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On Local Milman's Moduli

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Relations between local Milman's moduli for Banach spaces are established.

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1. Introduction

In [12], Milman gave a scheme of defining moduli which can be used as tools for studying geometry of Banach spaces. For instance they were used to characterize uniform convexity (see [4]), uniform smoothness (see [2]) and multi-dimensional counterparts of these properties (see [8, 10]). Infinite-dimensional Milman's moduli turned out to be related to the Kadec–Klee property (see [7]), nearly uniform convexity and nearly uniform smoothness (see [14]). They were successfully applied to some problems of nonlinear analysis, including differentiation of mappings on Banach spaces (see [5, 6]) and metric fixed point problems (see [7, 15]).

In this paper we study local versions of some Milman's moduli. In particular we show that a two-dimensional Milman's modulus is in a sense equivalent to the standard modulus of smoothness. This is a quantitative version of a result obtained in [2]. We also establish an inequality between local Milman's moduli corresponding to finite-dimensional and infinite-dimensional smoothness. As a tool we use a formula obtained in [7] which describes the latter modulus in terms of weakly null nets. In this paper we give an analogous formula for the modulus corresponding to infinite-dimensional convexity. An inequality between an infinite-dimensional Milman's modulus of convexity and the standard modulus of convexity was obtained in [6]. We give an example showing that its local version is not true.

2. Preliminaries

We consider only real Banach spaces. Given such space X, by B_X and S_X we denote its closed unit ball and unit sphere, respectively. Assuming that $\dim X > k$, by $\mathcal{B}^k$

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we denote the family of all k-dimensional subspaces of X. The family of all closed subspaces of X with finite codimension is denoted by $\mathcal{B}^0$. In this case we assume that $\dim X = \infty$. The local versions of Milman's moduli are defined in the following way. Given $x \in S_X$ and $\epsilon \geq 0$, we put

$$\widetilde{\beta}_X^k(\epsilon, x) = \sup_{E \in \mathcal{B}^k} \inf_{y \in S_E} \|x + \epsilon y\| - 1, \qquad \widetilde{\delta}_X^k(\epsilon, x) = \inf_{E \in \mathcal{B}^k} \sup_{y \in S_E} \|x + \epsilon y\| - 1$$

and

$$\widetilde{\beta}_X(\epsilon, x) = \sup_{E \in \mathcal{B}^0} \inf_{y \in S_E} \|x + \epsilon y\| - 1, \qquad \widetilde{\delta}_X(\epsilon, x) = \inf_{E \in \mathcal{B}^0} \sup_{y \in S_E} \|x + \epsilon y\| - 1.$$

The reader should be aware that also different notation can be found in the literature.

Let Φ be any of these moduli seen as a function of ϵ. Then Φ is a nonnegative function satisfying Lipschitz condition with constant 1 and $\Phi(\epsilon)/\epsilon$ is nondecreasing in the interval $(0, +\infty)$ (see [12, 11]). If $1 < p < \infty$, then

$$\widetilde{\beta}_{l_p}(\epsilon, x) = \widetilde{\delta}_{l_p}(\epsilon, x) = (1 + \epsilon^p)^{\frac{1}{p}} - 1$$

for every $x \in S_{l_p}$ and $\epsilon \geq 0$.

Let us recall that the modulus $\widetilde{\delta}_X^k$ is related to k-uniform convexity (see [9]) and $\widetilde{\beta}_X^k$ is related to k-uniform smoothness, which is the dual property (see [10]). The correspondence is reversed when passing to infinite-dimensional geometric properties. Namely, $\widetilde{\beta}_X$ is related to nearly uniform convexity while $\widetilde{\delta}_X$ is related to nearly uniform smoothness (see [11]).

3. Two-dimensional moduli

Let X be a Banach space, $x \in S_X$ and $\epsilon \geq 0$. Clearly,

$$\widetilde{\beta}_X^1(\epsilon, x) = \sup_{y \in S_X} \min\{\|x + \epsilon y\|, \|x - \epsilon y\|\} - 1$$

and we put

$$\beta_X^1(\epsilon) = \sup_{z \in S_X} \widetilde{\beta}_X^1(\epsilon, z).$$

We also set

$$\widetilde{\rho}_X(\epsilon, x) = \sup_{y \in S_X} \frac{1}{2}(\|x + \epsilon y\| + \|x - \epsilon y\|) - 1.$$

Then the formula

$$\rho_X(\epsilon) = \sup_{z \in S_X} \widetilde{\rho}_X(\epsilon, z)$$

gives us the standard modulus of smoothness (see [9]). Milman [12] claimed that the moduli ρ_X and β_X^1 coincide. Examples given in [4] and [1] show that this is not true in general. However, Banaś [2] proved that a Banach space X is uniformly smooth if and only if

$$\lim_{t \to 0} \frac{\beta_X^1(t)}{t} = 0,$$

so β_X^1 can be seen as a modulus of smoothness.

