Journal of Convex Analysis
Volume 17 (2010), No. 1, 13–33

On the Study of Quasipolyhedral Convex Functions

M. D. Fajardo
Department of Statistics and Operational Research,
Faculty of Sciences, University of Alicante, 03080 Alicante, Spain
md.fajardo@ua.es

Received: May 5, 2008
Revised manuscript received: October 10, 2008

Different methods are available to solve a constrained optimization problem where the objective function is convex and the constraint set is specified by a linear system of a finite number of linear inequalities. In particular, the problem can be formulated as an optimization problem with a unique constraint involving a polyhedral function. When the linear system has an arbitrary number of linear inequalities, the problem can also be transformed in such way that the constraint set is specified by a unique constraint involving a lower semi-continuous convex function. If this function is quasipolyhedral, it locally behaves like a polyhedral one, and this fact should allow to design an algorithm to resolve the optimization problem. In view of this new approach, this paper is devoted to study characterizations and properties of the class of quasipolyhedral functions, as well as their conjugate function and their subdifferential.

Keywords: Semi-infinite inequality systems, quasipolyhedral convex sets, subdifferential, conjugate function

1. Introduction

Consider the constrained optimization problem

$$(\mathrm{P}_1) \quad \begin{array}{l} \min g(x) \\ \text{s.t. } \langle a_i, x\rangle \le b_i,\ i = 1, ..., m, \end{array}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a convex function, $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, for all $i = 1, ..., m$, and $\langle ., .\rangle$ represents the usual inner product in $\mathbb{R}^n$. In order to design an algorithm to solve (P_1), we can replace it by the equivalent problem

$$(\mathrm{P}_1') \quad \begin{array}{l} \min g(x) \\ \text{s.t. } f(x) \le 0, \end{array}$$

where $f(x) := \max\{\langle a_i, x\rangle - b_i,\ i = 1, ..., m\}$ is a polyhedral function. A recent paper of Osborne [7] describes an active set algorithm for solving (P_1'), which concentrates on describing the local structure of the polyhedral function.

From this point of view, it raises in a natural way to transform the following convex semi-infinite programming (CSIP) problem

$$(\mathrm{P}) \quad \begin{array}{l} \min g(x) \\ \text{s.t. } \langle a_t, x\rangle \le b_t,\ t \in T, \end{array}$$

ISSN 0944-6532 / $ 2.50

where T is an arbitrary infinite index set, into the problem

$$\begin{array}{ll} (\mathrm{P}') & \min g(x) \\ & \text{s.t. } f(x) \leq 0, \end{array}$$

where $f(x) := \sup \{\langle a_t, x\rangle - b_t,\ t \in T\}$. If we are dealing with a non-empty constraint set, f is a lower semi-continuous (lsc, in brief) proper convex function, since its *effective domain*:

$$\operatorname{dom} f := \{x \in \mathbb{R}^n \mid f(x) < +\infty\},$$

is non-empty, and its *epigragh*:

$$\operatorname{epi} f := \left\{ \begin{pmatrix} x \\ \alpha \end{pmatrix} \in \mathbb{R}^{n+1} \middle| \ f(x) \leq \alpha \text{ and } x \in \operatorname{dom} f \right\},$$

is convex and closed. Moreover, $\operatorname{epi} f$ is the solution set of the system

$$\sigma = \{\langle a_t, x\rangle - x_{n+1} \leq b_t,\ t \in T\}. \tag{1}$$

Then we say that σ is a *representation* of f by means of those affine minorants associated with the inequalities of σ.

In [4] representations for a finite-valued convex function are classified in three types, according to three families of linear semi-infinite systems with good geometrical properties, as it is shown in [5, Chapter 5]. The existence of a locally polyhedral representation gives rise to the concept of quasipolyhedral function. In [4], conditions for the conjugate of a quasipolyhedral function to be also quasipolyhedral are obtained, as well as characterizations of the subdiferential and the $\varepsilon-$subdiferential of a quasipolyhedral function. Moreover, when a finite-valued convex function and its conjugate are quasipolyhedral, each of both admits a specific representation in terms of the extreme points of the epigraph of the another one.

In this paper we focus on the study of characterizations and geometrical properties of quasipolyhedral functions which are not finite-valued. Our objective is to derive results which can be useful for designing an algorithm to resolve (P′).

The paper is organized as follows. For a finite-valued convex function, from every linear inequality system whose solution set is the epigraph of the function, it can be derived a representation in the form of (1), but this is not true if the function is not finite-valued. Therefore, in Section 2, the concept of representation of a lsc convex function in terms of a linear inequality system is extended, including the possibility for the function to be improper, for the sake of completness. The existence of a locally polyhedral (LOP) representation of the function turns out to be equivalent to the quasipolyhedrality of its epigraph, in which case the function will be *quasipolyhedral*. A characterization of improper quasipolyhedral functions is also presented, whereas Section 3 is devoted to characterize proper quasipolyhedral functions. Finally Section 4 is dedicated to the study of the subdifferential and the conjugate of a quasipolyhedral function. As we will see, its subdiffential set at each point of its effective domain is a non-empty polyhedral set.

Let us introduce the necessary notation. Given a non-empy set X of the Euclidean space $\mathbb{R}^n$, the *convex* (*conical*, *affine*, *linear*) *hull* of X is denoted by $\operatorname{conv} X$ ($\operatorname{cone} X$, $\operatorname{aff} X$, $\operatorname{span} X$, respectively), and X° represents the *polar cone* of a convex cone, X,

$$X^\circ = \{y \in \mathbb{R}^n \mid \langle y, x\rangle \leq 0 \text{ for all } x \in X\}.$$

It is assumed that $\operatorname{cone} X$ always contains the origin and, consequently, $\operatorname{cone}(\emptyset) = \{0_n\}$, where 0_n is the null-vector in $\mathbb{R}^n$. We represent by $\dim X$ the dimension of $\operatorname{aff} X$. In particular, for a convex function f, $\dim(\operatorname{dom} f)$ is called the *dimension* of f, and it is represented by $\dim f$.

We recall that the *recession cone* of a convex set X is

$$0^+X = \{y \in \mathbb{R}^n \mid x + \lambda y \in X, \text{ for all } \lambda \geq 0 \text{ and all } x \in X\},$$

and its *lineality space*, $\operatorname{lin} X := (0^+X) \cap (-0^+X)$, whose dimension is called the *lineality* of X. In particular, if f is a proper convex function on $\mathbb{R}^n$, the projection onto the first n coordinates of $\operatorname{lin}(\operatorname{epi} f)$ is called the *lineality space* of f, and its dimension is the *lineality* of f.

Recall that, for any function f, its conjugate is defined as

$$f^*(u) := \sup\{\langle u, x\rangle - f(x) \mid x \in \operatorname{dom} f\}.$$

In particular, if f is a lsc proper convex function, then $f = f^{**}$.

If f is a proper convex function on $\mathbb{R}^n$, the set (possibly empty)

$$\partial f(x) := \{v \in \mathbb{R}^n \mid f(y) \geq f(x) + \langle v, y - x\rangle \text{ for all } y \in \operatorname{dom} f\}$$

is the *subdifferential* of f at $x \in \operatorname{dom} f$.

For a set $C \subset \mathbb{R}^n$, the *indicator function* δ_C is defined as $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ if $x \notin C$. Its conjugate is the *support function* of C:

$$\delta_C^*(u) = \sup\{\langle u, x\rangle, x \in C\}.$$

If C is non-empty, closed and convex, then δ_C is a proper lcs convex function.

From the topological side, $\operatorname{int} X$, $\operatorname{cl} X$, and $\operatorname{bd} X$ represent the *interior*, the *closure*, and the *boundary* of X, respectively, whereas $\operatorname{rint} X$ and $\operatorname{rbd} X$ represent the *relative interior* and the *relative boundary* of X (relatively to $\operatorname{aff} X$), respectively. In particular, if f is a proper convex function, we have

$$\operatorname{rint} \operatorname{epi} f = \left\{ \begin{pmatrix} x \\ \mu \end{pmatrix} \middle| \; x \in \operatorname{rint} \operatorname{dom} f \text{ and } f(x) < \mu \right\},$$

according to [9, Lemma 7.3].

The Euclidean norm (respectively, the Chebyshev norm) is represented by $\|.\|$ (respectively, $\|.\|_\infty$), whereas $\mathbb{B}$ (respectively, $\mathbb{B}_\infty$) is the corresponding open unit ball centered at the origin.

2. Quasipolyhedral functions

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a lsc convex function, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. If f is not identically equal to $+\infty$, epi f is a non-empty closed convex set in $\mathbb{R}^{n+1}$, and there exists a system of linear inequalities,

$$\{\langle c_t, x\rangle + \gamma_t x_{n+1} \leq d_t,\ t \in T\},$$

whose solution set is epi f. As a consequence of $\binom{0_n}{1} \in 0^+(\operatorname{epi} f)$, it holds $\gamma_t \leq 0$, for all $t \in T$. Let

$$T_1 := \{t \in T \mid \gamma_t < 0\},$$
$$T_2 := \{t \in T \mid \gamma_t = 0\}.$$

Let $a_t := c_t / |\gamma_t|$ and $b_t := d_t / |\gamma_t|$ for all $t \in T_1$. Hence epi f is the solution set of

$$\{\langle a_t, x\rangle - x_{n+1} \leq b_t,\ t \in T_1; \langle c_t, x\rangle \leq d_t,\ t \in T_2\}.$$

We can assume w.l.o.g. that every index in T_2 is *proper*, in the sense that $c_t \neq 0_n$, for all $t \in T_2$.

It is easy to show that we can write $f(x) = h(x) + \delta_C(x)$, where

$$h(x) := \sup\{\langle a_t, x\rangle - b_t,\ t \in T_1\},\qquad (2)$$
$$C := \{x \in \mathbb{R}^n \mid \langle c_t, x\rangle \leq d_t,\ t \in T_2\},$$

taking $C = \mathbb{R}^n$ when $T_2 = \emptyset$, and $h(x) = -\infty$ if $x \in C$, $h(x) = +\infty$ if $x \notin C$, when $T_1 = \emptyset$. Evidently, $T_1 = T_2 = \emptyset$ if and only if $f \equiv -\infty$, and epi $f = \mathbb{R}^{n+1}$.

Definition 2.1. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a lsc convex function, neither identically equal to $+\infty$ nor $-\infty$. If epi f is the solution set of the system

$$\sigma = \{\langle a_t, x\rangle - x_{n+1} \leq b_t,\ t \in T_1; \langle c_t, x\rangle \leq d_t,\ t \in T_2\},$$

we say that σ is a *representation* of f, and $f(x) = h(x) + \delta_C(x)$, where h and C are defined in (2).

