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Functional Inequalities Derived from the Brunn–Minkowski Inequalities for Quermassintegrals*

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We use Brunn–Minkowski inequalities for quermassintegrals to deduce a family of inequalities of Poincaré type on the unit sphere and on the boundary of smooth convex bodies in the n–dimensional Euclidean space.

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1. Introduction

The main idea of this paper is to use Brunn–Minkowski inequalities for quermassintegrals to derive a family of inequalities of Poincaré type on the unit sphere and on the boundary of convex bodies in the n–dimensional Euclidean space. This type of research was initiated in [4] where the case of the classical Brunn–Minkowski inequality is considered.

Let $K \subset \mathbb{R}^n$ be a convex body, i.e. a (non–empty) compact convex set. The quermassintegrals of K, denoted by $W_0(K), W_1(K), \dots, W_n(K)$, arise naturally in the polynomial expression of the volume of the outer parallel bodies of K given by the well known *Steiner formula*:

$$\mathcal{H}^n(K + tB^n) = \sum_{i=0}^{n} t^i \binom{n}{i} W_i(K), \quad t \geq 0.$$

where B^n is the unit ball of $\mathbb{R}^n$, $K + tB^n = \{x + ty : x \in K, y \in B^n\}$ is the outer parallel body of K at distance $t \geq 0$ and $\mathcal{H}^n$ is the volume, i.e., n–dimensional Lebesgue measure. For a detailed study of quermassintegrals we refer to [11, §4.2]. Some of the quermassintegrals have familiar geometric meaning: $W_0(K)$ is the volume of K, while

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$W_1(K)$ is, up to a dimensional factor, the surface area of K. Each quermassintegral W_i, $i < n$, satisfies a Brunn–Minkowski type inequality: for every K and L convex bodies and for every $t \in [0,1]$ we have

$$W_k((1-t)K + tL)^{1/(n-k)} \geq (1-t)W_k(K)^{1/(n-k)} + tW_k(L)^{1/(n-k)}. \tag{1}$$

If $t \in (0,1)$ then equality holds for $0 \leq k < n-1$ if and only if either K and L are homothetic or they lie in parallel $(n-i-1)$-planes. When $k = 0$ this is the classical Brunn–Minkowski inequality. In general, the above inequalities can be obtained as consequences of the Aleksandrov–Fenchel inequalities (see for instance [11, §6.4]). Inequality (1) claims that the functional $W_k^{1/(n-k)}$ is concave in the class of convex bodies; heuristically, this implies that the *second variation* of this functional, whenever it exists, must be negative semi–definite. In this paper we try to make this argument more precise and we study its consequences.

Throughout the paper we use the notion of elementary symmetric functions of (the eigenvalues of) symmetric matrices. In our notation, if A is a $N \times N$ real symmetric matrix, for $r \in \{0, 1, \dots, N\}$, $S_r(A)$ is the r–th elementary symmetric function of the eigenvalues of A and $(S_r^{ij}(A))$ is the *r–cofactor matrix* of A; these notions and their properties are recalled in §2.

As we will be working on the unit sphere and on the boundary of smooth convex bodies, the derivatives of a (smooth) function f defined on such sets will be always covariant derivatives. If an orthonormal frame has been fixed, the covariant derivatives of f with respect to the coordinates will be denoted by f_i, f_{ij}, and so on, and ∇f will denote the vector $(f_1, \dots, f_{n-1})$.

If $K \subset \mathbb{R}^n$ is a convex body of class C^2_+ (see §2 for the definition) then, for $k < n$,

$$W_k(K) = c(n,k) \int_{\mathbb{S}^{n-1}} h_K \, S_{n-k-1}((h_K)_{ij} + h_K \delta_{ij}) \, d\mathcal{H}^{n-1}, \tag{2}$$

where $c(n,k)$ is a constant and $(h_K)_{ij}$ are the second covariant derivatives of the support function h_K of K (see formula (5.3.11) in [11] for the value of $c(n,k)$ and §2 for precise definitions). This integral representation formula allows to compute explicitly the first and second directional derivatives of quermassintegrals. Then, imposing the Brunn–Minkowski inequality (1) we obtain the following results.

Theorem 1.1. *Let $K \subset \mathbb{R}^n$ be a convex body of class C^2_+, ν be its Gauss map and $k \in \{1, \dots, n-1\}$. For every $\psi \in C^1(\partial K)$, if*

$$\int_{\partial K} \psi S_{k-1}(D\nu) d\mathcal{H}^{n-1} = 0 \tag{3}$$

then

$$k \int_{\partial K} \psi^2 S_k(D\nu) d\mathcal{H}^{n-1} \leq \int_{\partial K} \left\langle (S_k^{ij}(D\nu))\nabla\psi, \left(S_{n-1}^{ij}(D\nu)\right) \nabla\psi \right\rangle \frac{1}{\kappa} \, d\mathcal{H}^{n-1} \tag{4}$$

where κ denotes the Gauss curvature.

Theorem 1.2. *Let h be the support function of a convex body $K \subset \mathbb{R}^n$ of class C^2_+ and $l \in \{1, \dots, n-1\}$. For every $\phi \in C^1(\mathbb{S}^{n-1})$, if*

$$\int_{\mathbb{S}^{n-1}} \phi S_l(h_{ij} + h\delta_{ij}) d\mathcal{H}^{n-1} = 0 \tag{5}$$

then

$$(n-l) \int_{\mathbb{S}^{n-1}} \phi^2 S_{l-1}(h_{ij} + h\delta_{ij}) d\mathcal{H}^{n-1} \le \int_{\mathbb{S}^{n-1}} \left\langle (S_l^{ij}(h_{ij} + h\delta_{ij})) \nabla\phi, \nabla\phi \right\rangle d\mathcal{H}^{n-1}. \tag{6}$$

Theorems 1.1 and 1.2 are the two faces of the same coin; they can be obtained one from each other by the change of variable provided by the Gauss map. The cases $k = 1$ of Theorem 1.1 and $l = n-1$ of Theorem 1.2 were already proved in [4], as consequences of the classic Brunn–Minkowski inequality. Another proof of Theorems 1.1 and 1.2 in these special cases, based on a functional inequality due to Brascamp and Lieb (see [2]), was communicated to us by Cordero–Erausquin ([5]).