We will establish quantitative relations between ρ_X and β_X^1. For this purpose we need the following lemma, which is a modification of a result due to Phelps [13].

Lemma 3.1. *Let X be a Banach space with $\dim X \geq 2$. If numbers $a, b \in \mathbb{R}$ and functionals $x^*, y^* \in X^*$ satisfy the following conditions: $1 - a \leq \|x^*\| \leq 1$, $\|y^*\| = 1$, $\|x^* + y^*\| > b$, $\|x^* - y^*\| > b$, then there exists $y \in \ker x^* \cap S_X$ such that $|y^*(y)| > \frac{b-a}{2}$.*

Proof. The conclusion obviously holds if $a > b$. Let us therefore assume that there exist numbers $a \leq b$ and functionals $x^*, y^* \in X^*$ such that $1 - a \leq \|x^*\| \leq 1$, $\|y^*\| = 1$, $\|x^* + y^*\| > b$, $\|x^* - y^*\| > b$ and $|y^*(y)| \leq \frac{1}{2}(b - a)$ for every $y \in \ker x^* \cap S_X$. By the Hahn–Banach theorem there is $z^* \in X^*$ such that $z^*(y) = y^*(y)$ for all $y \in \ker x^*$ and $\|z^*\| \leq \frac{1}{2}(b - a)$. Since $(y^* - z^*)(y) = 0$ for all $y \in \ker x^*$, $y^* - z^* = \alpha x^*$ for some $\alpha \in \mathbb{R}$. Observe that

$$|1 - |\alpha|\|x^*\|| = |\|y^*\| - \|y^* - z^*\|| \leq \|z^*\| \leq \frac{b-a}{2}.$$

If $\alpha \geq 0$, then

$$\begin{aligned}\|x^* - y^*\| &= \|(1-\alpha)x^* - z^*\| \leq |1 - \alpha|\|x^*\| + \|z^*\| \\ &\leq |1 - \alpha\|x^*\|| + |\|x^*\| - 1| + \|z^*\| \leq b.\end{aligned}$$

If in turn $\alpha \leq 0$, then

$$\begin{aligned}\|x^* + y^*\| &= \|(1+\alpha)x^* + z^*\| \leq |1 + \alpha|\|x^*\| + \|z^*\| \\ &\leq |1 + \alpha\|x^*\|| + |\|x^*\| - 1| + \|z^*\| \leq b.\end{aligned}$$

This shows that $\|x^* - y^*\| \leq b$ or $\|x^* + y^*\| \leq b$ which contradicts our assumptions. □

Corollary 3.2. *Let X be a Banach space with $2 \leq \dim X < \infty$. If numbers $a, b \in \mathbb{R}$ and functionals $x^*, y^* \in X^*$ satisfy the following conditions: $1 - a \leq \|x^*\| \leq 1$, $\|y^*\| = 1$, $\|x^* + y^*\| \geq b$, $\|x^* - y^*\| \geq b$, then there is $y \in \ker x^* \cap S_X$ such that $|y^*(y)| \geq \frac{b-a}{2}$.*

Theorem 3.3. *Let X be a Banach space. For every $x \in S_X$, $\epsilon \geq 0$ and $t \geq 0$ the following inequalities hold:*

$$\widetilde{\rho}_X(\epsilon, x) \geq \widetilde{\beta}_X^1(\epsilon, x), \tag{1}$$

$$\widetilde{\beta}_X^1(t\epsilon, x) \geq \left(2 + \frac{t}{2} + \frac{t\epsilon}{2}\right)\widetilde{\rho}_X(\epsilon, x) - \epsilon\left(2 + \frac{t\epsilon}{2}\right). \tag{2}$$

Proof. Inequality (1) is obvious. To establish estimate (2) it suffices to consider the case when X is a two-dimensional space. Let $x \in S_X$ and $t \geq 0$. Inequality (2) obviously holds if $\epsilon = 0$. Assume therefore that $\epsilon > 0$. Then there exists $z \in S_X$ such that

$$\frac{1}{2}(\|x + \epsilon z\| + \|x - \epsilon z\|) - 1 = \widetilde{\rho}_X(\epsilon, x).$$