It is worth revising some concepts and results about inequality systems which are used throughout the paper.

Let $\{\langle c_j, z\rangle \leq d_j,\ j \in J\}$ be a system of linear inequalities, with z and c_j in $\mathbb{R}^p$, d_j in $\mathbb{R}$, and J being an arbitrary (possibly infinite) index set. The solution set of the system is denoted by F, and the system is consistent if F is non-empty.

For any $\overline{z} \in \mathbb{R}^p$, an indice $j \in J$ is *active* at $\overline{z}$ if $\langle c_j, \overline{z}\rangle = d_j$. Hence, the *set of active indices* at $\overline{z}$ is

$$J(\overline{z}) := \{j \in J \mid \langle c_j, \overline{z}\rangle = d_j\},$$

and the so-called *active cone* at $\overline{z}$ is

$$A(\overline{z}) := \operatorname{cone}\{c_j,\ j \in J(\overline{z})\}.$$

The *cone of feasible directions* for F at $\overline{z} \in F$ is

$$D_F(\overline{z}) := \{w \in \mathbb{R}^p \mid \overline{z} + \lambda w \in F, \text{ for some } \lambda > 0\},$$

and its general relationship with the active cone is

$$A(\overline{z}) \subset D_F(\overline{z})^{\circ}, \quad \text{for every } \overline{z} \in F.$$

$D_F(\overline{z})^{\circ}$ is nothing else but the *normal cone* to F at $\overline{z}$, represented by $N_F(\overline{z})$.

Definition 2.2. A consistent system $\{\langle c_j, z\rangle \le d_j,\ j \in J\}$ is said to be *locally polyhedral* (LOP, in brief) if

$$A(\overline{z})^{\circ} = D_F(\overline{z}), \quad \text{for all } \overline{z} \in F.$$

Recall that a *polyhedral* convex set is a set which can be expressed as the intersection of some finite collection of closed half-spaces. Bounded polyhedral sets are called *polytopes*. A result which will be used sometimes is that the projection onto the first n coordinates of a polytope in $\mathbb{R}^{n+1}$ will be also a polytope in $\mathbb{R}^n$, according to [9, Th. 19.3].

Corollary 5.6.1 of [5] establishes that the solution set F of every LOP system is a *quasipolyhedral set*; i.e., F is a subset of $\mathbb{R}^p$ whose non-empty intersections with polytopes are polytopes. This property, evidently, is held by the set $\emptyset$.

Definition 2.3. A consistent system $\{\langle c_j, z\rangle \le d_j,\ j \in J\}$ is *tight* if, for every $\overline{z} \in \operatorname{bd} F$, $\dim A(\overline{z}) > 0$.

Theorem 5.5(ii) in [5] establishes that if $\{\langle c_j, z\rangle \le d_j,\ j \in J\}$ is tight and its feasible set F is a full-dimensional quasipolyhedral set, then the system is LOP.

Now, let f be a lsc convex function neither identically equal to $+\infty$ nor $-\infty$, and consider a representation of f,

$$\sigma = \{\langle a_t, x\rangle - x_{n+1} \le b_t,\ t \in T_1; \langle c_t, x\rangle \le d_t,\ t \in T_2\}.$$

We have, for every $\binom{x}{x_{n+1}} \in \operatorname{epi} f$,

$$T_1\begin{pmatrix} x \\ x_{n+1} \end{pmatrix} = \{t \in T_1 \mid \langle a_t, x\rangle - x_{n+1} = b_t\}.$$

Then

$$t \in T_1\begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \Rightarrow f(x) \ge \langle a_t, x\rangle - b_t = x_{n+1} \ge f(x).$$

Hence, if $T_1\binom{x}{x_{n+1}} \neq \emptyset$, it must be $x_{n+1} = f(x)$. We denote the set $T_1\binom{x}{f(x)}$, for $x \in \operatorname{dom} f$, by

$$T_1(x) = \{t \in T_1 \mid f(x) = \langle a_t, x\rangle - b_t\}.$$

On the other hand, if $\binom{x}{x_{n+1}} \in \operatorname{epi} f$

$$T_2\begin{pmatrix} x \\ x_{n+1} \end{pmatrix} = \{t \in T_2 \mid \langle c_t, x\rangle = d_t\},$$

and this set does not depend on x_{n+1}. Accordingly, we denote the set $T_2\binom{x}{x_{n+1}}$, for $x \in \operatorname{dom} f$ and all $x_{n+1} \geq f(x)$ by

$$T_2(x) = \{t \in T_2 \mid \langle c_t, x\rangle = d_t\}.$$

The active cone at $\binom{x}{x_{n+1}} \in \operatorname{epi} f$ associated with σ will be

$$A_\sigma\begin{pmatrix} x \\ x_{n+1}\end{pmatrix} = \begin{cases} \operatorname{cone}\left\{\begin{pmatrix} a_t \\ -1\end{pmatrix},\ t \in T_1(x);\ \begin{pmatrix} c_t \\ 0\end{pmatrix},\ t \in T_2(x)\right\}, & \text{if } x_{n+1} = f(x), \\ \operatorname{cone}\left\{\begin{pmatrix} c_t \\ 0\end{pmatrix},\ t \in T_2(x)\right\}, & \text{if } x_{n+1} > f(x). \end{cases}$$

It will be called *cone of active minorants* at $\binom{x}{x_{n+1}}$ associated with σ.

Moreover, $D_{\operatorname{epi} f}\binom{x}{x_{n+1}}$ and $N_{\operatorname{epi} f}\binom{x}{x_{n+1}}$ will represent the cone of feasible directions and the normal cone to $\operatorname{epi} f$ at $\binom{x}{x_{n+1}} \in \operatorname{epi} f$, respectively. For every $\binom{x}{x_{n+1}} \in \operatorname{epi} f$, we have

$$A_\sigma\begin{pmatrix} x \\ x_{n+1}\end{pmatrix} \subset D_{\operatorname{epi} f}\begin{pmatrix} x \\ x_{n+1}\end{pmatrix}^{\circ} \equiv N_{\operatorname{epi} f}\begin{pmatrix} x \\ x_{n+1}\end{pmatrix}. \tag{3}$$

Definition 2.4. A lsc convex function f neither identically equal to $+\infty$ nor $-\infty$ is *quasipolyhedral* if $\operatorname{epi} f$ is a quasipolyhedral set.

The concept of quasipolyhedral function generalizes that one of *polyhedral* function; i.e., a convex function whose epigraph is a polyhedral set. The class of polyhedral sets (respectively, functions) is closed under well known operations, as it is shown in [9]. The analysis of such operations applied to quasipolyhedral sets and functions can be found in [3].

Let us observe that if $f \equiv +\infty$, $\operatorname{epi} f = \emptyset$ and if $f \equiv -\infty$, $\operatorname{epi} f = \mathbb{R}^{n+1}$. In both cases, f is quasipolyhedral but they have no interest at all, hence from now on, we always consider that f is neither identically equal to $+\infty$ nor $-\infty$.

If σ is a LOP representation of f, the solution set of σ, which is $\operatorname{epi} f$, is a quasipolyhedral set, by Corollary 5.6.1 in [5]. Conversely, if $\operatorname{epi} f$ is a quasipolyhedral set, there exists a LOP representation of f, according to Theorem 5.11 in [5]. Hence, f is quasipolyhedral if and only if there exists a LOP representation of f.

Let σ be a representation of a lsc convex f and K be a non-empty set contained in $\operatorname{dom} f$. We shall use the notation

$$T_i(K) := \bigcup\{T_i(x), x \in K\}, \quad i = 1, 2.$$

The following three lemmas will be used in Sections 3 and 4. Moreover, Lemma 2.7 provides a characterization of improper quasipolyhedral functions.

Lemma 2.5. *Let $K \subset \mathbb{R}^n$ be a non-empty $k-$dimensional convex set, $k \leq n$, and $\overline{x} \in \operatorname{rint} K$. Then there exists a $k-$dimensional polytope P, such that $\overline{x} \in \operatorname{rint} P$ and $P \subset \operatorname{rint} K$.*

Proof. Since $\overline{x} \in \operatorname{rint} K$, $(\overline{x} + \varepsilon \mathbb{B}_\infty) \cap \operatorname{aff} K \subset K$, for some $\varepsilon > 0$. Take any $0 < \delta < \varepsilon$. We have

$$(\overline{x} + \delta \operatorname{cl} \mathbb{B}_\infty) \cap \operatorname{aff} K \subset (\overline{x} + \varepsilon \mathbb{B}_\infty) \cap \operatorname{aff} K \subset K,$$

and the polytope $P := (\overline{x} + \delta \operatorname{cl} \mathbb{B}_\infty) \cap \operatorname{aff} K \subset \operatorname{rint} K$ verifies that $\operatorname{aff} P = \operatorname{aff} K$, hence P is $k-$dimensional. Moreover, $\overline{x} \in (\overline{x} + \delta \mathbb{B}_\infty) \cap \operatorname{aff} P \subset P$ and $\overline{x} \in \operatorname{rint} P$. □

Lemma 2.6. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper quasipolyhedral function. Then, for every non-empty polytope $P \subset \operatorname{dom} f$, the function $f + \delta_P$ is polyhedral.*

Proof. Since $P \subset \operatorname{dom} f$, according to [9, Th. 10.2], f is continuous relative to P, hence f is bounded on P. We can take $\alpha \leq f(x) \leq \beta$, for all $x \in P$, and

$$\widehat{P} := \{P \times [\alpha, \beta]\} \cap \operatorname{epi} f$$

is a polytope in $\mathbb{R}^{n+1}$, since it is the non-empty intersection of a polytope and a quasipolyhedral set. It is easy to see that $\operatorname{epi}(f + \delta_p) = \widehat{P} \cup \{P \times [\beta, +\infty[\}$. Consequently, $\operatorname{epi}(f + \delta_p)$, which is a closed convex set, is the union of two polyhedral sets, and according to Theorem 19.6 in [9], $\operatorname{epi}(f + \delta_p)$ is a polyhedral set and $f + \delta_p$ is a polyhedral function. □

Lemma 2.7. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a lsc convex function. The following condition is necessary in order that f be quasipolyhedral: for all $x \in (\operatorname{rbd} \operatorname{dom} f) \cap \operatorname{dom} f$ there exists a polytope P such that $x \in P \subset \operatorname{dom} f$ and $D_P(x) = D_{\operatorname{dom} f}(x)$. Moreover, if f is improper, this condition is also sufficient.*

Proof. Given $\overline{x} \in (\operatorname{rbd} \operatorname{dom} f) \cap \operatorname{dom} f$, according to [9, Lemma 7.3], we can take $\beta \in \mathbb{R}$ such that $\binom{\overline{x}}{\beta} \in \operatorname{rbd} \operatorname{epi} f$ ($\beta = f(\overline{x})$ if $f(\overline{x}) > -\infty$ and, for instance, $\beta = 0$ if $f(\overline{x}) = -\infty$).