One way to look at (3)–(4) and (5)–(6) is as inequalities of Poincaré type, where a weighted L^2–norm of a function is bounded by a weighted L^2–norm of its gradient, under a zero–mean type condition. In particular, choosing $K = B^n$ in Theorem 1.1, or equivalently $h \equiv 1$ in Theorem 1.2, we recover the usual Poincaré inequality on $\mathbb{S}^{n-1}$ with the optimal constant:

$$\int_{\mathbb{S}^{n-1}} \phi \, d\mathcal{H}^{n-1} = 0 \;\Rightarrow\; \int_{\mathbb{S}^{n-1}} \phi^2 d\mathcal{H}^{n-1} \le \frac{1}{n-1} \int_{\mathbb{S}^{n-1}} |\nabla\phi|^2 d\mathcal{H}^{n-1}. \tag{7}$$

We also note that inequalities (4) and (6), under side conditions (3) and (5) respectively, are optimal. This fact, proved in Remark 4.5, §4, is a simple consequence of the invariance of quermassintegrals under translations.

When $l = 1$ we can remove the smoothness assumption on K in Theorem 1.2. Indeed we have $S_{l-1} = S_0 \equiv 1$ and $S_1^{ij}(h_{ij} + h\delta_{ij}) = \delta_{ij}$. Moreover $S_1(h_{ij} + h\delta_{ij}) d\mathcal{H}^{n-1} = [\Delta\, h + (n-1)h] d\mathcal{H}^{n-1}$ can be replaced by $dA_1(K, \cdot)$, where $A_1(K, \cdot)$ denotes the *area measure of order one* of K (see §5 for the definition).

Theorem 1.3. *Let $K \subset \mathbb{R}^n$ be a convex body with interior points and let $A_1(K, \cdot)$ be its area measure of order one. For every $\phi \in C^1(\mathbb{S}^{n-1})$, if*

$$\int_{\mathbb{S}^{n-1}} \phi(x) \, dA_1(K, x) = 0\,, \tag{8}$$

then

$$\int_{\mathbb{S}^{n-1}} \phi^2(x) d\mathcal{H}^{n-1}(x) \le \frac{1}{n-1} \int_{\mathbb{S}^{n-1}} |\nabla\phi(x)|^2 d\mathcal{H}^{n-1}(x)\,.$$

Hence Theorem 1.3 extends the usual Poincaré inequality (7) on $\mathbb{S}^{n-1}$ when the zero–mean condition is replaced by (8). For $n = 2$ this leads to an extension of the well known *Wirtinger inequality*, stated in Corollary 5.1 of §5. In higher dimension Theorem 1.3 together with some recent developments on the Christoffel problem ([7], [10]) leads to the following result.

Theorem 1.4. *Let $K \subset \mathbb{R}^n$ be a convex body containing the origin in its interior, such that*

$$\int_{\mathbb{S}^{n-1}} x\rho_K(x)\, d\mathcal{H}^{n-1}(x) = 0\,, \tag{9}$$

where ρ_K is the radial function of K. Then, for every $\phi \in C^1(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} \phi(x)\rho_K(x)\, d\mathcal{H}^{n-1}(x) = 0$$
$$\Rightarrow \int_{\mathbb{S}^{n-1}} \phi^2(x) d\mathcal{H}^{n-1}(x) \le \frac{1}{n-1} \int_{\mathbb{S}^{n-1}} |\nabla\phi(x)|^2 d\mathcal{H}^{n-1}(x)\,.$$

Note that condition (9) is fulfilled when K is centrally symmetric. Moreover, for every K there exists a point $\bar{x}$ such that if the origin is placed in $\bar{x}$ then (9) holds; see Remark 5.3 for the proof.

2. Preliminaries

2.1. Elementary symmetric functions

Let N be an integer; for a $N \times N$ symmetric matrix $A = (a_{ij})$ having eigenvalues $\lambda_1, \dots, \lambda_N$, and for $k \in \{0, 1, \dots, N\}$ we define the k–th elementary symmetric function of the eigenvalues of A as follows

$$S_k(A) = \sum_{1 \le i_1 < \cdots < i_k \le N} \lambda_{i_1} \cdots \lambda_{i_k}\,, \quad \text{if } k \ge 1\,, \tag{10}$$

and $S_0(A) = 1$. In particular $S_1(A)$ and $S_N(A)$ are the trace and the determinant of A, respectively. If A and k are as above and $i, j \in \{1, \dots, N\}$, we set

$$S_k^{ij}(A) = \frac{\partial S_k(A)}{\partial a_{ij}}\,.$$

The matrix $(S_k^{ij}(A))$ is also symmetric. The usual cofactor matrix happens when $k = N$ in $(S_k^{ij}(A))$, so $(S_k^{ij}(A))$ can be considered as a *k–th cofactor matrix of A*. Note that $(S_1^{ij}(A))$ is the identity matrix. In the sequel we will use some properties of elementary symmetric functions of matrices that, for convenience, we gather in the following statement; for the proof we refer the reader to [8] and [9, Chapter 1].