We find $u^*, v^* \in S_{X^*}$ so that $u^*(x + \epsilon z) = \|x + \epsilon z\|$ and $v^*(x - \epsilon z) = \|x - \epsilon z\|$. Then

$$\begin{aligned}2 + 2\widetilde{\rho}_X(\epsilon, x) &= u^*(x + \epsilon z) + v^*(x - \epsilon z) \\ &= (u^* + v^*)(x) + \epsilon(u^* - v^*)(z).\end{aligned} \tag{3}$$

Since $(u^* + v^*)(x) \leq \|u^* + v^*\| \leq 2$,

$$\|u^* - v^*\| \geq (u^* - v^*)(z) \geq \frac{2\widetilde{\rho}_X(\epsilon, x)}{\epsilon}.$$

Applying the inequality $\epsilon(u^* - v^*)(z) \leq \epsilon\|u^* - v^*\| \leq 2\epsilon$, from (3) we obtain

$$\|u^* + v^*\| \geq (u^* + v^*)(x) \geq 2 + 2\widetilde{\rho}_X(\epsilon, x) - 2\epsilon.$$

But $v^*(x) \leq 1$, so

$$u^*(x) \geq 1 + 2\widetilde{\rho}_X(\epsilon, x) - 2\epsilon$$

and similarly,

$$v^*(x) \geq 1 + 2\widetilde{\rho}_X(\epsilon, x) - 2\epsilon.$$

Observe that assumptions of Corollary 3.2 are fulfilled with $x^* = \frac{1}{2}(u^* + v^*)$, $y^* = u^*$, $a = \epsilon - \widetilde{\rho}_X(\epsilon, x)$ and $b = \frac{1}{\epsilon}\widetilde{\rho}_X(\epsilon, x)$. Indeed, since

$$2 \geq \|u^* - v^*\| \geq \frac{2\widetilde{\rho}_X(\epsilon, x)}{\epsilon},$$

we get

$$1 \geq \frac{\widetilde{\rho}_X(\epsilon, x)}{\epsilon}.$$

Thus

$$\|x^* + y^*\| = \left\|2u^* + \frac{1}{2}(v^* - u^*)\right\| \geq 2\|u^*\| - \frac{1}{2}\|v^* - u^*\| \geq 1 \geq \frac{\widetilde{\rho}_X(\epsilon, x)}{\epsilon}.$$

Applying Corollary 3.2, we therefore obtain $y \in S_X$ such that $(u^* + v^*)(y) = 0$, $|u^*(y)| \geq \frac{1}{2\epsilon}\widetilde{\rho}_X(\epsilon, x) - \frac{1}{2}(\epsilon - \widetilde{\rho}_X(\epsilon, x))$. Since $u^*(y) = -v^*(y)$, either $u^*(y) \geq 0$ or $v^*(y) \geq 0$. Assume that $u^*(y) \geq 0$. Then

$$\begin{aligned}\|x + t\epsilon y\| &\geq u^*(x + t\epsilon y) = u^*(x) + t\epsilon u^*(y) \\ &\geq 1 + \left(2 + \frac{t}{2} + \frac{t\epsilon}{2}\right)\widetilde{\rho}_X(\epsilon, x) - \epsilon\left(2 + \frac{t\epsilon}{2}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\|x - t\epsilon y\| &\geq v^*(x - t\epsilon y) = v^*(x) - t\epsilon v^*(y) = v^*(x) + t\epsilon u^*(y) \\ &\geq 1 + \left(2 + \frac{t}{2} + \frac{t\epsilon}{2}\right)\widetilde{\rho}_X(\epsilon, x) - \epsilon\left(2 + \frac{t\epsilon}{2}\right).\end{aligned}$$

Hence

$$\widetilde{\beta}^1_X(t\epsilon, x) \geq \left(2 + \frac{t}{2} + \frac{t\epsilon}{2}\right)\widetilde{\rho}_X(\epsilon, x) - \epsilon\left(2 + \frac{t\epsilon}{2}\right).$$

The case when $v^*(y) \geq 0$ is similar. $\square$

Corollary 3.4. *Let X be a Banach space. Then*

$$\rho_X(\epsilon) \geq \beta_X^1(\epsilon),$$

$$\beta_X^1(t\epsilon) \geq \left(2+\frac{t}{2}+\frac{t\epsilon}{2}\right)\rho_X(\epsilon) - \epsilon\left(2+\frac{t\epsilon}{2}\right).$$

for every $\epsilon \geq 0$, $t \geq 0$.

From (2) we see that

$$\frac{\widetilde{\beta}_X^1(t\epsilon, x)}{t\epsilon} \geq \frac{1}{2}\frac{\widetilde{\rho}_X(\epsilon, x)}{\epsilon} - \frac{2}{t} - \frac{\epsilon}{2}.$$

for every $t > 0$ and $\epsilon > 0$. This and (1) give us the following corollary.