Take any $k > 0$, and consider the polytope, in $\mathbb{R}^{n+1}$, $\left\{ \binom{\overline{x}}{\beta} + k \operatorname{cl} \mathbb{B}_\infty \right\}$. Then

$$\widehat{P} := \left\{ \begin{pmatrix} \overline{x} \\ \beta \end{pmatrix} + k \operatorname{cl} \mathbb{B}_\infty \right\} \cap \operatorname{epi} f \neq \emptyset,$$

and, since $\operatorname{epi} f$ is quasipolyhedral, $\widehat{P}$ is a polytope. Moreover, $\dim \widehat{P} = \dim \operatorname{epi} f$ and

$$D_{\widehat{P}} \begin{pmatrix} \overline{x} \\ \beta \end{pmatrix} = D_{\operatorname{epi} f} \begin{pmatrix} \overline{x} \\ \beta \end{pmatrix}. \tag{4}$$

Let P be the projection onto the first n coordinates of $\widehat{P}$. Then P is a polytope, such that $\overline{x} \in P \subset \operatorname{dom} f$. We will obtain $D_P(\overline{x}) = D_{\operatorname{dom} f}(\overline{x})$ from (4). Actually, we only have to show that $D_{\operatorname{dom} f}(\overline{x}) \subset D_P(\overline{x})$.

Let $w \in D_{\operatorname{dom} f}(\overline{x})$ and $\gamma > 0$ verifying $\overline{x} + \gamma w \in \operatorname{dom} f$. Choose whatever $\delta \in \mathbb{R}$ such that $\binom{\overline{x} + \gamma w}{\delta} \in \operatorname{epi} f$. Then

$$\begin{pmatrix} \overline{x} + \gamma w \\ \delta \end{pmatrix} = \begin{pmatrix} \overline{x} \\ \beta \end{pmatrix} + \gamma \begin{pmatrix} w \\ \frac{\delta - \beta}{\gamma} \end{pmatrix},$$

and $\binom{w}{\frac{\delta-\beta}{\gamma}} \in D_{\mathrm{epi}\, f}\binom{\overline{x}}{\beta}$. From (4), we have $\binom{\overline{x}}{\beta} + \rho\binom{w}{\frac{\delta-\beta}{\gamma}} \in \widehat{P}$, for some $\rho > 0$, hence $\overline{x} + \rho w \in P$ and $w \in D_P(\overline{x})$.

Let us observe that this necessary condition for a function to be quasipolyhedral implies that $D_{\mathrm{dom}\, f}(x)$ is polyhedral, for all $x \in (\mathrm{rbd}\,\mathrm{dom}\, f) \cap \mathrm{dom}\, f$, acccording to [1, Th. VII.1.6].

Conversely, if we assume that f is improper, since $\mathrm{dom}\, f$ is closed, we have that, for all $x \in \mathrm{dom}\, f$, $D_{\mathrm{dom}\, f}(x)$ is polyhedral, and applying [1, Th. VII.1.6], we conclude that $\mathrm{dom}\, f$ is quasipolyhedral. According to [5, Th. 5.11], there exists a LOP system $\sigma = \{\langle c_j, z\rangle \leq d_j,\ j \in J\}$ whose solution set is $\mathrm{dom}\, f$.

Consider

$$\widetilde{\sigma} := \{\langle c_j, z\rangle + 0z_{n+1} \leq d_j,\ j \in J\},$$

with solution set $\mathrm{dom}\, f \times \mathbb{R} = \mathrm{epi}\, f$. Then, for all $\binom{x}{x_{n+1}} \in \mathrm{epi}\, f$, it holds

$$A_{\widetilde{\sigma}}\begin{pmatrix} x \\ x_{n+1}\end{pmatrix}^{\circ} = [A_\sigma(x) \times \{0\}]^\circ = D_{\mathrm{dom}\, f}(x) \times \mathbb{R},$$

because σ is LOP. It is easy to see that $D_{\mathrm{dom}\, f}(x) \times \mathbb{R} = D_{\mathrm{epi}\, f}\binom{x}{x_{n+1}}$, which allow us to conclude that $\widetilde{\sigma}$ is also LOP. Hence, in virtue of [5, Cor. 5.6.1], $\mathrm{epi}\, f$ is a quasipolyhedral set and f is a quasipolyhedral function. □

3. Characterizing proper quasipolyhedral functions

First, we will give a necessary condition for a proper lsc convex function to be quasipolyhedral. The study is divided into two cases, depending on the dimension of the effective domain of the function.

Proposition 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function,* $\dim f = n$. *If f is quasipolyhedral, then there exists a representation of f, $\sigma = \{\langle a_t, x\rangle - x_{n+1} \leq b_t,\ t \in T_1; \langle c_t, x\rangle \leq d_t,\ t \in T_2\}$, such that for every non-empty polytope $P \subset \mathrm{dom}\, f$, $T_1(P)$ (respectively, $T_2(P)$) is a non-empty finite index subset (respectively, finite). Moreover, σ is* LOP.

Proof. For each $r \in \mathbb{N}$, consider

$$C_r := \mathrm{epi}\, f \cap \{r\, \mathrm{cl}\, \mathbb{B}_\infty\}.$$

It is clear that $C_r \subseteq C_{r+1}$, for all $r \in \mathbb{N}$ and $\mathrm{epi}\, f = \bigcup_{r=1}^{\infty} C_r$, then we may assume w.l.o.g. that $C_r \neq \emptyset$ for all $r \in \mathbb{N}$.

Since $r\, \mathrm{cl}\, \mathbb{B}_{n+1,\infty}$ is a polytope and $\mathrm{epi}\, f$ is quasipolyhedral, C_r is a polytope, for every $r \in \mathbb{N}$. Moreover, since $\dim \mathrm{epi}\, f = n+1$, we may assume that $\dim C_r = n+1$, for all $r \in \mathbb{N}$.

Let D_r be the projection onto the first n coordinates of C_r. Then D_r is a polytope. Moreover, $D_r \subseteq D_{r+1}$, for all $r \in \mathbb{N}$ and $\mathrm{dom}\, f = \bigcup_{r=1}^{\infty} D_r$. We may also assume that $\dim D_r = n$, for all $r \in \mathbb{N}$.

We define the convex function, for all $r \in \mathbb{N}$,

$$f_r(x) := f(x) + \delta_{D_r}(x).$$

According to Lemma 2.6, f_r is a polyhedral function, for every $r \in \mathbb{N}$. Moreover, since $\operatorname{dom} f_r = D_r$, we have $\dim \operatorname{epi} f_r = n+1$.

Consider a minimal representation of $\operatorname{epi} f_r$ (i.e., a representation without redundant inequalities),

$$\sigma^r := \left\{ \left\langle a_r^i, x \right\rangle - x_{n+1} \leq b_r^i,\ i = 1,2,...,k_r, \left\langle c_r^j, x \right\rangle \leq d_r^j,\ j = 1,2,...,v_r \right\}. \tag{5}$$

We may also assume that, for all $j \in \{1,2,...,v_r\}$, $\|c_r^j\| = 1$. Then

$$\begin{gathered} f_r(x) = \max\left\{ \left\langle a_r^i, x \right\rangle - b_r^i,\ i = 1,2,...,k_r \right\}, \\ x \in D_r \Longleftrightarrow \left\langle c_r^j, x \right\rangle \leq d_r^j,\ j = 1,2,...,v_r. \end{gathered}$$

Hence $\{\langle c_r^j, x\rangle \leq d_r^j,\ j = 1,2,...,v_r\}$ will be a minimal representation of the full-dimensional polytope D_r. Define

$$A_r := \left\{ \begin{pmatrix} a_r^i \\ b_r^i \end{pmatrix},\ i = 1,2,...,k_r \right\},$$

for every $r \in \mathbb{N}$. We have that $\operatorname{epi} f_r$ is a full-dimensional polyhedral set in $\mathbb{R}^{n+1}$, and, by Theorem 8.2 in [2], if

$$H_r^i := \left\{ \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1} \middle| \left\langle a_r^i, x \right\rangle - x_{n+1} = b_r^i \right\},$$

then $H_r^i \cap \operatorname{epi} f_r$ is a facet (i.e., a face of dimension n) of $\operatorname{epi} f_r$, for all $i \in \{1,2,...,k_r\}$. We shall prove that $H_r^i \cap \operatorname{epi} f_{r+1}$ is a facet of $\operatorname{epi} f_{r+1}$.

Let denote $F_r^i := H_r^i \cap \operatorname{epi} f_r$. From the fact that the projection onto the first n coordinates of F_r^i is full-dimensional, it is clear that this projection intersects with $\operatorname{int} D_r$. Hence, we can take $y^0 \in \operatorname{int} D_r$ such that $\binom{y^0}{f(y^0)} \in F_r^i$ and $a_r^i \in \partial f_r(y^0)$, since H_r^i is a non-vertical support hyperplane to $\operatorname{epi} f_r$. On the other hand, $y^0 \in \operatorname{int} D_r \subset \operatorname{int} D_{r+1}$ hence $f_r \equiv f_{r+1}$ in a neighborhood of y^0, $\{y^0 + \varepsilon\mathbb{B}\}$, with $\varepsilon > 0$. According to the identities

$$\partial f_r\left(y^0\right) = \partial\left(f_r + \delta_{\{y^0+\varepsilon\mathbb{B}\}}\right)\left(y^0\right) = \partial\left(f_{r+1} + \delta_{\{y^0+\varepsilon\mathbb{B}\}}\right)\left(y^0\right) = \partial f_{r+1}\left(y^0\right),$$

we have $a_r^i \in \partial f_{r+1}(y^0)$ and H_r^i is a non-vertical support hyperplane to $\operatorname{epi} f_{r+1}$, which implies that $H_r^i \cap \operatorname{epi} f_{r+1}$ is a facet of $\operatorname{epi} f_{r+1}$, since $F_r^i \subset H_r^i \cap \operatorname{epi} f_{r+1}$. Applying again Theorem 8.2 in [2], we conclude that $A_r \subset A_{r+1}$, for every $r \in \mathbb{N}$. Let $A := \bigcup_{r=1}^{\infty} A_r$.

