Journal of Convex Analysis
Volume 17 (2010), No. 1, 301–308

Lagrange Multipliers and Lower Bounds for Integral Functionals

Emmanuel Giner
Laboratoire MIP Université Paul Sabatier, UFR MIG,
118 Route de Narbonne, 31062 Toulouse cedex 04, France
giner@mip.ups-tlse.fr

Received: June 3, 2008
Revised manuscript received: March 4, 2009

We present a large class of examples with the remarkable property pointed out by Professor B. Ricceri (see [13], [14], [15]), about lower bounds of integral functionals. We use only a Lagrange duality result contained in [5].

Keywords: Decomposability, richness, essential infimum, measurable integrand, integral functional, duality, Lagrange multipliers

2000 Mathematics Subject Classification: 26A51, 26B20, 26E15, 28B15, 28B20, 46E30, 46N10, 49K40

1. Introduction

Except in the classical case of the interchange between infimum and integration considered by R. T. Rockafellar in [16], [17], [18], an exact computation of the lower bound for a minimization problem involving integral functionals is not in general an easy problem. In [13], [14] and [15], B. Ricceri considers the minimisation of elements of a class of integral functionals (possibly continuous) on a equality constraint defined by a special Lipschitzian integral functional and shows that the lower bound on the constraints is equal to the lower bound on the whole underlying decomposable space; the variety of the possible cost functions and constraints being surprising. We give an explanation of this phenomenon. When the measured space is nonatomic, even if the problem involving integral functionals is nonconvex, some convexity properties may appear, for example the existence of a duality formula with Lagrange multipliers, [5] Theorem 5.1, Theorem 2.4. This last duality result is true for the problems of B. Ricceri. We prove that the only Lagrange multiplier associated to the dual of these problems is the null multiplier and since every minimisation problem with this particular feature has the lower bound property put in light by B. Ricceri, we produce another class of examples.

2. Preliminaries

In the sequel we note $\overline{I\!R} = I\!R \cup \{\pm\infty\}$. Given a set X, $\overline{I\!R}$-valued functions f and g defined on X we denote $\operatorname{dom} f = \{x \in X : f(x) < \infty\}$, and $\{f = g\}$ is the set $\{x \in X : f(x) = g(x)\}$. If X is a normed vector space recall that the Fenchel

ISSN 0944-6532 / \$ 2.50

subdifferential $\partial f(x_0)$ of f at a point $x_0 \in \operatorname{dom} f$ is the closed convex subset of the topological dual X^* of X defined by:

$$\partial f(x_0) = \{x^* \in X^* : \forall x \in X,\ f(x) \geq f(x_0) + \langle x^*, x - x_0\rangle\}$$

and the Fenchel conjugate function f^* is given by: $f^*(x^*) = \sup_{x\in X}\langle x^*, x\rangle - f(x)$.
Let $(\Omega, \mathcal{T}, \mu)$ be a measured space by a σ-finite positive nonatomic measure μ, with tribe $\mathcal{T}$. For a measurable set $A \in \mathcal{T}$, we set $A^c = \{\omega \in \Omega : \omega \notin A\}$ and 1_A stands for the characteristic function of A, $1_A(\omega) = 1$ if $\omega \in A$, 0 if $\omega \notin A$. Let $(E, \|.\|)$ be a Banach space with Borel tribe $\mathcal{B}(E)$. Denote by $L_0(\Omega, E)$ the space of classes of measurable functions (for μ-almost everywhere equality) defined on Ω and with values in E, and by $L_p(\Omega, E)$, $1 \leq p \leq \infty$ the classical Lebesgue space of (classes of) functions endowed with the usual norm. Given a family $(u_i)_{i\in I}$ of elements of $L_0(\Omega, \overline{\mathbb{R}})$, we will use the essential infimum of this family, $\operatorname{ess\,inf}_I u_i$, introduced by J. Neveu in [11], II.4. The following notion is classic.

Definition 2.1 (See [7], Section 3). A subset X of $L_0(\Omega, E)$ is said to be decomposable if, for every x, y in X and every measurable set A, the function $x1_A + y1_{A^c}$ is an element of X.

Let M be a multifunction defined on Ω with values in E, and let $L_p(M)$ be the set of measurable selections of M (almost everywhere) which are in $L_p(\Omega, E)$.

Definition 2.2 ([5], Section 3). If X and Y are two subsets of $L_0(\Omega, E)$, X is said rich in Y if X is a subset of Y and if for any y in Y, there exist an increasing covering $(\Omega_n)_n$ of Ω by measurable sets of finite measure and a sequence $(x_n)_n$ of elements of X verifying for all $n \in \mathbb{N}$, $y1_{\Omega_n} = x_n 1_{\Omega_n}$.