Corollary 3.5. *Let X be a Banach space. Then*

$$\lim_{\epsilon\to 0}\frac{\widetilde{\rho}_X(\epsilon, x)}{\epsilon} \geq \lim_{\epsilon\to 0}\frac{\widetilde{\beta}_X^1(\epsilon, x)}{\epsilon} \geq \frac{1}{2}\lim_{\epsilon\to 0}\frac{\widetilde{\rho}_X(\epsilon, x)}{\epsilon}.$$

for every $x \in S_X$ and

$$\lim_{\epsilon\to 0}\frac{\rho_X(\epsilon)}{\epsilon} \geq \lim_{\epsilon\to 0}\frac{\beta_X^1(\epsilon)}{\epsilon} \geq \frac{1}{2}\lim_{\epsilon\to 0}\frac{\rho_X(\epsilon)}{\epsilon}.$$

4. Multi-dimensional moduli

In [7], it was shown that

$$\widetilde{\delta}_X(\epsilon, x) = \sup\left\{\limsup_{\alpha\in A}\|x+\epsilon y_\alpha\| - 1\right\} \tag{4}$$

for every $\epsilon \geq 0$, where the supremum is taken over all weakly null nets $(y_\alpha)_{\alpha\in A}$ in B_X. Let X be a Banach space such that the set $\mathcal{N}_1$ of all weakly null nets $(x_\alpha)_{\alpha\in A}$ in S_X is nonempty. It is easy to see that the supremum in formula (4) can be taken over all nets from $\mathcal{N}_1$ as well.

In some cases this formula is more suitable than the original one. In particular, it shows that $\widetilde{\delta}_X(\epsilon, x)$ is a convex function of ϵ. Using the idea from [7], we will establish an analogous formula for the modulus $\widetilde{\beta}_X$. In the proof we will apply the following lemma.

Lemma 4.1. *Let X be a Banach space and E be a closed subspace of X with codimension k. Then there exist vectors $x_1, \ldots, x_k \in S_X$ and functionals $x_1^*, \ldots, x_k^* \in X^*$ such that $E = \bigcap_{i=1}^k \ker x_i^*$, $x_i^*(x_i) = 1$ for every $i = 1, \ldots, k$ and $x_i^*(x_j) = 0$ for $i \neq j$.*

Proof. There exist vectors $y_1, \ldots, y_k$ in X/E and functionals $f_1^*, \ldots, f_k^* \in (X/E)^*$ which satisfy the following condition

$$f_i^*(y_j) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

But the space $(X/E)^*$ is canonically isometric to $E^\perp = \{x^* \in X^* : x^*(x) = 0$ for every $x \in E\}$, so there exist functionals $y_1^*, \dots, y_k^* \in E^\perp$ such that $\|y_i^*\| = \|f_i^*\|$ and $f_i^*(y_j) = y_i^*(x)$ for every $x \in y_j$ (see [3, Theorem 2.7]). We take vectors $u_1 \in y_1, \dots, u_k \in y_k$ and put $x_i = \frac{1}{\|u_i\|} u_i$, $x_i^* = \|u_i\| y_i^*$ for $i = 1, \dots, k$. Since the subspace E is closed, we have

$$E = (E^\perp)_\perp = \left\{x \in X : x^*(x) = 0 \text{ for every } x^* \in E^\perp\right\}$$

(see [3, p. 55]). But $\dim E^\perp = k$, so $E^\perp = \operatorname{span}\{x_1^*, \dots, x_k^*\}$. This yields $(E^\perp)_\perp = \bigcap_{i=1}^k \ker x_i^*$. The remaining conditions are obvious. □

Theorem 4.2. *Let X be a Banach space, $x \in S_X$ and $\epsilon \geq 0$. Then*

$$\widetilde{\beta}_X(\epsilon, x) = \inf \left\{ \limsup_{\alpha \in A} \|x + \epsilon y_\alpha\| - 1 : (y_\alpha)_{\alpha \in A} \in \mathcal{N}_1 \right\}. \tag{5}$$

Proof. Given $\epsilon \geq 0$, we first show that

$$\widetilde{\beta}_X(\epsilon, x) \geq \inf_{(y_\alpha)_{\alpha \in A} \in \mathcal{N}_1} \limsup_{\alpha \in A} \|x + \epsilon y_\alpha\| - 1. \tag{6}$$

In $\mathcal{B}^0$ we define an order as follows: $E_1 \leq E_2$ if and only if $E_2 \subset E_1$. It is easy to see that $\mathcal{B}^0$ is a directed set. Let us fix $\gamma > 0$. For each $E \in \mathcal{B}^0$ we find $y_E \in S_E$ so that

$$\|x + \epsilon y_E\| - 1 \leq \widetilde{\beta}_X(\epsilon, x) + \gamma.$$

We have $(y_E)_{E \in \mathcal{B}^0} \in \mathcal{N}_1$. Indeed, considering any $x^* \in S_{X^*}$, we have $x^*(y_E) = 0$, provided that $E \geq \ker x^*$. In consequence the net $(y_E)_{E \in \mathcal{B}^0}$ is weakly null.