Proposition 2.1. *In the notation introduced above the following facts hold:*

i) *For every k, if A is diagonal then $(S_k^{ij}(A))$ is diagonal;*

ii) *the eigenvalues of $(S_k^{ij}(A))$ are given by*

$$\Lambda_s = S_{k-1}(\mathrm{diag}(\lambda_1, \dots, \lambda_{s-1}, \lambda_{s+1}, \dots, \lambda_N))\,, \quad s = 1, \dots, N-1\,,$$

where $\lambda_1, \dots, \lambda_N$ are the eigenvalues of A;

iii) *if A is non–singular then*

$$\frac{1}{\det(A)} S_k(A) = S_{N-k}(A^{-1})\,;$$

iv)

$$S_k(A) = \frac{1}{k} \sum_{i,j=1}^{N} S_k^{ij}(A) a_{ij}\,; \tag{11}$$

v)

$$\mathrm{trace}(S_k^{ij}(A)) = (N-(k-1)) S_{k-1}(A)\,. \tag{12}$$

2.2. Convex bodies and quermassintegrals

We denote by $\mathcal{K}^n$ the set of convex bodies in $\mathbb{R}^n$. In this paper we will use several results concerning convex bodies, for the proof of these results we refer the reader to [11]. To every $K \in \mathcal{K}^n$ we can associate its *support function* h_K

$$h_K\,:\,\mathbb{S}^{n-1} \to \mathbb{R}\,, \qquad h_K(u) = \sup\big\{\langle x, u\rangle : x \in K\big\},$$

(see e.g. [11, §1.7]). Note that in the present paper the support function is defined on the unit sphere $\mathbb{S}^{n-1}$ and we do not consider its homogeneous extension to the whole space $\mathbb{R}^n$. K is said to be of class C^2_+ if $\partial K \in C^2$ and the Gauss curvature is strictly positive at each point of ∂K. If K is of class C^2_+ we denote by ν_K its Gauss map: for every $x \in \partial K$, $\nu_K(x)$ is the outer unit normal vector to K at x. When the body K is clear from the context, we just write h and ν instead of h_K and ν_K respectively. If K is of class C^2_+, then ν_K establishes a diffeomorphism between ∂K and $\mathbb{S}^{n-1}$ and its differential $D\nu_K$ is the *Weingarten map* of ∂K. The matrix associated with the linear map $D(\nu_K^{-1})$ is $(h_{ij} + h\delta_{ij})$ where for $i, j = 1, \dots, n-1$, h_i and h_{ij} denote respectively the first and second covariant derivatives of h with respect to an orthonormal frame on $\mathbb{S}^{n-1}$ and δ_{ij} is the standard Kronecker symbol.

In other words $(h_{ij} + h\delta_{ij})$ is the matrix of the reverse second fundamental form of ∂K.

For brevity, in the sequel we will adopt the notation:

$$(h_{ij} + h\delta_{ij}) = \Xi^{-1}\,.$$

In particular, if K is of class C^2_+ then Ξ^{-1} is positive definite on $\mathbb{S}^{n-1}$ and its eigenvalues are the principal radii of curvature of K. Conversely, if $h \in C^2(\mathbb{S}^{n-1})$ and the matrix $(h_{ij} + h\delta_{ij})$ is positive definite at each point of $\mathbb{S}^{n-1}$, then h is the support function of a (uniquely determined) convex body K of class C^2_+. Hence the set

$$\mathcal{C} = \{h \in C^2(\mathbb{S}^{n-1}) : (h_{ij} + h\delta_{ij}) > 0 \text{ on } \mathbb{S}^{n-1}\}$$

consists of support functions of convex bodies of class C^2_+.

When K is of class C^2_+, the quermassintegrals of K can be expressed as integrals involving the support function h of K. In fact, for $k \in \{0, 1, \dots, n-1\}$,

$$W_k(K) = \frac{1}{n} \binom{n-1}{n-k-1}^{-1} \int_{\mathbb{S}^{n-1}} h\, S_{n-k-1}(\Xi^{-1})\, d\mathcal{H}^{n-1} \tag{13}$$

(see formula (5.3.11) in [11]). Note that for $K, L \in \mathcal{K}^n$ and $t \in [0,1]$ we have

$$h_{(1-t)K+tL} = (1-t)h_K + t h_L\,.$$

From the above facts and inequality (1) we deduce the following result.

Proposition 2.2. *For $i \in \{0,1,\dots,n-1\}$ define the functional*

$$F_i : \mathcal{C} \to \mathbb{R}_+ , \qquad F_i(h) = \int_{\mathbb{S}^{n-1}} h\, S_{n-i-1}(\Xi^{-1})\, d\mathcal{H}^{n-1} .$$

Then $(F_i)^{1/(n-i)}$ is concave in $\mathcal{C}$.

3. A lemma concerning Hessian operators on the sphere

This section is devoted to prove the following result, which will be used in the proofs of Theorems 1.1 and 1.2.

Lemma 3.1. *Let $u \in C^2(\mathbb{S}^{n-1})$, $k \in \{1,\dots,n-1\}$ and let $\{E_1,\dots,E_{n-1}\}$ be a local orthonormal frame of vector fields on $\mathbb{S}^{n-1}$. Then, for every $i \in \{1,\dots,n-1\}$,*

$$\mathrm{div}_j(S_k^{ij}(\nabla^2 u + uI)) := \sum_{j=1}^{n-1} \frac{\partial}{\partial E_j} S_k^{ij}(\nabla^2 u + uI) = 0 ,$$

where $\frac{\partial}{\partial E_j}$ denotes the covariant differential acting on E_j and I denotes the $(n-1)\times(n-1)$ identity matrix.

The case $k = n-1$ of the preceding lemma was proved by Cheng and Yau in [3, p. 504]. We also note that an analogous result is valid in the Euclidean setting, with $(\nabla^2 u + uI)$ replaced by $\nabla^2 u$ (see for instance [8, Proposition 2.1] and [9, §2.3]). Our proof follows the argument of [9] for the Euclidean case and uses some standard tools from differential geometry on $\mathbb{S}^{n-1}$.