Since $\operatorname{dom} f = \bigcup_{r=1}^{\infty} D_r$, for all $x \in \operatorname{dom} f$, there exists $r \in \mathbb{N}$ such that $x \in D_r$. In particular, if $x^0 \in (\operatorname{bd} \operatorname{dom} f) \cap \operatorname{dom} f$, there exists $r \in \mathbb{N}$ such that $\left\| \binom{x^0}{f(x^0)} \right\|_\infty < r$ and $x^0 \in \operatorname{bd} D_r$ (in fact, $x^0 \in \operatorname{bd} D_s$ for all $s \geq r$). Recall that $\{\langle c_r^j, x\rangle \leq d_r^j, j = 1,2,...,v_r\}$ is a minimal representation of D_r, hence there exists $j \in \{1,2,...,v_r\}$ verifying $\langle c_r^j, x^0\rangle =$

d_r^j, according to [2, Th. 8.2]. Denoting $T_2^r := \{1, 2, ..., v_r\}$, we conclude that $T_2^r(x^0) \neq \emptyset$. Let

$$B_r := \left\{ \begin{pmatrix} c_r^j \\ d_r^j \end{pmatrix} \middle| \, j \in T_2^r\left(x^0\right), \text{ for some } x^0 \in (\operatorname{bd}\operatorname{dom} f) \cap D_r, \ \left\| \begin{pmatrix} x^0 \\ f\left(x^0\right) \end{pmatrix} \right\|_\infty < r \right\}. \tag{6}$$

($B_r = \emptyset$ for all $r \in \mathbb{N}$ if $\operatorname{dom} f$ is an open set).

We shall show that $B_r \subset B_{r+1}$, for every $r \in \mathbb{N}$.

Take $\binom{c_r^j}{d_r^j} \in B_r$. Then $\langle c_r^j, x^0 \rangle = d_r^j$ for a certain $x^0 \in (\operatorname{bd}\operatorname{dom} f) \cap D_r$, verifying $\left\| \binom{x^0}{f(x^0)} \right\|_\infty < r$. Then $\binom{x^0}{f(x^0)} \in \operatorname{int}\{s \operatorname{cl} \mathbb{B}_\infty\}$, for all $s \geq r$, and, for all $\binom{v}{v_{n+1}} \in \mathbb{R}^{n+1}$, we can find $\lambda > 0$ such that $\binom{x^0}{f(x^0)} + \lambda \binom{v}{v_{n+1}} \in s \operatorname{cl} \mathbb{B}_\infty$. It implies that, for all $s \geq r$, it holds

$$D_{\operatorname{epi} f_s} \begin{pmatrix} x^0 \\ f\left(x^0\right) \end{pmatrix} = D_{\operatorname{epi} f} \begin{pmatrix} x^0 \\ f\left(x^0\right) \end{pmatrix}.$$

As it was done in the proof of Lemma 2.7, we get, for all $s \geq r$,

$$D_{D_s}\left(x^0\right) = D_{\operatorname{dom} f}\left(x^0\right). \tag{7}$$

Now, from the minimality of the representation of D_r, $x^0 \in \operatorname{bd} D_r$ and $\langle c_r^j, x^0 \rangle = d_r^j$, we have that

$$D_r \cap \left\{ y \in \mathbb{R}^n \middle| \left\langle c_r^j, y \right\rangle = d_r^j \right\}$$

is a facet of D_r containing x^0, applying Theorem 8.2(iii) in [2]. Then the vector c_r^j generates an extreme ray in $D_{D_r}\left(x^0\right)^\circ$, according to Theorem 8.2(iv) in [2]. Now, from (7), it follows that c_r^j will generate an extreme ray in $D_{D_{r+1}}\left(x^0\right)^\circ$. By Theorem 8.2(v) in [2], we have that

$$D_{r+1} \cap \left\{ y \in \mathbb{R}^n \middle| \left\langle c_r^j, y \right\rangle = d_r^j \right\}$$

is a facet of D_{r+1} containing x^0. Hence there exist $\lambda > 0$ and $l \in \{1, 2, ..., v_{r+1}\}$ such that $c_r^j = \lambda c_{r+1}^l$ and $d_r^j = \lambda d_{r+1}^l$. Since we are asumming that $\|c_r^j\| = \left\|c_{r+1}^l\right\| = 1$, we have $\lambda = 1$ and $B_r \subset B_{r+1}$. Let $B := \bigcup_{r=1}^\infty B_r$. It is clear that, for all $x^0 \in (\operatorname{bd}\operatorname{dom} f) \cap \operatorname{dom} f$, there exists $\binom{c}{d} \in B$ such that $\langle c, x^0 \rangle = d$.

Let us introduce the system

$$\sigma := \left\{ \langle a, x \rangle - x_{n+1} \leq b, \ \begin{pmatrix} a \\ b \end{pmatrix} \in A; \ \ \langle c, x \rangle \leq d, \ \begin{pmatrix} c \\ d \end{pmatrix} \in B \right\}.$$

Denote by F its solution set. We shall prove that $F = \operatorname{epi} f$.

Let $\binom{\overline{x}}{\overline{x}_{n+1}} \in \operatorname{epi} f$, and $\overline{x} \in D_r$. Then $\binom{\overline{x}}{\overline{x}_{n+1}} \in \operatorname{epi} f_s$, for all $s \geq r$. Hence $\langle a_s^i, \overline{x} \rangle - \overline{x}_{n+1} \leq b_s^i$, for all $i \in \{1, 2, ..., k_s\}$ and for all $s \geq r$. Since $A_s \subset A_r$, for all $s \leq r$, we conclude that $\langle a, \overline{x} \rangle - \overline{x}_{n+1} \leq b$, for all $\binom{a}{b} \in A$.

On the other hand, since $\overline{x} \in D_s$, for all $s \geq r$, it holds $\langle c_s^j, \overline{x} \rangle \leq d_s^j$, for all $j = 1, 2, ..., v_s$, for all $s \geq r$. Since $B_s \subset B_r$, for all $s \leq r$, we conclude that $\langle c, x \rangle \leq d$, for all $\binom{c}{d} \in B$. Hence $F \supseteq \operatorname{epi} f$.

According to [9, Th. 18.8], $\operatorname{epi} f$ is the intersection of the closed half-spaces tangent to it. We shall show that $F \subseteq \operatorname{epi} f$ verifying that every point in F belongs to that intersection.

Let us consider an hyperplane in $\mathbb{R}^{n+1}$,

$$H := \left\{ \begin{pmatrix} y \\ y_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1} \,\middle|\, \langle \widehat{a}, y \rangle + \widehat{b} y_{n+1} = \widehat{c} \right\},$$

such that $\langle \widehat{a}, x \rangle + \widehat{b} x_{n+1} \leq \widehat{c}$, for all $\binom{x}{x_{n+1}} \in \operatorname{epi} f$ and $\langle \widehat{a}, x^0 \rangle + \widehat{b} x^0_{n+1} = \widehat{c}$, for a certain $\binom{x^0}{x^0_{n+1}} \in \operatorname{bd} \operatorname{epi} f$. Since, from some $s \in \mathbb{N}$, $x^0 \in D_s$, take $r \geq s$ such that

$$\left\| \begin{pmatrix} x^0 \\ f(x^0) \end{pmatrix} \right\|_\infty < r. \tag{8}$$

The fact that $\binom{x^0}{x^0_{n+1}} \in \operatorname{bd} \operatorname{epi} f$ implies that $\binom{x^0}{x^0_{n+1}} \in \operatorname{bd} \operatorname{epi} f_r$.

We have $\binom{\widehat{a}}{\widehat{b}} \in N_{\operatorname{epi} f_r} \binom{x^0}{x^0_{n+1}}$. The minimal representation σ^r of $\operatorname{epi} f_r$ is LOP (since every finite system is LOP), and, denoting $T_1^r := \{1, 2, ..., k_r\}$, it holds, by [5, Th. 5.6],

$$\begin{pmatrix} \widehat{a} \\ \widehat{b} \end{pmatrix} = \sum_{i \in T_1^r(x^0)} \lambda_i \begin{pmatrix} a_r^i \\ -1 \end{pmatrix} + \sum_{j \in T_2^r(x^0)} \beta_j \begin{pmatrix} c_r^j \\ 0 \end{pmatrix}, \tag{9}$$

with $\lambda_i \geq 0$, for all $i \in T_1^r(x^0)$ and $\beta_j \geq 0$, for all $j \in T_2^r(x^0)$. Both sets $T_1^r(x^0)$ and $T_2^r(x^0)$ can not be simultaneously empty, because σ^r is tight. Taking the inner product of both members in the above equation with $\binom{x^0}{x^0_{n+1}}$, we get

$$\widehat{c} = \sum_{i \in I} \lambda_i b_r^i + \sum_{j \in J} \beta_j d_r^j. \tag{10}$$

Evidently, $T_1^r(x^0) \subset A$. If $T_2^r(x^0) \neq \emptyset$, $x^0 \in \operatorname{bd} D_r$ (we have applied [2, Th. 8.2(i)] to the minimal representation of D_r). If $x^0 \in \operatorname{int} \operatorname{dom} f$, from (8), we can take $\varepsilon > 0$ such that $|f(x^0) + \varepsilon| < r$ and

$$\begin{pmatrix} x^0 \\ f(x^0) + \varepsilon \end{pmatrix} \in (\operatorname{int} \operatorname{epi} f) \cap \operatorname{int} \{ r \operatorname{cl} \mathbb{B}_\infty \} = \operatorname{int} C_r.$$

According to [9, Th. 6.6], $x^0 \in \operatorname{int} D_r$, which is a contradiction. Hence $x^0 \in (\operatorname{bd} \operatorname{dom} f) \cap \operatorname{dom} f$, and recalling (8), we have $T_2^r(x^0) \subset B$. From (9) and (10) we obtain

$$\begin{pmatrix} \widehat{a} \\ \widehat{b} \\ \widehat{c} \end{pmatrix} \in \operatorname{cone} \left\{ \begin{pmatrix} a \\ -1 \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \in A; \begin{pmatrix} c \\ 0 \\ d \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \in B \right\},$$

which, in virtue of Extended Farkas' Lemma, implies that $\langle \widehat{a}, y \rangle + \widehat{b} y_{n+1} \leq \widehat{c}$ is a consequence of σ; i.e., for all $\binom{x}{x_{n+1}} \in F$, $\langle \widehat{a}, x \rangle + \widehat{b} x_{n+1} \leq \widehat{c}$.