We obtain

$$\widetilde{\beta}_X(\epsilon, x) + \gamma \geq \sup_{E \in \mathcal{B}^0} \|x + \epsilon y_E\| - 1 \geq \inf_{(y_\alpha)_{\alpha \in A} \in \mathcal{N}_1} \limsup_{\alpha \in A} \|x + \epsilon y_\alpha\| - 1.$$

Since $\gamma > 0$ is arbitrary, this gives us inequality (6).

For the opposite inequality we fix $\gamma \in (0, 1)$ and choose $E \in \mathcal{B}^0$ so that

$$\|x + \epsilon y\| - 1 \geq \widetilde{\beta}_X(\epsilon, x) - \gamma$$

for each $y \in S_E$. Let $k = \operatorname{codim} E$. Applying Lemma 4.1, we find $z_1, \dots, z_k \in S_X$ and $z_1^*, \dots, z_k^* \in X^*$ satisfying the conditions $E = \bigcap_{i=1}^k \ker z_i^*$, $z_i^*(z_i) = 1$ for $i = 1, \dots, k$ and $z_i^*(z_j) = 0$ for $i, j = 1, \dots, k$, $i \neq j$. Now we consider an arbitrary net $(x_\alpha)_{\alpha \in A} \in \mathcal{N}_1$. We put

$$y_\alpha = x_\alpha - \sum_{i=1}^k z_i^*(x_\alpha) z_i.$$

There exists $\alpha_0 \in A$ such that $|z_i^*(x_\alpha)| \leq \frac{\gamma}{2k}$ $(i = 1, \dots, k)$ whenever $\alpha \geq \alpha_0$. Let us fix $\alpha \geq \alpha_0$. We obtain

$$\begin{aligned}\left\| y_\alpha - \frac{y_\alpha}{\|y_\alpha\|} \right\| &= |1 - \|y_\alpha\|| = |\|x_\alpha\| - \|y_\alpha\|| \\ &\leq \|x_\alpha - y_\alpha\| = \left\| \sum_{i=1}^k z_i^*(x_\alpha) z_i \right\| \leq \frac{\gamma}{2}.\end{aligned}$$

Note that $z_i^*(y_\alpha) = 0$ for $i = 1, \ldots, k$, so $\frac{1}{\|y_\alpha\|} y_\alpha \in S_E$, and in consequence

$$\left\| x + \epsilon \frac{y_\alpha}{\|y_\alpha\|} \right\| - 1 \geq \widetilde{\beta}_X(\epsilon, x) - \gamma$$

for each y_α. Since

$$\begin{aligned} \|x + \epsilon x_\alpha\| - 1 &\geq \left\| x + \epsilon \frac{y_\alpha}{\|y_\alpha\|} \right\| - 1 - \epsilon \left\| x_\alpha - \frac{y_\alpha}{\|y_\alpha\|} \right\| \\ &\geq \widetilde{\beta}_X(\epsilon, x) - \gamma - \epsilon \left(\|x_\alpha - y_\alpha\| + \left\| y_\alpha - \frac{y_\alpha}{\|y_\alpha\|} \right\| \right) \\ &\geq \widetilde{\beta}_X(\epsilon, x) - \gamma - \epsilon\gamma, \end{aligned}$$

we conclude that $\limsup_{\alpha \in A} \|x + \epsilon x_\alpha\| - 1 \geq \widetilde{\beta}_X(\epsilon, x) - \gamma - \epsilon\gamma$. As γ and $(x_\alpha)_{\alpha \in A}$ are arbitrary, we obtain the desired inequality. $\square$

Remark 4.3. Due to the fact that each bounded net of real numbers has a convergent subsequence it is easy to check that in (4) and (5) "lim sup" can be replaced with "lim inf". Moreover, for some spaces nets can be replaced by sequences in these formulas (see [11]).

In [6], it was shown that

$$\sup_{x \in S_X} \widetilde{\delta}_X(\epsilon, x) \leq 2\rho_X(\epsilon)$$

for every $\epsilon \in (0, 1)$. Relation between $\widetilde{\delta}_X$ and $\widetilde{\beta}_X^k$ is described by the following theorem.