Proof. For $k \in \{0,1,\dots,N\}$, the k–th elementary symmetric function of a symmetric $N\times N$ matrix $A = (a_{ij})$ can be written in the following way (see, for instance, [8])

$$S_k(A) = \frac{1}{k} \sum \delta\begin{pmatrix} i_1,\dots,i_k \\ j_1,\dots,j_k \end{pmatrix} a_{i_1 j_1} \cdots a_{i_k j_k}$$

where the sum is taken over all possible indices $i_r, j_r \in \{1,\dots,N\}$ for $r = 1,\dots,k$ and the Kronecker symbol $\delta\binom{i_1,\dots,i_k}{j_1,\dots,j_k}$ equals 1 (respectively, -1) when $i_1,\dots,i_k$ are distinct and $(j_1,\dots,j_k)$ is an even (respectively, odd) permutation of $(i_1,\dots,i_k)$; otherwise it is 0. Using the above equality we have

$$S_k^{ij}(A) = \frac{1}{(k-1)!} \sum \delta\begin{pmatrix} i, i_1,\dots,i_{k-1} \\ j, j_1,\dots,j_{k-1} \end{pmatrix} a_{i_1 j_1} \cdots a_{i_{k-1} j_{k-1}} .$$

Hence we can write

$$
\begin{aligned}
&(k-1)!\sum_{j=1}^{n-1}\frac{\partial}{\partial E_j}S_k^{ij}(\nabla^2 u+uI) \qquad (14)\\
&=\sum_{j=1}^{n-1}\sum\delta\binom{i,i_1,\dots,i_{k-1}}{j,j_1,\dots,j_{k-1}}\frac{\partial}{\partial E_j}((u_{i_1j_1}+u\delta_{i_1j_1})\cdots(u_{i_{k-1}j_{k-1}}+u\delta_{i_{k-1}j_{k-1}}))\\
&=\sum_{j=1}^{n-1}\sum\delta\binom{i,i_1,\dots,i_{k-1}}{j,j_1,\dots,j_{k-1}}[(u_{i_1j_1j}+u_j\delta_{i_1j_1})(u_{i_2j_2}+u\delta_{i_2j_2})\cdots(u_{j_{k-1}i_{k-1}}+u\delta_{i_{k-1}j_{k-1}})\\
&\qquad+\cdots+(u_{i_1j_1}+u\delta_{i_1j_1})(u_{i_2j_2}+u\delta_{i_2j_2})\cdots(u_{i_{k-1}j_{k-1}j}+u_j\delta_{i_{k-1}j_{k-1}})].
\end{aligned}
$$

In the last sum, for fixed $i_1,\dots i_{k-1},j_1,\dots j_{k-1},j$, let us consider the terms

$$A=\delta_1(u_{i_1j_1j}+u_j\delta_{i_1j_1})C \quad\text{and}\quad B=\delta_2(u_{i_1jj_1}+u_{j_1}\delta_{i_1j})C\,,$$

where

$$\delta_1=\delta\begin{pmatrix}i,i_1,i_2,\dots,i_{k-1}\\ j,j_1,j_2,\dots,j_{k-1}\end{pmatrix},\qquad \delta_2=\delta\begin{pmatrix}i,i_1,i_2,\dots,i_{k-1}\\ j_1,j,j_2,\dots,j_{k-1}\end{pmatrix},$$

and

$$C=(u_{i_2j_2}+u\delta_{i_2j_2})\cdots(u_{j_{k-1}i_{k-1}}+u\delta_{i_{k-1}j_{k-1}})\,.$$

Clearly $\delta_1=-\delta_2$. Moreover we have the following relation concerning covariant derivatives on $\mathbb{S}^{n-1}$ (see, for instance, [3])

$$u_{rst}+u_t\delta_{rs}=u_{rts}+u_s\delta_{rt}\,,\quad \forall\, r,s,t=1,\cdots,n-1\,.$$

Hence $A+B=0$. We have proved that to the term A in the last sum in (14) it corresponds another term B, uniquely determined, which cancels out with A. The same argument can be repeated for any other term of the sum and this concludes the proof. $\square$

4. Proof of Theorems 1.1 and 1.2

In this section K is a fixed convex body of class C^2_+ and h is its support function; in particular $h\in\mathcal{C}$. We recall that $\Xi^{-1}=(h_{ij}+h\delta_{ij})$ and, for $k\in\{0,\dots,n-1\}$,

$$F_k(h)=\int_{\mathbb{S}^{n-1}}h\,S_{n-k-1}(\Xi^{-1})\,d\mathcal{H}^{n-1}\,.$$

Note that if $\phi\in C^\infty(\mathbb{S}^{n-1})$ and $\epsilon>0$ is sufficiently small, then $h+s\phi\in\mathcal{C}$ for $|s|\le\epsilon$. We will denote by Ξ_s^{-1} the matrix $((h_s)_{ij}+h_s\delta_{ij})$.

Proposition 4.1. *Let $k\in\{0,\dots,n-1\}$, $h\in\mathcal{C}$, $\phi\in C^\infty(\mathbb{S}^{n-1})$ and $\epsilon>0$ be such that $h_s=h+s\phi\in\mathcal{C}$ for every $s\in(-\epsilon,\epsilon)$. Let $f(s)=F_k(h_s)$. Then*

$$f'(s)=(n-k)\int_{\mathbb{S}^{n-1}}\phi\,S_{n-k-1}(\Xi_s^{-1})d\mathcal{H}^{n-1}\,,\quad s\in(-\epsilon,\epsilon)\,.$$

Proof.

$$\begin{aligned} f'(s) &= \int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial s}\left[h_s\, S_{n-k-1}(\Xi_s^{-1})\right] d\mathcal{H}^{n-1} \\ &= \int_{\mathbb{S}^{n-1}} \left[\phi\, S_{n-k-1}(\Xi_s^{-1}) + h_s \frac{\partial}{\partial s}(S_{n-k-1}(\Xi_s^{-1}))\right] d\mathcal{H}^{n-1} \qquad (15) \\ &= \int_{\mathbb{S}^{n-1}} \left[\phi\, S_{n-k-1}(\Xi_s^{-1}) + h_s \sum_{i,j=1}^{n-1} S_{n-k-1}^{ij}(\Xi_s^{-1})(\phi_{ij}+\phi\delta_{ij})\right] d\mathcal{H}^{n-1}. \end{aligned}$$

Integrating by parts twice and using Lemma 3.1 we obtain

$$\int_{\mathbb{S}^{n-1}} h_s \sum_{i,j=1}^{n-1} S_{n-k-1}^{ij}(\Xi_s^{-1})\phi_{ij} d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} \phi \sum_{i,j=1}^{n-1} S_{n-k-1}^{ij}(\Xi_s^{-1})(h_s)_{ij} d\mathcal{H}^{n-1}. \qquad (16)$$

On the other hand, by (11)

$$\sum_{i,j=1}^{n-1} S_{n-k-1}^{ij}(\Xi_s^{-1})((h_s)_{ij} + h_s\delta_{ij}) = (n-k-1)\, S_{n-k-1}(\Xi_s^{-1}). \qquad (17)$$

The proof is completed inserting (16) and (17) in (15). □

The proof of the next result is a straightforward consequence of Proposition 4.1.