Then $\operatorname{epi} f$ is the solution set of σ. From the minimality of each σ^r and the way B is chosen, we can conclude that σ is tight. According to Theorem 5.5 in [5], this system is LOP.

The representation σ verifies the property we are looking for since, for any non-empty polytope $P \subset \operatorname{dom} f$, there will exist $r \in \mathbb{N}$ such that $\left\| \binom{x}{f(x)} \right\|_\infty \leq r$, for all $x \in P$, and $P \subset D_r$. We conclude that $T_1(P) \subset \{1, ..., k_r\}$ and $T_2(P) \subset \{1, ..., v_r\}$. Moreover, because of the tightness of σ, for every $x \in \operatorname{dom} f$, $T_1(x) \neq \emptyset$, then $T_1(P) \neq \emptyset$. □

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper quasipolyhedral function, $\dim f = k < n$ and $\{b_0, b_1, ..., b_k\}$ be a maximal set of affinely independent points in $\operatorname{dom} f$. We can write, for all $x \in \operatorname{dom} f$,

$$x = \sum_{i=1}^{k} \lambda_i (b_i - b_0) + b_0, \tag{11}$$

and the coefficients in such an expression of x are unique. We consider the affine transformation S from $\mathbb{R}^k$ to $\mathbb{R}^n$,

$$Sy := Ay + b_0,$$

where A is the linear transformation $Ay := \sum_{i=1}^k y_i (b_i - b_0)$. Since $\{b_1 - b_0, ..., b_k - b_0\}$ are linearly independent, $\ker A = \{0_k\}$ and S is injective.

We define $f_1 := f \circ S$, and $f_1 : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$. This is a proper lsc convex function, verifying that $S(\operatorname{dom} f_1) = \operatorname{dom} f$. This implies that $\operatorname{aff} \operatorname{dom} f = S(\operatorname{aff} \operatorname{dom} f_1)$, hence $\dim f_1 = k$.

On the other hand, defining the affine tranformation $\widehat{S} : \binom{y}{y_{k+1}} \to \binom{Sy}{y_{k+1}}$ from $\mathbb{R}^{k+1}$ to $\mathbb{R}^{n+1}$, it holds that $\widehat{S}(\operatorname{epi} f_1) = \operatorname{epi} f$ and $\widehat{S}$ is injective. Note that $\widehat{S}\binom{y}{y_{k+1}} = \widehat{A}\binom{y}{y_{k+1}} + \widehat{b_0}$, where $\widehat{A}\binom{y}{y_{k+1}} = \binom{Ay}{y_{k+1}}$ and $\widehat{b_0} = \binom{b_0}{0}$. Since $\ker \widehat{A} = \{0_{k+1}\}$, we can write

$$\operatorname{epi} f_1 = \widehat{A}^{-1} \left(\operatorname{epi} f - \widehat{b_0} \right). \tag{12}$$

Obviously, a translation of a quasipolyhedral set will be a quasipolyhedral set too, and, according to [3, Prop. 2.1], $\operatorname{epi} f_1$ is quasipolyhedral and f_1 is a quasipolyhedral function.

Proposition 3.2. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function,* $\dim f = k < n$*. If f is quasipolyhedral, then there exists a representation of f, $\sigma = \{\langle a_t, x\rangle - x_{n+1} \leq b_t,\ t \in T_1; \langle c_t, x\rangle \leq d_t,\ t \in T_2\}$ such that for every non-empty polytope P in* $\operatorname{dom} f$, $T_1(P)$ (*respectively,* $T_2(P)$) *is a non-empty finite index subset* (*respectively, finite*). *Moreover, σ is* LOP.

Proof. Let us observe that the function $f_1 : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$ described above verifies the conditions of Proposition 3.1, because it is quasipolyhedral and $\dim f_1 = k$. Hence there exists a LOP representation of f_1,

$$\widetilde{\sigma} = \left\{ \langle \widetilde{a}_t, y\rangle - y_{k+1} \leq \widetilde{b}_t,\ t \in \widetilde{T}_1; \langle \widetilde{c}_t, y\rangle \leq \widetilde{d}_t,\ t \in \widetilde{T}_2 \right\},$$

such that for every non-empty polytope P_1 in $\operatorname{dom} f_1$, $\widetilde{T}_1(P_1)$ is non-empty and finite, whereas $\widetilde{T}_2(P_1)$ is finite.

Let $c_1, ..., c_{n-k} \in \mathbb{R}^n$ such that

$$\operatorname{span}\{c_1, ..., c_{n-k}\} = (\operatorname{span}\{b_1 - b_0, ..., b_k - b_0\})^{\perp}.$$

Let $a_t \in \mathbb{R}^n$ be a solution of $A^* a_t = \widetilde{a}_t$, where A^* denotes the transpose matrix of A, and $b_t := \widetilde{b}_t + \langle a_t, b_0 \rangle$.

Similarly, let $c_t \in \mathbb{R}^n$ be a solution of $A^* c_t = \widetilde{c}_t$ and $d_t := \widetilde{d}_t + \langle c_t, b_0 \rangle$. Rename $T_1 = \widetilde{T}_1$ and $T_2 = \widetilde{T}_2$ and define

$$\sigma := \left\{ \begin{array}{l} \langle a_t, x \rangle - x_{n+1} \leq b_t,\ t \in T_1; \langle c_t, x \rangle \leq d_t,\ t \in T_2; \\ \langle c_j, x \rangle \leq \langle c_j, b_0 \rangle, \langle -c_j, x \rangle \leq \langle -c_j, b_0 \rangle, j = 1, ..., n-k \end{array} \right\}.$$

First, we shall see that σ is a representation of f; i.e. $\operatorname{epi} f$ is its solution set.

Let $\binom{\overline{x}}{\overline{x}_{n+1}} \in \operatorname{epi} f$. Then $\overline{x} \in \operatorname{dom} f$ and, from (11), it holds that $\langle c_j, \overline{x} - b_0 \rangle = 0$, for all $j = 1, ..., n-k$.

On the other hand, there exists $\binom{\overline{y}}{\overline{y}_{k+1}} \in \operatorname{epi} f_1$ such that $\widehat{S}\binom{\overline{y}}{\overline{y}_{k+1}} = \binom{A\overline{y}+b_0}{\overline{y}_{k+1}} = \binom{\overline{x}}{\overline{x}_{n+1}}$ and

$$\begin{aligned} \langle \widetilde{a}_t, \overline{y} \rangle - \overline{y}_{k+1} \leq \widetilde{b}_t, &\quad \text{for all } t \in T_1, \\ \langle \widetilde{c}_t, \overline{y} \rangle \leq \widetilde{d}_t, &\quad \text{for all } t \in T_2. \end{aligned}$$

For each $t \in T_1$, we have $\langle A^* a_t, \overline{y} \rangle - \overline{y}_{k+1} \leq b_t - \langle a_t, b_0 \rangle$, then $\langle a_t, A\overline{y} + b_0 \rangle - \overline{y}_{k+1} \leq b_t$, hence $\langle a_t, \overline{x} \rangle - \overline{x}_{n+1} \leq b_t$.

We can use the same argument for concluding that $\langle c_t, \overline{x} \rangle \leq d_t$, for each $t \in T_2$ and $\binom{\overline{x}}{\overline{x}_{n+1}}$ is a solution of σ.

Now, let $\binom{\overline{x}}{\overline{x}_{n+1}}$ be a solution of σ. Then $\langle c_j, \overline{x} - b_0 \rangle = 0$, for all $j = 1, ..., n-k$, and $\overline{x} - b_0 \in \operatorname{span}\{b_1 - b_0, ..., b_k - b_0\}$. We write $\overline{x} = A\overline{y} + b_0$, with $\overline{y} \in \mathbb{R}^k$. Since $\langle a_t, \overline{x} \rangle - \overline{x}_{n+1} \leq b_t$, for all $t \in T_1$, we obtain

$$\langle A^* a_t, \overline{y} \rangle - \overline{x}_{n+1} \leq b_t - \langle a_t, b_0 \rangle,$$

hence, for all $t \in T_1$, it holds $\langle \widetilde{a}_t, \overline{y} \rangle - \overline{x}_{n+1} \leq \widetilde{b}_t$. With the same argument, we obtain $\langle \widetilde{c}_t, \overline{y} \rangle \leq \widetilde{d}_t$, for all $t \in T_2$. Then $\binom{\overline{y}}{\overline{x}_{n+1}}$ is a solution of $\widetilde{\sigma}$, which implies that $\binom{\overline{y}}{\overline{x}_{n+1}} \in \operatorname{epi} f_1$, and $\binom{\overline{x}}{\overline{x}_{n+1}} = \widehat{S}\binom{\overline{y}}{\overline{x}_{n+1}} \in \operatorname{epi} f$.

Next we shall see that σ is LOP. Let us observe that, since $\operatorname{dom} f = S(\operatorname{dom} f_1)$, $\overline{x} \in \operatorname{dom} f$ if and only if $A\overline{y} + b_0 = \overline{x}$, where $\overline{y} \in \operatorname{dom} f_1$. Moreover, $f(\overline{x}) = f_1(\overline{y})$, hence

$$T_1(\overline{x}) = \widetilde{T}_1(\overline{y}) \quad \text{and} \quad T_2(\overline{x}) = \widetilde{T}_2(\overline{y}). \tag{13}$$

As a consequence of these equalities, since $\widetilde{T}_1(\overline{y}) \neq \emptyset$, for all $\overline{y} \in \operatorname{dom} f_1$, we obtain $T_1(\overline{x}) \neq \emptyset$, for all $\overline{x} \in \operatorname{dom} f$.

The system σ is LOP if, for all $\binom{x}{x_{n+1}} \in \operatorname{epi} f$,

$$A_\sigma \begin{pmatrix} x \\ x_{n+1} \end{pmatrix}^\circ \subset D_{\operatorname{epi} f} \begin{pmatrix} x \\ x_{n+1} \end{pmatrix},$$

because the other inclusion always holds.