Theorem 4.4. *Let X be a Banach space, $k \in \mathbb{N}$, $\epsilon \geq 0$ and $x \in S_X$. Then*

$$\widetilde{\delta}_X\left(\frac{\epsilon}{2k}, x\right) \leq \widetilde{\beta}_X^k(\epsilon, x).$$

Proof. Clearly, it suffices to consider only the case when $\widetilde{\delta}_X\left(\frac{\epsilon}{2k}, x\right) > 0$. We take

$$\gamma \in \left(0, \frac{1}{3\epsilon} \widetilde{\delta}_X\left(\frac{\epsilon}{2k}, x\right)\right).$$

In view of formula (4) and the last remark we can find a net $(y_\alpha)_{\alpha \in A} \in \mathcal{N}_1$ so that

$$\liminf_{\alpha \in A} \left\| x + \frac{\epsilon}{2k} y_\alpha \right\| - 1 > \widetilde{\delta}_X\left(\frac{\epsilon}{2k}, x\right) - \gamma\epsilon.$$

By the Hahn–Banach theorem there exist $x^* \in S_{X^*}$ such that $x^*(x) = 1$ and a net $(y_\alpha^*)_{\alpha \in A} \subset S_{X^*}$ satisfying for each $\alpha \in A$ the condition $y_\alpha^*\left(x + \frac{\epsilon}{2k} y_\alpha\right) = \left\| x + \frac{\epsilon}{2k} y_\alpha \right\|$. Applying the Alaoglu theorem, we find a subnet $(y_\alpha^*)_{\alpha \in B}$ weakly* convergent to some $y^* \in X^*$ and next a subnet $(y_\alpha)_{\alpha \in C}$ of the net $(y_\alpha)_{\alpha \in B}$ such that $|y^*(y_\alpha)| \leq \frac{\gamma}{2}$ for all $\alpha \in C$.

There exists $\eta \in C$ such that

$$\inf_{\alpha\in C,\,\alpha\geq\eta}\left\|x+\frac{\epsilon}{2k}y_\alpha\right\| > \liminf_{\alpha\in A}\left\|x+\frac{\epsilon}{2k}y_\alpha\right\| - \gamma\epsilon.$$

Consequently,

$$\inf_{\alpha\in C,\,\alpha\geq\eta}\left\|x+\frac{\epsilon}{2k}y_\alpha\right\| > \widetilde{\delta}_X\left(\frac{\epsilon}{2k},x\right)+1-2\gamma\epsilon > 1+\gamma\epsilon. \tag{7}$$

By induction, we find vectors $y_{\alpha_1},\dots,y_{\alpha_{2k}}$ and functionals $y^*_{\alpha_1},\dots,y^*_{\alpha_{2k}}$, where $\alpha_i \in C$, $\alpha_i \geq \eta$ for $i=1,\dots,2k$, satisfying conditions $y_{\alpha_i} \notin \operatorname{span}\{y_{\alpha_1},\dots,y_{\alpha_{i-1}}\}$ for $i=2,\dots,2k$ and $|(y^*_{\alpha_i}-y^*)(y_{\alpha_j})| \leq \frac{\gamma}{2}$ for $i \neq j$.

We set $\alpha_1=\eta$. Assume that we have already constructed $\alpha_1,\dots,\alpha_{i-1}$. There exists $\mu \in C$ such that $y_\alpha \notin \operatorname{span}\{y_{\alpha_1},\dots,y_{\alpha_{i-1}}\}$ for all $\alpha \geq \mu$. Otherwise, finitely dimensional space $\operatorname{span}\{y_{\alpha_1},\dots,y_{\alpha_{i-1}}\}$ would contain a subnet of $(y_\alpha)_{\alpha\in C}$, which is impossible, because $(y_\alpha)_{\alpha\in C}$ tends weakly to zero, but none of its subnets converges strongly to zero. Since $(y_\alpha)_{\alpha\in C}$ is a weakly null net and $(y^*_\alpha - y^*)_{\alpha\in C}$ is a weakly* null net, there exists $\alpha_i \in C$ such that $\alpha_i \geq \mu$, $|(y^*_{\alpha_j}-y^*)(y_{\alpha_i})| \leq \frac{\gamma}{2}$ and $|(y^*_{\alpha_i}-y^*)(y_{\alpha_j})| \leq \frac{\gamma}{2}$ for each $j=1,\dots,i-1$.