Proposition 4.2. *In the assumptions and notations of Proposition* 4.1

$$f''(s) = (n-k)\int_{\mathbb{S}^{n-1}} \phi \sum_{i,j=1}^{n-1} S_{n-k-1}^{ij}(\Xi_s^{-1})(\phi_{ij}+\phi\delta_{ij}) d\mathcal{H}^{n-1}. \qquad (18)$$

We are now ready to prove Theorems 1.1 and 1.2; we begin with the latter.

Proof of Theorem 1.2. Without loss of generality we may assume that $\phi \in C^\infty(\mathbb{S}^{n-1})$. Fix $\epsilon > 0$ such that $h + s\phi \in \mathcal{C}$ for $s \in (-\epsilon, \epsilon)$ and let $k = n - l - 1$. As above, we set $f(s) = F_k(h + s\phi)$ and define $g(s) = f^{\frac{1}{n-k}}(s)$. We know from Proposition 2.2 that g is a concave function and so

$$g''(0) = \frac{1}{n-k}\left[\left(\frac{1}{n-k} - 1\right) f(0)^{\frac{1}{n-k}-2}(f'(0))^2 + (f(0))^{\frac{1}{n-k}-1} f''(0)\right] \le 0.$$

Notice that, by Proposition 2.1, the assumption (5) gives exactly $f'(0) = 0$, so the condition $g''(0) \le 0$ becomes $(f(0))^{\frac{1}{n-k}-1} f''(0) \le 0$. Since $f(0) = n\binom{n-1}{n-k-1} W_k(K) > 0$ it follows $f''(0) \le 0$. Now (18) gives us

$$\int_{\mathbb{S}^{n-1}} \phi^2 \sum_{i,j=1}^{n-1} S_l^{ij}(\Xi^{-1})\delta_{ij} d\mathcal{H}^{n-1} \le -\int_{\mathbb{S}^{n-1}} \phi \sum_{i,j=1}^{n-1} S_l^{ij}(\Xi^{-1})\phi_{ij} d\mathcal{H}^{n-1}.$$

Integrating by parts in the right hand–side and using Lemma 3.1 we obtain

$$\int_{\mathbb{S}^{n-1}} \phi \sum_{i,j=1}^{n-1} S_l^{ij}(\Xi^{-1})\phi_{ij} d\mathcal{H}^{n-1} = -\int_{\mathbb{S}^{n-1}} \sum_{i,j=1}^{n-1} S_l^{ij}(\Xi^{-1})\phi_i\phi_j d\mathcal{H}^{n-1}$$

and we are done with the aid of part v) of Proposition 2.1. □

Remark 4.3. Notice that the $(n-1)$-st quermassintegral W_{n-1} (which is proportional to the *mean width*) is linear, i.e. for every K and L convex bodies

$$W_{n-1}(K+L) = W_{n-1}(K) + W_{n-1}(L)\,.$$

This implies in particular that if we take $k = n-1$ in Proposition 2.2 we get $f'' \equiv 0$.

For the proof of Theorem 1.1 we need the following auxiliary result.

Lemma 4.4. *Let $\phi \in C^\infty(\mathbb{S}^{n-1})$ and $\psi(x) = \phi(\nu(x))$, $x \in \partial K$, where ν is the Gauss map of K. Fix $r \in \{1, \dots, n-1\}$. Then for every $y \in \mathbb{S}^{n-1}$*

$$\begin{aligned}
&\frac{1}{\det(\Xi^{-1}(y))} \left\langle (S_r^{ij}(\Xi^{-1}(y)))\nabla\phi(y), \nabla\phi(y)\right\rangle \\
&= \left\langle ((\Xi^{-1}(y))(\nabla\psi(x)), S_{n-r}^{ij}(\Xi(x))\nabla\psi(x))\right\rangle\,,
\end{aligned}$$

where $x = \nu^{-1}(y)$ and $\Xi(x) = D\nu(x) = (h_{ij} + h\delta_{ij})_{ij}^{-1}(x)$.

Proof. We may assume that $\Xi^{-1}(y)$ is diagonal:

$$\Xi^{-1}(y) = \mathrm{diag}(\lambda_1, \dots, \lambda_{n-1})\,, \quad \lambda_i > 0\,,\ i = 1, \dots, n-1\,.$$

Then

$$D\nu(x) = \mathrm{diag}(\mu_1, \dots, \mu_{n-1})\,, \quad \mu_i = \frac{1}{\lambda_i}\,,\ i = 1, \dots, n-1\,.$$

In particular

$$\nabla\psi(x) = D\nu(x)\nabla\phi(\nu(x)) = (\mu_1\phi_1(y), \dots, \mu_{n-1}\phi_{n-1}(y))\,. \tag{19}$$

By Proposition 2.1 the matrix $(S_r^{ij}(\Xi^{-1}(y)))$ is also diagonal and its eigenvalues are given by

$$\Lambda_s = S_{r-1}(\mathrm{diag}(\lambda_1, \dots, \lambda_{s-1}, \lambda_{s+1}, \dots, \lambda_{n-1}))\,, \quad s = 1, \dots, n-2\,.$$

Using again Proposition 2.1 we get

$$\begin{aligned}
&\frac{\sum_{i,j=1}^{n-1} S_r^{ij}(\Xi^{-1}(y))\phi_i(y)\phi_j(y)}{\det(\Xi^{-1}(y))} \\
&= \sum_{i=1}^{n-1} \frac{\Lambda_i}{\det(\Xi^{-1}(y))}\,\phi_i^2(y) \\
&= \sum_{i=1}^{n-1} \mu_i S_{n-r-1}(\mathrm{diag}(\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_{n-1}))\,\phi_i^2(y) \\
&= \sum_{i=1}^{n-1} \mu_i S_{n-r}^{ii}(D\nu(x))\phi_i^2(y) \\
&= \langle \nabla\psi(x), (S_{n-r}^{ij}(D\nu(x)))\nabla\phi(y)\rangle\,.
\end{aligned}$$