In the sequel $f : \Omega \times E \to \overline{\mathbb{R}}$, (respectively $g : \Omega \times E \to \overline{\mathbb{R}}^d$) is a $\mathcal{T} \otimes \mathcal{B}(E)$-measurable integrand. For an element x of $L_0(\Omega, E)$, the function $f(x)$ (respectively $g(x)$) is defined by: $f(x)(\omega) = f(\omega, x(\omega))$ (respectively $g(x)(\omega) = g(\omega, x(\omega))$. The interest of the notion of richness may be the following lemma:

Lemma 2.3. *Let X and Y be two decomposable subsets and f be a measurable integrand, if X is rich in Y, then* $\operatorname{ess\,inf}_X f(x) = \operatorname{ess\,inf}_Y f(y)$.

Proof of Lemma 2.3. Since $X \subset Y$, then $\operatorname{ess\,inf}_X f(x) \geq \operatorname{ess\,inf}_Y f(y)$. Conversely, let $u = \operatorname{ess\,inf}_X f(x)$. If $y \in Y$, there exists an increasing covering $(\Omega_n)_n$ of Ω by measurable sets of finite measure and a sequence $(x_n)_n$ of elements of X verifying for all $n \in \mathbb{N}$, $y1_{\Omega_n} = x_n 1_{\Omega_n}$. Hence for all n: $f(y)1_{\Omega_n} = f(x_n)1_{\Omega_n} \geq u1_{\Omega_n}$, taking the limit in n we deduce that $f(y) \geq u$. Since the last inequality is valid for every $y \in Y$, we obtain $\operatorname{ess\,inf}_Y f(y) \geq u$. The proof of Lemma 2.3 is complete.

Given a measurable function v element of $L_0(\Omega, \overline{\mathbb{R}})$ (respectively $L_0(\Omega, \overline{\mathbb{R}}^d)$), the upper integral I_v of v is given by:

$$I_v = \int_\Omega^* v d\mu = \inf\left\{\int_\Omega u d\mu, u \in L_1(\Omega, \mathbb{R}), v \leq u, \mu - \text{a.e.}\right\}$$

(respectively $I_v = (I_{v_i})_i$, if $v = (v_i)_i$, $1 \leq i \leq d$); classically the integral functional associated to f (respectively g) is defined on $L_0(\Omega, E)$ (respectively $L_0(\Omega, \overline{I\!R}^d)$) by: $I_f(x) = I_{f(x)}$ (respectively $I_g(x) = I_{g(x)}$). The domain of the functional I_f is the set:

$$\operatorname{dom} I_f = \{x \in L_0(\Omega, E) : \ I_f(x) < \infty\}.$$

Hereafter X is a decomposable subset of $L_0(\Omega, E)$. Let us define the following sets: $\operatorname{Dom} g(\omega, .) = \{e \in E : g(\omega, e) \in I\!R^d\}$, $\operatorname{Dom} I_g = \{x \in X : I_g(x) \in I\!R^d\}$ and $r_{I_f}(I_g) = \{I_g(x),\ x \in \operatorname{dom} I_f \cap \operatorname{Dom} I_g\}$. For a convex subset C of $I\!R^d$ we will denote its relative interior by riC.
In the sequel, we consider an optimization problem of the following type:

$$(\mathcal{P}) \qquad \inf\{I_f(x) + h(I_g(x)),\ x \in \operatorname{Dom} I_g\},$$

where $h : I\!R^d \mapsto \overline{I\!R}$ is a convex function. We will suppose that $\inf(\mathcal{P})$ is finite.
The performance function of the problem $\mathcal{P}$ is defined by:

$$p(y) = \inf\{I_f(x) + h(I_g(x) + y), x \in \operatorname{Dom} I_g\}.$$

The following result is a slight strengthening of the formulation of [5], Theorem 5.1.

Theorem 2.4. *When $0 \in ri(\operatorname{dom} h - r_{I_f}(I_g))$, we have the Lagrange duality formula:*

$$\inf(\mathcal{P}) = \max_{y^* \in I\!R^d} \inf_{x \in \operatorname{Dom} I_g} I_f(x) + \langle y^*, I_g(x)\rangle - h^*(y^*),$$

and the set of maximisers of the dual problem is exactly $\partial p(0)$. We have also:

$$\inf_{x \in \operatorname{Dom} I_g} I_f(x) + \langle y^*, I_g(x)\rangle = I_{u_{y^*}}$$

with: $u_{y^*} = \operatorname{ess\,inf}_{\operatorname{Dom} I_g} f(x) + \langle y^*, g(x)\rangle$. Moreover if the tribe $\mathcal{T}$ is complete, the Banach space E is separable, and if there exists a multifunction M with measurable graph such that X is rich in $L_0(M)$, then for (almost every) $\omega \in \Omega$:

$$u_{y^*}(\omega) = \inf\{f(\omega, e) + \langle y^*, g(\omega, e)\rangle,\ e \in M(\omega) \cap \operatorname{Dom} g(\omega, .)\}.$$