For such $y_{\alpha_1},\dots,y_{\alpha_{2k}}$ and $y^*_{\alpha_1},\dots,y^*_{\alpha_{2k}}$ we have

$$|y^*_{\alpha_i}(y_{\alpha_j})| \leq |(y^*_{\alpha_i}-y^*)(y_{\alpha_j})| + |y^*(y_{\alpha_j})| \leq \gamma, \tag{8}$$

whenever $i \neq j$. For $i \in \{1,\dots,k\}$ we put $x_i = \frac{1}{2}(y_{\alpha_{2i}} - y_{\alpha_{2i-1}})$. Vectors $x_1,\dots,x_k$ are linearly independent, so the subspace $E=\operatorname{span}\{x_1,\dots,x_k\}$ has dimension k. Observe that for arbitrary $a_1,\dots,a_k \in \mathbb{R}$ we have

$$\left\|\sum_{i=1}^k a_i x_i\right\| \leq \sum_{i=1}^k |a_i|\|x_i\| \leq k \max_{i\in\{1,\dots,k\}} |a_i|.$$

Any two norms in a finite dimensional space are equivalent, so we obtain a constant $c>0$ such that

$$\max_{i\in\{1,\dots,k\}} |a_i| \leq c\left\|\sum_{i=1}^k a_i x_i\right\|.$$

Consider the set

$$M=\left\{\sum_{i=1}^k a_i x_i : (a_i)\in\mathbb{R}^k,\ \frac{1}{k} \leq \max_{i\in\{1,\dots,k\}} |a_i| \leq c\right\}.$$

If $\left\|\sum_{i=1}^k a_i x_i\right\| = 1$, then $\frac{1}{k} \leq \max_{i\in\{1,\dots,k\}} |a_i| \leq c$, so $S_E \subset M$. We have

$$\widetilde{\beta}^k_X(\epsilon,x) \geq \inf_{y\in S_E}\|x+\epsilon y\| - 1 \geq \inf_{y\in M}\|x+\epsilon y\| - 1. \tag{9}$$

Obviously, M is a compact set and therefore in the last expression the infimum is achieved for some $y_0=\sum_{i=1}^k a_i x_i \in M$. Putting $m=\max_{i\in\{1,\dots,k\}}|a_i|$, we find $i_0 \in$

$\{1, \ldots, k\}$ such that $|a_{i_0}| = m$. If $a_{i_0} = m$, then using (8), we get

$$\begin{aligned} \left\| x + \frac{\epsilon}{km} y_0 \right\| &= \left\| x + \frac{\epsilon}{2k} \sum_{i=1}^{k} \frac{a_i}{m} (y_{\alpha_{2i}} - y_{\alpha_{2i-1}}) \right\| \\ &\geq y^*_{\alpha_{2i_0}} \left(x + \frac{\epsilon}{2k} \sum_{i=1}^{k} \frac{a_i}{m} (y_{\alpha_{2i}} - y_{\alpha_{2i-1}}) \right) \\ &= y^*_{\alpha_{2i_0}}(x) + \frac{\epsilon}{2k} \sum_{i=1}^{k} \frac{a_i}{m} (y^*_{\alpha_{2i_0}}(y_{\alpha_{2i}}) - y^*_{\alpha_{2i_0}}(y_{\alpha_{2i-1}})) \\ &\geq y^*_{\alpha_{2i_0}}(x) + \frac{\epsilon}{2k} y^*_{\alpha_{2i_0}}(y_{\alpha_{2i_0}}) - \gamma\epsilon \\ &= \|x + \frac{\epsilon}{2k} y_{\alpha_{2i_0}}\| - \gamma\epsilon > 1. \end{aligned} \tag{10}$$

If $\alpha_{i_0} = -m$, then in the above calculation we substitute $y^*_{\alpha_{2i_0-1}}$ for $y^*_{\alpha_{2i_0}}$. In this case we obtain the inequality

$$\left\| x + \frac{\epsilon}{km} y_0 \right\| \geq \left\| x + \frac{\epsilon}{2k} y_{\alpha_{2i_0-1}} \right\| - \gamma\epsilon > 1. \tag{11}$$

Since $\left\| x + \frac{\epsilon}{km} y_0 \right\| > 1$ and $km \geq 1$,

$$\left\| x + \frac{\epsilon}{km} y_0 \right\| \leq \frac{1}{km} \|x + \epsilon y_0\| + \left(1 - \frac{1}{km} \right) \|x\| \leq \|x + \epsilon y_0\|$$

Thus

$$\inf_{y \in M} \|x + \epsilon y\| \geq \left\| x + \frac{\epsilon}{km} y_0 \right\|. \tag{12}$$

From (9), (10), (11) and (12) we therefore obtain

$$\begin{aligned} \widetilde{\beta}^k_X(\epsilon, x) &\geq \min \left\{ \left\| x + \frac{\epsilon}{2k} y_{\alpha_{2i_0}} \right\|, \left\| x + \frac{\epsilon}{2k} y_{\alpha_{2i_0-1}} \right\| \right\} - 1 - \gamma\epsilon \\ &\geq \inf_{\alpha \in C,\, \alpha \geq \eta} \left\| x + \frac{\epsilon}{2k} y_\alpha \right\| - 1 - \gamma\epsilon. \end{aligned}$$