The conclusion of the lemma follows from the first equality in (19) and the symmetry of the matrix $(S_{n-r}^{ij}(D\nu(x)))$. □

Proof of Theorem 1.1. We set $\phi(y) = \psi(\nu^{-1}(y))$, $y \in \mathbb{S}^{n-1}$. Consider the map $\nu^{-1} : \mathbb{S}^{n-1} \to \partial K$; its Jacobian is given by

$$\det(D(\nu^{-1})(y)) = \det(\Xi^{-1}(y)) > 0\,, \quad \forall y \in \mathbb{S}^{n-1}\,.$$

Moreover, by Proposition 2.1 we have that for every $r \in \{0, 1, \dots, n-1\}$,

$$S_r(D\nu(\nu^{-1}(y))) = \frac{S_{n-r-1}(\Xi^{-1}(y))}{\det(\Xi^{-1}(y))}\,, \quad \forall y \in \mathbb{S}^{n-1}\,.$$

Hence we can write

$$\int_{\partial K} \psi S_{k-1}(D\nu) d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} \phi S_{n-k}(\Xi^{-1}) d\mathcal{H}^{n-1}\,,$$
$$\int_{\partial K} \psi^2 S_k(D\nu) d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} \phi^2 S_{n-k-1}(\Xi^{-1}) d\mathcal{H}^{n-1}\,.$$

And, by Lemma 4.4,

$$\int_{\partial K} \left\langle S_k^{ij}(D\nu)\nabla\psi, (D\nu)^{-1}\nabla\psi \right\rangle d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} \left\langle (S_{n-k}^{ij}(\Xi^{-1}))\nabla\phi, \nabla\phi \right\rangle d\mathcal{H}^{n-1}\,.$$

The proof is completed applying Theorem 1.2 with $l = n - k$ and recalling that

$$(D\nu)^{-1} = \frac{1}{\det(D\nu)}(S_{n-1}^{ij}(D\nu)) = \frac{1}{\kappa}(S_{n-1}^{ij}(D\nu))\,.$$

□

Remark 4.5. *With the notation of the proof of Theorem* 1.2*, let* $\phi(y) = \langle y_0, y \rangle$*, where* $y_0 \in \mathbb{S}^{n-1}$ *is fixed. Note that condition* (5) *is verified since*

$$\int_{\mathbb{S}^{n-1}} y S_l(h_{ij}(y) + h\delta_{ij}(y)) d\mathcal{H}^{n-1}(y) = \int_{\mathbb{S}^{n-1}} y \, dA_l(K, y)\,,$$

where $A_l(K, \cdot)$ *is the* l*–th area measure of* K *(see* [11] *or the next section for the definition), and the latter integral is zero by standard properties of area measures. Moreover, for every* s*,* $h + s\phi$ *is the support function of a translate of* K*. Since quermassintegrals are invariant with respect to translations, the function* f *is constant in particular* $f''(0) = 0$*. This proves that if* ϕ *is as above we have equality in* (6)*. Analogously, choosing* $\psi(x) = \langle x_0, \nu(x) \rangle$ *where* $0 \neq x_0 \in \mathbb{R}^n$ *is fixed, we see that condition* (3) *of Theorem* 1.1 *is fulfilled and* (4) *becomes an equality.*

5. The case $l = 1$: proof of Theorems 1.3 and 1.4

We start this section recalling the definition of area measures; for a detailed presentation of this topic we refer the reader to [11, Chapter 5]. If $K_1, \dots, K_m$, $m \in \mathbb{N}$, are convex bodies in $\mathbb{R}^n$ and $\lambda_1, \dots, \lambda_m$ are non–negative real numbers, then we have:

$$\mathcal{H}^n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\dots,i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1}, \dots, K_{i_n})\,.$$

The coefficients of the polynomial at the right–hand side are called *mixed volumes*. Moreover, if we fix $n-1$ convex bodies $K_2,\dots,K_n$, there exists a unique non–negative Borel measure $A(K_2,\dots,K_n,\cdot)$ (called *mixed area measure*) such that for every convex body K_1

$$V(K_1,K_2,\dots,K_n)=\frac{1}{n}\int_{\mathbb{S}^{n-1}} h_{K_1}(x)\,dA(K_2,\dots,K_n,x)\,.$$

For $j=1,\dots,n-1$, the *area measure* of order j of a convex body K is obtained in the following way: $A_j(K,\cdot)=A(K,\dots,K,B^n,\dots,B^n,\cdot)$, where K is repeated j times and B^n $n-j-1$ times. An alternative definition of area measures is based on a local version of the Steiner formula (see [11, Chapter 4]). In particular, the area measure of order one of K is $A_1(K,\cdot)=A(K,B^n\dots,B^n,\cdot)$. If K is of class C^2_+, then it can be proved that

$$dA_1(K,\cdot)=\frac{1}{n-1}S_1((h_K)_{ij}+h_K\delta_{ij})d\mathcal{H}^{n-1}\,. \tag{20}$$

Hence condition (5) is equivalent to (8) when h is the support function of a convex body of class C^2_+ and $l=1$.