Proof of Theorem 2.4. From [5], Theorem 4.1 (valid even if the tribe is not complete and the Banach space E is non separable), the map (I_f, I_g) is epi-convex (see [5], Definition 2.14). Thus due to [5], Theorem 2.19 the problem $(\mathcal{P})$ is stable (see [5], Definition 2.5). A simple computation gives: $p^*(y^*) = h^*(y^*) - \inf_{x \in DomI_g}\langle y^*, I_g(x)\rangle + I_f(x)$. The problem $\mathcal{P}$ being stable, for $z^* \in \partial p(0)$ we have: $0 \in \partial p^*(z^*)$, and:

$$\inf(\mathcal{P}) = p(0) = -p^*(z^*) = \max_{y^* \in I\!R^d} -p^*(y^*),$$

this gives the Lagrange duality formula and shows that z^* is a maximiser of the dual problem. Conversely, let z^* be a maximiser of the dual problem, then $0 \in \partial p^*(z^*)$, hence $z^* \in \partial p^{**}(0)$, but since p is subdifferentiable at 0, we have $p(0) = p^{**}(0)$ and $\partial p^{**}(0) = \partial p(0)$, thus $z^* \in \partial p(0)$. We have proved that the set of maximisers of the dual problem is exactly $\partial p(0)$. Moreover, from [5], Lemma 3.10, $\operatorname{Dom} I_g$ $(= X \cap L^1_g,$

with the notations of [5]) is a decomposable set. Therefore [5], Theorem 3.1 (valid even if the tribe is not complete and the Banach E is non separable), gives:

$$\inf_{x\in \mathrm{Dom}\, I_g} I_f(x) + \langle y^*, I_g(x)\rangle = I_{\mathrm{ess\,inf}_{\mathrm{Dom}\, I_g} f(x)+\langle y^*, g(x)\rangle} = I_{u_{y^*}},$$

and the proof of the first part of Theorem 2.4 is complete. The second part is exactly [5], Theorem 5.1.

3. About Ricceri's Property

Theorem 3.1. *Let X be a decomposable subset of $L_0(\Omega, E)$, and let C be a convex subset of $\mathbb{R}^d$. Let us suppose that the following assumptions are fulfilled:*

(H_1) $0 \in ri(C - r_{I_f}(I_g))$,

(H_2) $y^* \in \mathbb{R}^d$, $y^* \neq 0$, $\sup_{c\in C}\langle y^*, c\rangle < +\infty \Rightarrow \inf_{x\in \mathrm{Dom}\, I_g} I_f(x) + \langle y^*, I_g(x)\rangle = -\infty$.

Then, setting $\mathcal{X} = \{x \in X : \ I_g(x) \in C\}$, the following equalities hold:

$$\inf_{x\in\mathcal{X}} I_f(x) = \inf_{x\in \mathrm{Dom}\, I_g} I_f(x) = I_{\mathrm{ess\,inf}_{\mathrm{Dom}\, I_g} f(x)},$$

provided the above left hand side is finite. Moreover if $\mathrm{Dom}\, I_g$ *is rich in X, then*

$$\inf_{x\in\mathcal{X}} I_f(x) = \inf_{x\in X} I_f(x) = I_{\mathrm{ess\,inf}_X f(x)}.$$

Proof of Theorem 3.1. We consider the problem $\mathcal{P}$ associated to the convex function $h = i_C$, where $i_C(e) = 0$, if $e \in C$, $i_C(e) = \infty$, if not, then $\mathrm{dom}\, h = C$, and $h^*(y^*) = \sup_{c\in C}\langle y^*, c\rangle$ vanishes at the origin and dont take the value $-\infty$. Due to assumption (H_1) and the first part of Theorem 2.4 with the convex function h defined above, we have

$$\inf_{x\in\mathcal{X}} I_f(x) = \inf(\mathcal{P}) = \max_{y^*\in\mathbb{R}^d} \inf_{x\in \mathrm{Dom}\, I_g} I_f(x) + \langle y^*, I_g(x)\rangle - h^*(y^*) \qquad (1).$$

Since $\inf(\mathcal{P})$ is finite, property (H_2) gives:

$$y^* \neq 0,\ h^*(y^*) < +\infty \Rightarrow -\infty = \inf_{x\in \mathrm{Dom}\, I_g} I_f(x) + \langle y^*, I_g(x)\rangle - h^*(y^*) < \inf(\mathcal{P}).$$