In view of (7) this shows that

$$\widetilde{\beta}^k_X(\epsilon, x) \geq \widetilde{\delta}_X \left(\frac{\epsilon}{2k}, x \right) - 3\gamma\epsilon.$$

Passing to the limit with $\gamma \to 0$, we obtain the desired inequality. □

Let X be a Banach space. By δ_X we denote the standard modulus of convexity of X, i.e.,

$$\delta_X(\epsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in S_X,\ \|x - y\| \geq \epsilon \right\}$$

where $\epsilon \in [0, 2]$. In [6], it was shown that

$$\delta_X(\epsilon) \leq \beta_X(\epsilon) = \inf_{x \in S_X} \widetilde{\beta}_X(\epsilon, x)$$

for every $\epsilon \in (0,1)$. Figiel [4] obtained the following formula

$$\widetilde{\delta}_X^1\left(\frac{\epsilon}{2(1-\delta_X(\epsilon))}\right) = \frac{\delta_X(\epsilon)}{1-\delta_X(\epsilon)}$$

for all $\epsilon \in [0,2)$. It follows that

$$\delta_X(\epsilon) = (1-\delta_X(\epsilon))\,\widetilde{\delta}_X^1\left(\frac{\epsilon}{2(1-\delta_X(\epsilon))}\right) \geq \widetilde{\delta}_X^1\left(\frac{\epsilon}{2}\right),$$

which gives us the estimate

$$\beta_X(\epsilon) \geq \widetilde{\delta}_X^1\left(\frac{\epsilon}{2}\right)$$

for every $\epsilon \in (0,1)$. In contrast to Theorem 4.4, the local version of this estimate is not true, as the following example shows.

Example 4.5. Let $X = c_0$ be endowed with the norm

$$\|(x_n)\|_1 = \max\left\{\sup_{n\in\mathbb{N}} |x_1 - x_n|, \sup_{n\in\mathbb{N}} |x_n|\right\}.$$

The norm $\|\cdot\|_1$ is equivalent to the standard norm of c_0. We claim that there exists $x \in X$ such that $\widetilde{\beta}_X(1,x) = 0$ and $\widetilde{\delta}_X^1(\epsilon,x) \geq \frac{\epsilon}{2}$ for all $\epsilon > 0$. In this case the following formula holds

$$\widetilde{\beta}_X(\epsilon,x) = \inf\left\{\limsup_{n\to\infty} \|x + \epsilon y_n\| - 1\right\}$$

where the infimum is taken over all weakly null sequences (y_n) in S_X (see [11]).

Let (e_n) be the standard basis of c_0. The sequence (e_n) is weakly null, $\|e_n\|_1 = 1$ for every $n \in \mathbb{N}$ and $\|e_1 + e_n\|_1 = 1$ for $n > 1$. Therefore

$$\widetilde{\beta}_X(1,e_1) \leq \limsup_{n\to\infty} \|e_1 + e_n\|_1 - 1 = 0,$$

which shows that $\widetilde{\beta}_X(1,e_1) = 0$.

Let $\epsilon > 0$ and $z = (\zeta_n) \in S_X$. In the case when $|\zeta_1| \geq \frac{1}{2}$ we have

$$\max\{\|e_1 + \epsilon z\|_1, \|e_1 - \epsilon z\|_1\} \geq \max\{|1+\epsilon\zeta_1|, |1-\epsilon\zeta_1|\} \geq 1 + \frac{\epsilon}{2}.$$

Otherwise $|\zeta_1 - \zeta_k| \geq \frac{1}{2}$ for some $k > 1$ and

$$\max\{\|e_1 + \epsilon z\|_1, \|e_1 - \epsilon z\|_1\} \geq \max\{|1+\epsilon\zeta_1 - \epsilon\zeta_k|, |1-\epsilon\zeta_1 + \epsilon\zeta_k|\} \geq 1 + \frac{\epsilon}{2}.$$

Thus

$$\max\{\|e_1 + \epsilon z\|_1, \|e_1 - \epsilon z\|_1\} - 1 \geq \frac{\epsilon}{2}.$$

This shows that

$$\widetilde{\delta}_X^1(\epsilon,e_1) = \inf_{y\in S_X} \max\{\|e_1 + \epsilon y\|_1, \|e_1 - \epsilon y\|_1\} - 1 \geq \frac{\epsilon}{2}.$$

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