Let $\binom{\overline{x}}{\overline{x}_{n+1}} \in \operatorname{epi} f$ and suppose that $\overline{x}_{n+1} = f(\overline{x})$. We have

$$A_\sigma \begin{pmatrix} \overline{x} \\ f(\overline{x}) \end{pmatrix} = \operatorname{cone} \left\{ \begin{pmatrix} a_t \\ -1 \end{pmatrix}, \ t \in T_1(\overline{x}), \begin{pmatrix} c_t \\ 0 \end{pmatrix}, \ t \in T_2(\overline{x}); \ \begin{pmatrix} \pm c_j \\ 0 \end{pmatrix}, , j = 1, ..., n-k \right\}.$$

Take $0_{n+1} \neq \binom{v}{v_{n+1}} \in A_\sigma \binom{\overline{x}}{f(\overline{x})}^\circ$. Then $v \in \operatorname{span}\{b_1 - b_0, ..., b_k - b_0\}$ and $v = Aw$, for a certain $w \in \mathbb{R}^k$. Let $\overline{y} := A^{-1}(\overline{x} - b_0)$. Taking into account (13), it is easy to see that $\binom{w}{v_{n+1}} \in A_{\tilde{\sigma}} \binom{\overline{y}}{f_1(\overline{y})}^\circ = D_{\operatorname{epi} f_1} \binom{\overline{y}}{f_1(\overline{y})}$, because $\widetilde{\sigma}$ is LOP. Hence there exists $\lambda > 0$ such that

$$\begin{pmatrix} \overline{y} \\ f_1(\overline{y}) \end{pmatrix} + \lambda \begin{pmatrix} w \\ v_{n+1} \end{pmatrix} \in \operatorname{epi} f_1,$$

and

$$\widehat{S}\left(\begin{pmatrix} \overline{y} \\ f_1(\overline{y}) \end{pmatrix} + \lambda \begin{pmatrix} w \\ v_{n+1} \end{pmatrix} \right) = \begin{pmatrix} \overline{x} \\ f(\overline{x}) \end{pmatrix} + \lambda \begin{pmatrix} v \\ v_{n+1} \end{pmatrix} \in \operatorname{epi} f.$$

We obtain $\binom{v}{v_{n+1}} \in D_{\operatorname{epi} f} \binom{\overline{x}}{f(\overline{x})}$.

The case $\overline{x}_{n+1} > f(\overline{x})$ can be done in a similar way, and it is simpler since

$$A_\sigma \begin{pmatrix} \overline{x} \\ \overline{x}_{n+1} \end{pmatrix} = \operatorname{cone} \left\{ \begin{pmatrix} c_t \\ 0 \end{pmatrix}, \ t \in T_2(\overline{x}); \begin{pmatrix} \pm c_j \\ 0 \end{pmatrix}, j = 1, ..., n-k \right\}.$$

Finally we shall see that σ verifies the desidered property. Let P be a non-empty polytope, $P \subset \operatorname{dom} f$. We have that $P \subset A(\operatorname{dom} f_1) + b_0$, and $P - b_0 \subset A(\operatorname{dom} f_1)$. Then $A^{-1}(P - b_0) \subset \operatorname{dom} f_1$ is a non-empty polytope in $\mathbb{R}^k$ and $\widetilde{T}_1(A^{-1}(P - b_0))$ is non-empty and finite, whereas $\widetilde{T}_2(A^{-1}(P - b_0))$ is finite. From the fact that, for all $x \in P$, $T_i(x) = \widetilde{T}_i(A^{-1}(x - b_0))$, $i = 1, 2$, we obtain $\widetilde{T}_i(A^{-1}(P - b_0)) = T_i(P)$, $i = 1, 2$. □

In view of Propositions 3.1 and 3.2, we can state the following theorem whose proof is a direct consequence of these propositions.

Theorem 3.3. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function. If f is quasipolyhedral, then there exists a representation of f, $\sigma = \{\langle a_t, x\rangle - x_{n+1} \leq b_t, \ t \in T_1; \langle c_t, x\rangle \leq d_t, \ t \in T_2\}$ such that for every non-empty polytope P in* dom f, *$T_1(P)$ (respectively, $T_2(P)$) is a non-empty finite index subset (respectively, finite). Moreover, σ is* LOP.

Next we look for a converse statement of this theorem. It is necessary to include an additional hypothesis for f in view of Lemma 2.7.

Theorem 3.4. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function. Suppose that, for all $x \in (\operatorname{rbd} \operatorname{dom} f) \cap \operatorname{dom} f$, there exists a polytope P such that $x \in P \subset \operatorname{dom} f$ and $D_P(x) = D_{\operatorname{dom} f}(x)$.*

If there exists a representation of f, $\sigma = \{\langle a_t, x\rangle - x_{n+1} \le b_t,\ t \in T_1; \langle c_t, x\rangle \le d_t,\ t \in T_2\}$, such that for every non-empty polytope $\widehat{P} \subset \operatorname{dom} f$, $T_1\left(\widehat{P}\right)$ is a non-empty finite index subset, then f is quasipolyhedral.

Proof. Denote $k = \dim f$. Since $\operatorname{epi} f$ is a non-empty convex set, it will be quasipolyhedral if and only if $D_{\operatorname{epi} f}\binom{x}{x_{n+1}}$ is polyhedral, for all $\binom{x}{x_{n+1}} \in \operatorname{epi} f$, according to [1, VII.1.6]. However, because $\operatorname{epi} f$ is also closed, in the case $\binom{x}{x_{n+1}} \in \operatorname{rint} \operatorname{epi} f$, it holds

$$D_{\operatorname{epi} f}\begin{pmatrix} x \\ x_{n+1} \end{pmatrix} = \begin{cases} \mathbb{R}^{n+1}, & \text{if } k = n, \\ \mathbb{R}^{k+1} \times \{0_{n-k}\}, & \text{if } k < n, \end{cases}$$

which is polyhedral.

Hence, take any $\binom{\overline{x}}{\overline{x}_{n+1}} \in \operatorname{rbd} \operatorname{epi} f$. According to Lemma 2.5 (in the case $\overline{x} \in \operatorname{rint} \operatorname{dom} f$) or by hypothesis (if $\overline{x} \in (\operatorname{rbd} \operatorname{dom} f) \cap \operatorname{dom} f$), there exists a polytope $P \subset \operatorname{dom} f$ verifying $\overline{x} \in P$ and $D_P(\overline{x}) = D_{\operatorname{dom} f}(\overline{x})$. Moreover, $T_1(P)$ is non-empty and finite.

Consider a representation of P

$$\{\langle a_i, x\rangle \le b_i, i = 1, ..., p\}.$$

Since f is continuous relative to P, let $\alpha \le f(x)$, for all $x \in P$. Define the non-empty convex set

$$\overline{P} := \{P \times [\alpha, +\infty[\} \cap \operatorname{epi} f.$$

We shall prove that $\overline{P}$ is the solution set of the (finite) system:

$$\sigma\left(\overline{P}\right) := \left\{ \begin{array}{l} \langle a_i, x\rangle \le b_i, i = 1, ..., p,\ -x_{n+1} \le -\alpha, \\ \langle a_t, x\rangle - x_{n+1} \le b_t,\ t \in T_1(P) \end{array} \right\},$$

then $\overline{P}$ will be a polyhedral set.

It is evident that $\overline{P}$ is contained in the solution set of $\sigma\left(\overline{P}\right)$, since it is the solution set of

$$\left\{ \begin{array}{l} \langle a_i, x\rangle \le b_i, i = 1, ..., p,\ -x_{n+1} \le -\alpha, \\ \langle a_t, x\rangle - x_{n+1} \le b_t,\ t \in T_1,\ \langle c_t, x\rangle \le d_t,\ t \in T_2 \end{array} \right\},$$

and $\sigma\left(\overline{P}\right)$ is contained in this system.

On the other hand, if $\binom{\overline{y}}{\overline{y}_{n+1}}$ is a solution of $\sigma\left(\overline{P}\right)$, then $\binom{\overline{y}}{\overline{y}_{n+1}} \in \{P \times [\alpha, +\infty[\}$. Taking any $t_0 \in T_1(\overline{y}) \subset T_1(P)$, it follows that

$$f(\overline{y}) = \langle a_{t_0}, \overline{y}\rangle - b_{t_0}.$$

Since $\binom{\overline{y}}{\overline{y}_{n+1}}$ satisfies the inequality associated with t_0, we have $\overline{y}_{n+1} \ge \langle a_{t_0}, \overline{y}\rangle - b_{t_0} = f(\overline{y})$. Hence, $\binom{\overline{y}}{\overline{y}_{n+1}} \in \operatorname{epi} f \cap \{P \times [\alpha, +\infty[\} = \overline{P}$.

Next we shall prove that $D_{\overline{P}}\binom{\overline{x}}{\overline{x}_{n+1}} = D_{\operatorname{epi} f}\binom{\overline{x}}{\overline{x}_{n+1}}$. Since inclusion "$\subset$" is evident, let $0_{n+1} \neq \binom{v}{v_{n+1}} \in D_{\operatorname{epi} f}\binom{\overline{x}}{\overline{x}_{n+1}}$ and $\lambda > 0$ such that

$$\begin{pmatrix} \overline{x} \\ \overline{x}_{n+1} \end{pmatrix} + \lambda \begin{pmatrix} v \\ v_{n+1} \end{pmatrix} \in \operatorname{epi} f.$$

Since $\overline{x} + \lambda v \in \operatorname{dom} f$, it holds $v \in D_{\operatorname{dom} f}(\overline{x}) = D_P(\overline{x})$, and we can assume that $\overline{x} + \lambda v \in P$. Moreover, $\alpha \leq f(\overline{x} + \lambda v) \leq \overline{x}_{n+1} + \lambda v_{n+1}$. We conclude

$$\begin{pmatrix} \overline{x} \\ \overline{x}_{n+1} \end{pmatrix} + \lambda \begin{pmatrix} v \\ v_{n+1} \end{pmatrix} \in \operatorname{epi} f \cap \{P \times [\alpha, +\infty[\} = \overline{P},$$

hence $\binom{v}{v_{n+1}} \in D_{\overline{P}}\binom{\overline{x}}{\overline{x}_{n+1}}$. Since $\overline{P}$ is polyhedral, it follows that $D_{\overline{P}}\binom{\overline{x}}{\overline{x}_{n+1}}$ will be a polyhedral cone, obtaining what we have aimed. □

4. Subdifferential and conjugate of a quasipolyhedral function

The subdifferential set at a point of a quasipolyhedral function verifies the same property as a polyhedral function. The case where the function is improper has no interest, since it is always $-\infty$ in every point of its effective domain.

Proposition 4.1. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper quasipolyhedral function and let $x \in \operatorname{dom} f$. Then $\partial f(x) \neq \emptyset$ and it is a polyhedral set. In particular, if $x \in \operatorname{rint} \operatorname{dom} f$, then $\partial f(x)$ is a polytope.*

Proof. Let $\overline{x} \in \operatorname{dom} f$. Applying Lemmas 2.5 and 2.7, there exists a polytope P such that $\overline{x} \in P$, $P \subset \operatorname{dom} f$ and $D_P(\overline{x}) = D_{\operatorname{dom} f}(\overline{x})$.