Proof of Theorem 1.3. We may assume that $\phi\in C^\infty(\mathbb{S}^{n-1})$. K can be approximated by a sequence K_r, $r\in\mathbb{N}$, such that for every r, K_r is of class C^2_+ and $(K_r)_{r\in\mathbb{N}}$ converges to K in the Hausdorff metric as r tends to infinity. Fix $r\in\mathbb{N}$ and let h_r be the support function of K_r. For s sufficiently small in absolute value, consider the function

$$f_r(s)=\int_{\mathbb{S}^{n-1}}(h_r+s\phi)S_1((h_r+s\phi)_{ij}+(h_r+s\phi)\delta_{ij})\,d\mathcal{H}^{n-1}\,.$$

By Proposition 2.2, $\sqrt{f_r}$ is concave so that $2f_r(0)f_r''(0)-(f_r'(0))^2\le 0$. Since the relation $(S_1^{ij}(A))=(\delta_{ij})$ holds for any matrix A, using (13) and Propositions 4.1 and 4.2 (with $k=n-2$), we obtain

$$\begin{aligned} &n(n-1)W_{n-2}(K_r)\int_{\mathbb{S}^{n-1}}\phi\left((n-1)\phi+\sum_{i=1}^{n-1}\phi_{ii}\right)\,d\mathcal{H}^{n-1}\\ &\le\left(\int_{\mathbb{S}^{n-1}}\phi S_1((h_r)_{ij}+h_r\delta_{ij})\,d\mathcal{H}^{n-1}\right)^2\,. \end{aligned}\tag{21}$$

From (20) we know that

$$\int_{\mathbb{S}^{n-1}}\phi S_1((h_r)_{ij}+h_r\delta_{ij})\,d\mathcal{H}^{n-1}=(n-1)\int_{\mathbb{S}^{n-1}}\phi(x)\,dA_1(K_r,x)\,.$$

Moreover, as r tends to infinity the sequence of measures $A_1(K_r,\cdot)$ converges weakly to $A_1(K,\cdot)$ (see [11, Theorem 4.2.1]). This implies

$$\lim_{r\to\infty}\int_{\mathbb{S}^{n-1}}\phi(x)\,dA_1(K_r,x)=\int_{\mathbb{S}^{n-1}}\phi(x)\,dA_1(K,x)=0\,. \tag{22}$$

On the other hand $W_{n-2}(K_r)$ converges to $W_{n-2}(K)$ as r tends to infinity (by standard continuity results on quermassintegrals) and $W_{n-2}(K)>0$ since K has interior points. The conclusion follows letting $r\to\infty$ in (21), using (22) and integrating by parts. □

As mentioned in the Introduction, Theorem 1.3 extends the usual (sharp) Poincaré inequality (7) on $\mathbb{S}^{n-1}$ when the usual zero–mean condition is replaced by (8). Clearly, in order to apply this result it would be useful to understand when a measure μ on $\mathbb{S}^{n-1}$ is the area measure of order one of some convex body. This amounts to solve the Christoffel problem for μ (see for instance [11, §4.3]). For $n = 2$ this problem coincides with the Minkowski problem and its solution is completely understood. Let μ be a non–negative Borel measure on $\mathbb{S}^1$ such that: i) μ is not the sum of two point–masses; ii)

$$\int_{\mathbb{S}^1} x\, d\mu(x) = 0\,.$$

Then there exists a convex body K in $\mathbb{R}^2$ such that $A_1(K,\cdot) = \mu(\cdot)$ (note that conditions i) and ii) are also necessary in order μ to be the area measure of order one of some convex body). Hence we have the following extension of the well known *Wirtinger inequality.*

Corollary 5.1. *Let μ be a non–negative Borel measure on $[0, 2\pi]$ such that μ is not the sum of two point–masses and*

$$\int_0^{2\pi} \sin\theta\, d\mu(\theta) = \int_0^{2\pi} \cos\theta\, d\mu(\theta) = 0\,.$$

Then, for every $\phi \in C^1([0, 2\pi])$ such that $\phi(0) = \phi(2\pi)$

$$\int_0^{2\pi} \phi(\theta)\, d\mu(\theta) = 0 \quad \Rightarrow \quad \int_0^{2\pi} (\phi(\theta))^2\, d\theta \le \int_0^{2\pi} (\phi'(\theta))^2\, d\theta\,.$$

In higher dimension the Christoffel problem is more complicated. Necessary and sufficient conditions for a measure μ to be the area measure of order one of some convex body were found by Firey [6] and Berg [1] (see also [11, §4.2]). On the other hand these conditions are not easy to use in practice. A considerable progress (in a larger class of problems) has been made by Guan and Ma in [7] and Sheng, Trudinger and Wang in [10] where a rather simple sufficient condition is found. Here we state this result in the case of area measures of order one.

Theorem 5.2 (Guan, Ma, Sheng, Trudinger, Wang). *Let $f \in C^{1,1}(\mathbb{S}^{n-1})$, $f > 0$ and let $g = 1/f$. If*

$$\int_{\mathbb{S}^{n-1}} x f(x) d\mathcal{H}^{n-1}(x) = 0\,,$$

and the matrix $(g_{ij} + g\delta_{ij})$ is positive semi–definite a.e. on $\mathbb{S}^{n-1}$, then there exists a convex body K, uniquely determined up to translations, such that

$$dA_1(L,\cdot) = f\, d\mathcal{H}^{n-1}\,,$$

i.e., f is the density of $A_1(K,\cdot)$ with respect to $\mathcal{H}^{n-1}$.

Using the above result and Theorem 1.3, we now proceed to show Theorem 1.4.

Proof of Theorem 1.4. We recall that the radial function ρ_K of K is defined as $\rho_K(x) = \max\{\lambda \geq 0 \,|\, \lambda x \in K\}$. Let H be the polar body of K:

$$H = \{x \in \mathbb{R}^n \,:\, \langle x, y\rangle \leq 1\,,\ \forall y \in K\}\,.$$

H is still a convex body and the origin belongs to its interior. Note that (see for instance [11, Remark 1.7.7])

$$\rho_K = \frac{1}{h_H}\,, \quad \text{on } \mathbb{S}^{n-1}\,,$$

so that condition (9) becomes

$$\int_{\mathbb{S}^{n-1}} x\,\frac{1}{h_H(x)} d\mathcal{H}^{n-1}(x) = 0\,. \tag{23}$$

We first show that we can find a sequence of convex bodies H_r, $r \in \mathbb{N}$, of class C^2_+, converging to H in the Hausdorff metric and such that

$$\int_{\mathbb{S}^{n-1}} x\,\frac{1}{h_{H_r}(x)} d\mathcal{H}^{n-1}(x) = 0. \tag{24}$$

