1} map the convex subsets of Y to convex subsets of X, and let F have a locally closed graph and a locally bounded domain. If F is nearly open at a point, then F is also open at that point.*

Proof. Let F be nearly open at the point $(x, y) \in$ graph(F). Since F has a locally closed graph, it follows there exist a neighborhood U^* of x and a neighborhood V^* of y such that

$$(U^* \times V^*) \cap \overline{\text{graph}(F)} \subseteq \text{graph}(F).$$

Since domain(F) is locally bounded, we can suppose (taking a smaller U^* if necessary) that the set domain(F) $\cap U^*$ is bounded. Further, denote by $\mathcal{U}$ the family of all closed, convex neighborhoods U of x such that $U \subseteq U^*$, denote by $\mathcal{V}$ the family of all convex neighborhoods V of y such that $V \subseteq V^*$, and note $\mathcal{U}$ and $\mathcal{V}$ are bases for the neighborhood systems of the points x and y respectively. Now, it is easy to prove openness of F at (x, y). Let $U \in \mathcal{U}$. Since F is nearly open at (x, y), it follows there exists $V \in \mathcal{V}$ such that the inclusion (2) holds. In view of Lemma 2.2, the inclusion (1) holds too, and openness of F at (x, y) follows. □

4. Counterexamples.

The counterexample below shows that Theorem 3.1 may fail is F does not have a locally relatively compact domain, whereas Theorem 3.2 may fail if F^{-1} does not map the convex subsets of Y to convex subsets of X.

First, recall that, if X and Y are metric spaces, then openness of F at (x, y) means that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$B(y, \delta) \subseteq (B(x, \epsilon)), \tag{8}$$

whereas near openness of F at (x, y) means that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$B(y, \delta) \subseteq \overline{F(B(x, \epsilon))}. \tag{9}$$

Here, $B(c, r)$ stands for the open ball with center c and radius r.

Further, consider the set Q of rational numbers and consider the Hilbert space $l^2(Q)$, which can be identified with the familiar space $l^2(N)$.

Further, let the multifunction $F : l^2(Q) \to R$ be defined through the equality

$$\text{graph}(F) = \{(q \cdot \kappa_q, q); q \in Q\}.$$

Here, $\kappa_q(q')$ stands for the "rational" Kronecker symbol, namely $\kappa_q(q') = 0$ if $q' \neq q$, whereas $\kappa_q(q') = 1$ if $q' = q$.

Clearly, $\kappa_q \in l^2(Q)$, $\|\kappa_q\| = 1$, and moreover, $\|q_1 \cdot \kappa_{q_1} - q_2 \cdot \kappa_{q_2}\| = \sqrt{|q_1|^2 + |q_2|^2}$ if $q_1 \neq q_2$. Accordingly, the set graph(F) is closed because all of its points are isolated points except for its point $(0, 0)$ which is an accumulation point. Moreover, F is nearly open at $(0, 0)$, but not open at $(0, 0)$ because $F(B_{l^2(Q)}(0, \epsilon)) = Q \cap B_R(0, \epsilon)$. Here, $B_M(c, r)$ stands for the open ball in the corresponding metric space M.

Finally, note $B(0, \epsilon) \cap$ domain(F) is not relatively compact for any $\epsilon > 0$, whereas F^{-1} does not map the convex subsets of R to convex subsets of $l^2(Q)$.

A related counterexample shows that openness of a multifunction at a point may not imply near openness at any other point, even if that multifunction has a closed graph.

Let the multifunction $F : l^2(R) \to R$ be defined through the equality

$$\text{graph}(F) = \{(r \cdot \kappa_r, r); r \in R\}.$$

This time $\kappa_r(r')$ stands for the "real" Kronecker symbol. Namely $\kappa_r(r') = 0$ if $r' \neq r$, whereas $\kappa_r(r') = 1$ if $r' = r$. Clearly, $\kappa_r \in l^2(R)$ and $\|\kappa_r\| = 1$. The set graph(F) is closed because all of its points are isolated points except for its point $(0, 0)$ which is an accumulation point. Moreover, F is open at $(0, 0)$ because $F(B_{l^2(R)}(0, \epsilon)) = B_R(0, \epsilon)$.

5. Relation to earlier work.

In case of Theorem 3.1, if F does not have a locally relatively compact domain, then a restrictive near openness of F implies the corresponding openness of F, namely *local uniform near openness* implies *local uniform openness*, provided that X and Y are metric spaces, X is complete, and F has a locally closed graph (see [13, p. 145, Theorem 4]).

Recall the "local uniform" terminology in [13, pp. 144, 145], which expound on the "uniform" terminology in [9, p. 505, Theorem 2.1]: F is said to be *uniformly open on a set* $W \subseteq \mathrm{graph}(F)$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $(x, y) \in W$ there holds the inclusion (8); F is said to be *locally uniformly open* if for every $(x, y) \in \mathrm{graph}(F)$ there exists a neighborhood W of (x, y) such that F is uniformly open on the set $W \cap \mathrm{graph}(F)$. Analogously: F is said to be *uniformly nearly open on a set* $W \subseteq \mathrm{graph}(F)$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $(x, y) \in W$ there holds the inclusion (9); F is said to be *locally uniformly nearly open* if for every $(x, y) \in \mathrm{graph}(F)$ there exists a neighborhood W of (x, y) such that F is uniformly nearly open on the set $W \cap \mathrm{graph}(F)$.

The basic lemma from which there follows the local uniform result above, essentially states that (see [13, p. 146, Theorem 7]) a metric version of the topological inclusion (4), namely

$$(B(x, \epsilon) \times B(y, \delta)) \cap \overline{\mathrm{graph}(F)} \subseteq \mathrm{graph}(F),$$

implies a family of metric versions of the topological inclusion (3), namely for every $\epsilon' \in (0, \epsilon)$ there holds the inclusion

$$B(y, \delta) \cap \overline{F(B(x, \epsilon'))} \subseteq F(B(x, \epsilon)),$$

provided that for every $\epsilon' \in (0, \epsilon)$ and for every $\delta' \in (0, \delta)$ the multifunction F is uniformly nearly open on the set

$$(B(x, \epsilon') \times B(y, \delta')) \cap \mathrm{graph}(F).$$

An elementary counterexample, namely

$$\mathrm{graph}(F) = \{(r, r) \in R^2; |r| < 1\},\ (x, y) = (0, 0),\ (\epsilon, \delta) = (1, 1),$$

shows that uniform near openness on each of the (ϵ', δ')-sets above does not imply uniform near openness on the set

$$(B(x, \epsilon) \times B(y, \delta)) \cap \mathrm{graph}(F),$$

whereas the ϵ'-inclusions above do not imply the inclusion

$$B(y, \delta) \cap \overline{F(B(x, \epsilon))} \subseteq F(B(x, \epsilon)).$$

The skeletal similarities and differences exhibited by these results are setting for the question whether there is some unifying result underlying all of them (see [8, p. 452]).

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