Consider the lsc convex function

$$g(x) := f(x) + \delta_P(x).$$

According to Lemma 2.6, g is a polyhedral function. Applying [9, Th. 23.10], $\partial g(\overline{x}) \neq \emptyset$ and it is a polyhedral set. Take $u \in \partial g(\overline{x})$. If $y \in P$, since $g(y) = f(y)$, we have

$$f(y) \geq f(\overline{x}) + \langle u, y - \overline{x} \rangle.$$

In the case $y \notin P$, but $y \in \operatorname{dom} f$, then $y - \overline{x} \in D_{\operatorname{dom} f}(\overline{x}) = D_P(\overline{x})$. Let $0 < \lambda < 1$ verifying $z := \overline{x} + \lambda(y - \overline{x}) \in P$. Then

$$f(z) \geq f(\overline{x}) + \langle u, z - \overline{x} \rangle.$$

Now $z = \lambda y + (1 - \lambda)\overline{x}$ and $f(z) \leq \lambda f(y) + (1 - \lambda) f(\overline{x})$, hence

$$\begin{aligned} f(y) &\geq \frac{1}{\lambda}\{f(z) - (1 - \lambda) f(\overline{x})\} \\ &\geq \frac{1}{\lambda}\{f(\overline{x}) + \langle u, z - \overline{x} \rangle - (1 - \lambda) f(\overline{x})\} \\ &= f(\overline{x}) + \left\langle u, \frac{1}{\lambda}(z - \overline{x}) \right\rangle = f(\overline{x}) + \langle u, y - \overline{x} \rangle. \end{aligned}$$

We conclude that $u \in \partial f(\overline{x})$ and $\partial f(\overline{x}) \neq \emptyset$.

Since δ_P is polyhedral and $\operatorname{dom} \delta_P \cap \operatorname{rint} \operatorname{dom} f \neq \emptyset$, according to [9, Th. 23.8] we have

$$\partial g(\overline{x}) = \partial f(\overline{x}) + \partial \delta_P(\overline{x}).$$

But $\partial \delta_P(\overline{x}) = N_P(\overline{x}) = N_{\operatorname{dom} f}(\overline{x}) = 0^+(\partial f(\overline{x}))$ because $\partial f(\overline{x}) \neq \emptyset$, hence $\partial f(\overline{x}) + \partial \delta_P(\overline{x}) = \partial f(\overline{x})$ and $\partial g(\overline{x}) = \partial f(\overline{x})$.

We conclude that for all $\overline{x} \in \operatorname{dom} f$, $\partial f(\overline{x}) \neq \emptyset$ and it is polyhedral. In the case $\overline{x} \in \operatorname{rint} \operatorname{dom} f$, it is also bounded, hence it is a polytope. □

We continue with the study of the conjugate of a quasipolyhedral function. In the case that f is polyhedral, its conjugate is also polyhedral (see [9, Th. 19.2]), but it can not be generalized for a quasipolyhedral function.

Example 4.2. Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be defined

$$f(x) := \sup\left\{-\frac{1}{r(r+1)}x + \frac{2r+1}{r(r+1)},\ r \in \mathbb{N}\right\} + \delta_C(x),$$
$$C := \{x \in \mathbb{R} \mid -x \leq 0\}.$$

It is easy to see that f is quasipolyhedral, but its conjugate,

$$f^*(u) = \sup\left\{-(r+1)u - \frac{1}{(r+1)},\ r \in \mathbb{N}\right\} + \delta_{C'}(u),$$
$$C' = \{u \in \mathbb{R} \mid u \leq 0\},$$

verifies that $\operatorname{epi} f^*$ has a convergent sequence of extreme points,

$$\left\{\begin{pmatrix} -\frac{1}{r(r+1)} \\ \frac{2r+1}{r(r+1)} \end{pmatrix},\ r \in \mathbb{N}\right\},$$

and hence it is not a quasipolyhedral set, according to [5, Th. 5.6(ii)].

Let us observe that if f is an improper quasipolyhedral function, then $f^* \equiv +\infty$, which is quasipolyhedral. Hence, from now on we will consider proper quasipolyhedral functions. The following property will be necessary for the quasipolyhedrality of its conjugate.

Definition 4.3. A proper lsc convex function f on $\mathbb{R}^n$ is called *co-finite* if

$$0^+(\operatorname{epi} f) = \operatorname{cone}\left\{\begin{pmatrix} 0_n \\ 1 \end{pmatrix}\right\}. \tag{14}$$

In the above Example, $0^+(\operatorname{epi} f) = \operatorname{cone}\left\{\binom{0}{1}, \binom{1}{0}\right\} \supsetneq \left\{\binom{0}{1}\right\}$, and f is not co-finite.

Corollary 13.3.1 in [9] establishes that a lsc proper convex function f on $\mathbb{R}^n$ is co-finite if and only if $\operatorname{dom} f^* = \mathbb{R}^n$; i.e., if and only if its conjugate is finite-valued. Moreover, in [4, Lemma 5.1], it is proved that f is co-finite if and only if it is *1-coercive*; i.e.,

$$\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty. \tag{15}$$

Proposition 4.4. *Let f be a proper quasipolyhedral function on $\mathbb{R}^n$ which is co-finite. Then f^* is quasipolyhedral.*

Proof. The quasipolyhedrality of f^* can be proven following the same steps as in the proof of Proposition 5.2 in [4]. In both cases, $\operatorname{dom} f^* = \mathbb{R}^n$, and for all $u \in \mathbb{R}^n$, $\partial f^*(u) \neq \emptyset$ and it is a compact set, according to Theorem 23.5 in [9].We only have to take care in two statements of that proof in which it is used the continuity of f in the whole space, and we have an alternative proof of them:

The first statement in the proof of Proposition 5.2 in [4] where the continuity of f is applied is the compactness of the set, in $\mathbb{R}^{n+1}$,

$$F(u) := \left\{ \begin{pmatrix} x \\ f(x) \end{pmatrix},\ x \in \partial f^*(u) \right\},$$

for each $u \in \mathbb{R}^n$. In our case, $F(u) \neq \emptyset$, because $\partial f^*(u) \neq \emptyset$ and for all $x \in \partial f^*(u)$ it holds $f(x) = \langle u, x\rangle - f^*(u) < \infty$.

Let us check that $F(u)$ is closed. Take $\left\{ \binom{x^r}{f(x^r)} \right\}_{r=1}^{\infty} \subset F(u)$, converging to $\binom{x^0}{\alpha}$. Then $\{x^r\}_{r=1}^{\infty} \subset \partial f^*(u)$ converges to x^0, and $x^0 \in \partial f^*(u)$. Since $f(x^0) = \langle u, x^0\rangle - f^*(u)$ and $f(x^r) = \langle u^r, x\rangle - f^*(u)$ for all $r \in \mathbb{N}$, we have

$$\alpha = \lim_{r\to\infty} f(x^r) = \lim_{r\to\infty} \langle u, x^r\rangle - f^*(u) = \left\langle u, x^0 \right\rangle - f^*(u) = f(x^0).$$

Hence $\binom{x^0}{\alpha} = \binom{x^0}{f(x^0)} \in F(u)$.

Now we shall prove that $F(u)$ is also bounded. Assuming the contrary, for all $r \in \mathbb{N}$ there would exist $\binom{x^r}{f(x^r)} \in F(u)$ such that $\left\| \binom{x^r}{f(x^r)} \right\|_\infty > r$. Since $\{x^r\} \subset \partial f^*(u)$, we can assume that $|f(x^r)| > r$, for all $r \in \mathbb{N}$. Now take a convergent subsequence of $\{x^r\}$, $\{x^{r_k}\}$ and let $x^0 = \lim_{k\to\infty} x^{r_k}$. Then

$$\lim_{k\to\infty} f\left(x^{r_k}\right) = \lim_{r\to\infty} \langle u, x^{r_k}\rangle - f^*(u) = \left\langle u, x^0 \right\rangle - f^*(u),$$

which implies that $\{|f(x^r)|\}_{r=1}^{\infty}$ has a finite accumulation point, and this is impossible. We conclude that $F(u)$ is compact.

The second statement in the proof of Proposition 5.2 in [4] where the continuity of f is applied is the finiteness of the set $T^*(C)$, where C is a non-empty compact set in $\mathbb{R}^n$.

In our case, let C be a non-empty compact set in $\mathbb{R}^n$ and suppose that $T^*(C)$ is not finite. Then there will exist, for all $r \in \mathbb{N}$, $u^r \in C$ and $x^r \in T^*(u^r)$ such that

$$f^*(u^r) = \langle u, x^r\rangle - f\left(x^r\right), \tag{16}$$

with $\binom{x^r}{f(x^r)}$ being an extreme point of $\operatorname{epi} f$.

$\left\{ \binom{x^r}{f(x^r)} \right\}_{r=1}^{\infty}$ is a sequence of non-repeated extreme points of $\operatorname{epi} f$, then it has no finite accumulation points, i.e.,

$$\lim_{r\to\infty} \left\| \begin{pmatrix} x^r \\ f\left(x^r\right) \end{pmatrix} \right\| = +\infty. \tag{17}$$

Since $\{u^r\}_{r=1}^{\infty} \subset C$, and this is a compact set, taking a subsequence, if it is necessary, we have $\lim_{r\to\infty} u^r = \overline{u} \in C$.

If $\{x^r\}_{r=1}^{\infty}$ were a convergent sequence and $\lim_{r\to\infty} x^r = x^0$, then taking limits in (16) we have

$$\lim_{r\to\infty} f(x^r) = \langle u, x^0 \rangle - f^*(\overline{u}),$$

because f^* is continuous in $\mathbb{R}^n$, contradicting (17). Then $\lim_{r\to\infty} \|x^r\| = +\infty$, and we can assume w.l.o.g. that $\lim_{r\to\infty} x^r (\|x^r\|)^{-1} = y$.