In order to do it we consider the function

$$F(y) = \int_{\mathbb{S}^{n-1}} \ln(h_H(x) - \langle x, y\rangle) d\mathcal{H}^{n-1}(x)\,,$$

where y is an interior point of H. Then

$$\nabla F(y) = -\int_{\mathbb{S}^{n-1}} x\,\frac{1}{h_H(x) - \langle x, y\rangle} d\mathcal{H}^{n-1}(x) \tag{25}$$

and

$$\left(\frac{\partial^2 F}{\partial y_i \partial y_j}(x)\right) = -\left(\int_{\mathbb{S}^{n-1}} \frac{x_i x_j}{(h_H(x) - \langle x, y\rangle)^2} d\mathcal{H}^{n-1}(x)\right)\,.$$

The last equality implies that F is strictly concave and then, by (23) and (25), $y = 0$ is the unique point where F attains its maximum. Now let $\tilde{H}_r$, $r \in \mathbb{N}$, be a sequence of convex bodies of class C^2_+ converging to H in the Hausdorff metric. We set $\tilde{h}_r = h_{\tilde{H}_r}$. For $r \in \mathbb{N}$ and y interior to $\tilde{H}_r$, consider the corresponding function for $\tilde{H}_r$:

$$F_r(y) = \int_{\mathbb{S}^{n-1}} \ln(\tilde{h}_r(x) - \langle x, y\rangle) d\mathcal{H}^{n-1}(x)\,.$$

As above, it can be shown that F_r is strictly concave in the interior of $\tilde{H}_r$. Moreover, since $\tilde{h}_r \to h_H$ uniformly on $\mathbb{S}^{n-1}$, F_r converges to F uniformly on compact subsets of H. Let $\delta > 0$ be such that the ball B_δ centred at 0 with radius δ is contained in the interior of H. Then

$$F(0) > \max_{y \in \partial B_\delta} F(y)\,.$$

By the uniform convergence, the same inequality holds if F is replaced by F_r, when r is sufficiently large. This implies that for every sufficiently large r there exists a uniquely determined point y_r, belonging both, to the interior of H, and to the interior or each H_r, such that $F_r(y_r) = \max_{H_r} F_r$; in particular

$$\nabla F_r(y_r) = -\int_{\mathbb{S}^{n-1}} x \frac{1}{\tilde{h}_r(x) - \langle x, y_r\rangle} d\mathcal{H}^{n-1}(x) = 0 .$$

for r sufficiently large.

Using again the uniform convergence we get that $y_r \to 0$ as $r \to \infty$; consequently the sequence $H_r = \tilde{H}_r - y_r$ converges to H in the Hausdorff metric. On the other hand $h_r := \tilde{h}_r(x) - \langle x, y_r\rangle$ is the support function of H_r. Hence for every r sufficiently large condition (24) is fulfilled.

For $r \in \mathbb{N}$, let h_r denote the support function of H_r; we have that

$$((h_r)_{ij} + h_r\delta_{ij}) > 0 \quad \text{on } \mathbb{S}^{n-1} \text{ for every } r \in \mathbb{N}. \tag{26}$$

Hence for every $r \in \mathbb{N}$ we can apply Theorem 5.2 with $f = f_r = 1/h_r$, obtaining a convex body L_r such that

$$dA_1(L_r, \cdot) = f_r \, d\mathcal{H}^{n-1} .$$

Since H is a convex body with interior points, we have that $c < h_H < C$ on $\mathbb{S}^{n-1}$, for suitable positive constants c and C. Using the uniform convergence we obtain that there exist $d, D > 0$ such that $d \le f_r(x) \le D$, $\forall\, x \in \mathbb{S}^{n-1}$, $\forall r \in \mathbb{N}$. Hence we may apply Lemma 3.1 in [7] to deduce that the sequence L_r is bounded and by the Blaschke selection theorem (see [11, Theorem 1.8.6]), up to a subsequence, it converges to a convex body L in the Hausdorff metric. As already noticed in the proof of Theorem 1.3, the sequence of measures $A_1(L_r, \cdot)$ converges weakly to $A_1(L, \cdot)$ as r tends to infinity. Consequently

$$dA_1(L, \cdot) = \frac{1}{h_H} \, d\mathcal{H}^{n-1} = \rho_K d\mathcal{H}^{n-1} .$$

The conclusion follows applying Theorem 1.3. □

Remark 5.3. Let us prove that for every convex body K with non–empty interior there is a translate of K such that condition (9) holds. First, assume that the origin is an interior point of K. For $y \in K$ let ρ_y be the radial function of K with respect to y and consider the vector–valued map

$$F(y) = \frac{1}{\int_{\mathbb{S}^{n-1}} \rho_y(x) \, d\mathcal{H}^{n-1}(x)} \int_{\mathbb{S}^{n-1}} x\rho_y(x) \, d\mathcal{H}^{n-1}(x) .$$

Let us prove that $F(y)$ is an interior point of K for every $y \in K$. Let $\bar{x} \in \mathbb{S}^{n-1}$ and let $z \in \partial K$ be such that $\bar{x}$ belongs to the normal cone of K at z. Then the set

$$\alpha = \{u \in \mathbb{R}^n \mid \langle u - z, \bar{x}\rangle \le 0\}$$

is the supporting half–space of K at z with outer unit normal vector $\bar{x}$. For every $x \in \mathbb{S}^{n-1}$ $x\rho_y(x) \in \partial K$ and then

$$\langle x\rho_y(x) - z, \bar{x}\rangle \le 0$$

and this quantity is strictly negative for some choices of x. From this fact and the definition of F we get that $F(y)$ is an interior point of α. Since α is an arbitrary supporting half–space to K this proves that $F(y)$ belongs to the interior of K. An application of the Schauder Fixed Point Theorem shows that there exists $\bar{y}$, interior to K, such that $F(\bar{y}) = \bar{y}$. If we choose the origin in $\bar{y}$ we obtain condition (9).

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