From (16) we obtain

$$\langle \overline{u}, y \rangle = \lim_{r\to\infty} \left\langle u^r, \frac{x^r}{\|x^r\|} \right\rangle = \lim_{r\to\infty} \frac{f^*(u^r)}{\|x^r\|} + \frac{f(x^r)}{\|x^r\|}. \tag{18}$$

Now $\lim_{r\to\infty} f^*(u^r) = f^*(\overline{u})$ and f is 1-coercive, hence we get, taking limits in (18), that $\langle \overline{u}, y \rangle = +\infty$. Then $T^*(C)$ is finite. □

Every quasipolyhedral set C has at most a countable set of extreme points, because if we consider the sequence of polytopes $\{\overline{y} + r \operatorname{cl} \mathbb{B}_\infty\}_{r=1}^{\infty}$, where $\overline{y} \in C$, we have that $\{\overline{y} + r \operatorname{cl} \mathbb{B}_\infty\} \cap C$ is a polytope, for all $r \in \mathbb{N}$. This implies that its number of extreme points is finite. Since every extreme point of C will be in some $\{\overline{y} + r \operatorname{cl} \mathbb{B}_\infty\}$, it will be an extreme point of $\{\overline{y} + r \operatorname{cl} \mathbb{B}_\infty\} \cap C$. We conclude that the set of extreme points of C will be contained in the union of all the sets of extreme points of $\{\overline{y} + r \operatorname{cl} \mathbb{B}_\infty\} \cap C$, for $r \in \mathbb{N}$, and this union has a cardinal which is countable. This property will be verified by the epigraph of a quasipolyhedral function. Following the same steps as the proof of Proposition 5.2 in [4], if f is quasipolyhedral and co-finite, the system

$$\sigma^* = \{\langle a_t, u \rangle - u_{n+1} \le b_t, t \in T^*\},$$

where $\left\{\binom{a_t}{b_t},\ t \in T^*\right\}$ is the (at most countable) set of extreme points of $\operatorname{epi} f$, is a representation of f^*. Next we shall find a representation of f in terms of the set of extreme points and of extreme directions of $\operatorname{epi} f^*$. Since any convex set whose lineality is non-zero has neither extreme points nor extreme directions, we only consider the case in which lineality $\operatorname{epi} f^* = 0$; i.e., $\operatorname{lin}(\operatorname{epi} f^*) = \{0_{n+1}\}$ which means that $\operatorname{epi} f^*$ does not contain lines. This is equivalent to lineality $f^* = 0$, and, according to [9, Th. 13.4], it holds if and only if $\dim f = n$.

Proposition 4.5. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper quasipolyhedral co-finite function, $\dim f = n$. Then there exists a representation of f,*

$$\sigma = \{\langle a_t, x \rangle - x_{n+1} \le b_t,\ t \in T_1; \langle c_t, x \rangle \le d_t,\ t \in T_2\},$$

where $\left\{\binom{a_t}{b_t},\ t \in T_1\right\}$ is the (non-empty) set of extreme points of $\operatorname{epi} f^*$ and $\left\{\binom{c_t}{d_t},\ t \in T_2\right\}$ is the (possibly empty) set of extreme directions of $\operatorname{epi} f^*$.

Proof. According to [9, Cor. 18.5.3], $T_1 \neq \emptyset$, hence define $h(x) := \sup\{\langle a_t, x \rangle - b_t,\ t \in T_1\}$ and $C := \{x \in \mathbb{R}^n \,|\, \langle c_t, x \rangle \le d_t,\ t \in T_2\}$, if $T_2 \neq \emptyset$ or $C := \mathbb{R}^n$, in the case $T_2 = \emptyset$. We shall see that

$$f(x) = h(x) + \delta_C(x);$$

i.e., $f(x) = h(x)$ for all $x \in \operatorname{dom} f$ and $\operatorname{dom} f = \operatorname{dom} h \cap C$.

Let $\overline{x} \in \operatorname{dom} f$. Then $\partial f(\overline{x}) \neq \emptyset$, applying Proposition 4.1. Define

$$G(\overline{x}) := \left\{ \begin{pmatrix} u \\ u_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1} \middle| \langle u, x \rangle - u_{n+1} = f(\overline{x}) \right\} \cap \operatorname{epi} f^*.$$

Actually, $G(\overline{x}) := \left\{ \binom{u}{f^*(u)} \in \mathbb{R}^{n+1} \middle| u \in \partial f(\overline{x}) \right\} \neq \emptyset$. Then $G(\overline{x})$ is a non-empty exposed face of $\operatorname{epi} f^*$ which does not contain lines, since $\operatorname{epi} f^*$ does not contain lines. According to [9, Cor. 18.5.3], $G(\overline{x})$ has at least an extreme point, which will be also an extreme point of $\operatorname{epi} f^*$. Hence we have proved that, for all $x \in \operatorname{dom} f$, there exists $u \in \mathbb{R}^n$ such that $\langle u, x \rangle - f^*(u) = f(x)$ and $\binom{u}{f^*(u)}$ is an extreme point of $\operatorname{epi} f^*$.

Since $f = f^{**}$, we have, for all $x \in \operatorname{dom} f$,

$$\langle u, x \rangle - f^*(u) = f(x) = \sup \{ \langle v, x \rangle - f^*(v),\ v \in \mathbb{R}^n \} \geq h(x) \geq \langle u, x \rangle - f^*(u).$$

We conclude that $\operatorname{dom} f \subset \operatorname{dom} h$ and

$$f(x) = h(x), \quad \text{for all } x \in \operatorname{dom} f.$$

Now, if $\operatorname{epi} f^*$ had extreme directions, they would be contained in $0^+(\operatorname{epi} f^*)$. According to [9, Th. 13.3], $0^+(\operatorname{epi} f^*) = \operatorname{epi} \delta^*_{\operatorname{dom} f}$, hence, for all $t \in T_2$, $\delta^*_{\operatorname{dom} f}(c_t) \leq d_t$. This implies, that, for all $t \in T_2$ and for all $x \in \operatorname{dom} f$, $\langle c_t, x \rangle \leq d_t$, and $\operatorname{dom} f \subset C$. Finally, we shall see that $C \cap \operatorname{dom} h \subset \operatorname{dom} f$.

Let $x \in C$ and $h(x) < +\infty$. We have,

$$\left\langle \begin{pmatrix} c_t \\ d_t \end{pmatrix}, \begin{pmatrix} x \\ -1 \end{pmatrix} \right\rangle \leq 0, \quad \text{for all } t \in T_2, \quad \text{and} \tag{19}$$
$$\left\langle \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \begin{pmatrix} x \\ -1 \end{pmatrix} \right\rangle \leq h(x), \quad \text{for all } t \in T_1.$$

Applying [9, Th. 18.5], we obtain

$$\operatorname{epi} f^* = \operatorname{conv} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in T_1 \right\} + \operatorname{cone} \left\{ \begin{pmatrix} c_t \\ d_t \end{pmatrix},\ t \in T_2 \right\}.$$

Then, for all $u \in \mathbb{R}^n$,

$$\begin{pmatrix} u \\ f^*(u) \end{pmatrix} = \sum_{t \in T_1} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{t \in T_2} \beta_t \begin{pmatrix} c_t \\ d_t \end{pmatrix},$$

where $\lambda_t \geq 0$, for all $t \in T_1$, $\sum_{t \in T_1} \lambda_t = 1$ and $\beta_t \geq 0$, for all $t \in T_2$. Taking the inner product of both members in the above equality with $\binom{x}{-1}$ and recalling (19), we get

$$\langle u, x \rangle - f^*(u) \leq h(x),$$

and $f(x) = \sup \{ \langle u, x \rangle - f^*(u),\ u \in \mathbb{R}^n \} < +\infty$. Therefore $x \in \operatorname{dom} f$ and $f(x) = h(x) + \delta_C(x)$, for all $x \in \mathbb{R}^n$. □

Example 4.6. Let $f(x) := \sup\{(2r+1)\,x - (r^2+r)\,,\ r \in \mathbb{N}\cup\{0\}\} + \delta_C\,(x)\,,$ where $C := \{x \in \mathbb{R}\,|{-x} \leq 0\}$. It is easy to show that f is co-finite and quasipolyhedral. The set of extreme points of epi f is

$$\left\{\begin{pmatrix} r \\ r^2 \end{pmatrix},\ r \in \mathbb{Z}\right\}.$$

Hence

$$f^*(u) = \sup\left\{ru - r^2,\ r \in \mathbb{N}\cup\{0\}\right\}.$$

The set of extreme points of epi f^* is

$$\left\{\begin{pmatrix} 2r+1 \\ r^2+r \end{pmatrix},\ \mathbb{N}\cup\{0\}\right\},$$

and $\binom{-1}{0}$ is the unique extreme direction of epi f^*.

Example 4.7. Let $f(x) := \sup\{-r\,(r+1)\,x+,\ r \in \mathbb{N}; (2s+1)\,x - (s^2+s)\,,\ s \in \mathbb{N}\}\,.$ It is easy to show that f is co-finite and quasipolyhedral. The set of extreme points of epi f is

$$\left\{\begin{pmatrix} \frac{1}{r} \\ r \end{pmatrix},\ r \in \mathbb{N}; \begin{pmatrix} s \\ s^2 \end{pmatrix},\ s \in \mathbb{N}\setminus\{1\}\right\}.$$

Hence

$$f^*(u) = \sup\left\{\frac{1}{r}u - r,\ r \in \mathbb{N}; su - s^2,\ s \in \mathbb{N}\setminus\{1\}\right\}.$$

The set of extreme points of epi f^* is

$$\left\{\begin{pmatrix} -r\,(r+1) \\ -\,(2r+1) \end{pmatrix},\ r \in \mathbb{N}; \begin{pmatrix} 2s+1 \\ s^2+r \end{pmatrix},\ s \in \mathbb{N}\right\},$$

and epi f^* has no extreme directions.

References

[1] J. Bair, R. Fourneau: Etude Géometrique des Espaces Vectoriels II: Polyèdres et Polytopes Convexes, Springer, Berlin (1980).

[2] A. Brønsted: An Introduction to Convex Polytopes, Springer, New York (1983).

[3] M. D. Fajardo, M. A. López: Some results on quasipolyhedral convexity, Top 15(1) (2007) 103–113.

[4] M. D. Fajardo, M. A. López, R. Puente: Linear representations and quasipolyhedrality of a finite-valued convex function, Optimization 57(2) (2008) 215–237.

[5] M. A. Goberna, M. A. López: Linear Semi-Infinite Optimization, Wiley, Chichester (1998).

[6] J. B. Hiriart-Urruty, C. Lemarechal: Convex Analysis and Minimization Algorithms. Part I: Fundamentals, Springer, Berlin (1991).

[7] M. R. Osborne: Polyhedral function constrained optimization problems, AZIAM J. 46 (2005) C196–C209.

[8] R. R. Phelps: Convex Functions, Monotone Operators and Differentiability, 2nd Ed., Springer, Berlin (1993).

[9] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton (1970).