This proves that the maximum in (1) is attained only for $y^* = 0$, therefore,

$$\inf_{x\in\mathcal{X}} I_f(x) = \inf(\mathcal{P}) = \inf_{x\in \mathrm{Dom}\, I_g} I_f(x) = I_{u_0} = I_{\mathrm{ess\,inf}_{\mathrm{Dom}\, I_g} f(x)}.$$

The last assertion of Theorem 3.1 is a consequence of the Lemma 2.3. Since $\mathrm{Dom}\, I_g$ is rich in X, then $\mathrm{ess\,inf}_X f(x) = \mathrm{ess\,inf}_{\mathrm{Dom}\, I_g} f(x)$, and we can write:

$$\inf_{x\in X} I_f(x) \le \inf_{\in\mathcal{X}} I_f(x) = \inf_{x\in \mathrm{Dom}\, I_g} I_f(x) = I_{\mathrm{ess\,inf}_{\mathrm{Dom}\, I_g} f(x)} = I_{\mathrm{ess\,inf}_X f(x)} \le \inf_X I_f(x),$$

and the proof of Theorem 3.1 is complete.

Let us prove that the Ricceri property [15], Theorem 2.1, is a consequence of the above result.
Let X be a decomposable linear subspace of $L_p(\Omega, E)$, $1 \le p < \infty$. We denote by $\mathcal{V}(X)$ the family of all sets $V \subseteq X$ of the following type:

$$V = \left\{ x \in X : \Psi(x) = \int_\Omega h(x) d\mu \right\},$$

where Ψ is a continuous linear functional on X, and $h : \Omega \times E \to \overline{\mathbb{R}}$, is a measurable integrand such the integral functional $x \to I_h(x)$ is (well defined and) Lipschitzian in X, with a Lipschitz constant strictly less than $\|\Psi\|_{X^*}$.

Corollary 3.2 (See [15], Theorem 2.1). *Let us consider a $\mathcal{T} \otimes \mathcal{B}(E)$-measurable integrand $f : \Omega \times E \to \overline{\mathbb{R}}$, such that there exist $\alpha \in L_1(\Omega, \mathbb{R})$, $\gamma_i \in]0, 1[$ and $\beta_i \in L_{p/(p-\gamma_i)}(\Omega, \mathbb{R})$, $i = 1...k$, satisfying:*

$$-\alpha(\omega) \le f(\omega, e) \le \alpha(\omega) + \sum_{i=1}^{k} \beta_i(\omega)|e_i|^{\gamma_i}.$$

Then, for every decomposable linear space X of $L_p(\Omega, E)$, $1 \le p < \infty$, and every $V \in \mathcal{V}(X)$, one has:

$$\inf_{x \in V} I_f(x) = \inf_{x \in X} I_f(x).$$

Proof of Corollary 3.2 (or more precisely, another proof of [15], Theorem 2.1). By the Hahn-Banach theorem, Ψ is the restriction to X of a continuous linear functional on $L_p(\Omega, E)$. So by a well-known representation theorem ([10], VII, 4, Theorem 7, and VII, 5, Theorem 9), there exists a scalarly measurable mapping $x^* : \Omega \to E^*$ such for every $x \in X$:

$$\Psi(x) = \int_\Omega \langle x^*, x \rangle d\mu.$$

The mapping $(\omega, e) \to \langle x^*(\omega), e \rangle$ is a Caratheodory integrand, hence $g(\omega, e) = \langle x^*(\omega), e \rangle - h(\omega, e)$ is a measurable integrand and $I_g = \Psi - I_h$. Moreover $\mathcal{X} = \{x \in X : I_g(x) = 0\}$ is equal to V. Let us prove that:

$$\inf_{x \in \mathcal{X}} I_f(x) = \inf_{x \in X} I_f(x).$$

Clearly with the notations of Theorem 3.1, here $d = 1$, $C = \{0\}$, and $\operatorname{Dom} I_g = X$. If $x \in L_p(\Omega, E)$, then $\|x\|^{\gamma_i} \in L_{p/\gamma_i}$; since $\gamma_i/p + (p - \gamma_i)/p = 1$, we deduce that $\beta_i \|x\|^{\gamma_i} \in L_1(\Omega, \mathbb{R})$, and:

$$\int \beta_i \|x\|^{\gamma_i} d\mu \le \|\beta_i\|_{L_{p/(p-\gamma_i)}} . \|x\|^{\gamma_i}_{L_p} \tag{2}$$

This proves that $\operatorname{dom} I_f = X$, thus:

$$r_{I_f}(I_g) = I_g(X). \tag{3}$$

We suppose first that $h(\omega, 0) = 0,$ a.e.
Let $c > 0$ be the Lipschitz constant of I_h. By assumptions there exists an element $x_0 \in X$ in the unit sphere such that:

$$\Psi(x_0) - c > 0. \tag{4}$$

This gives for $r > 0$:

$$I_g(rx_0) = \Psi(rx_0) - I_h(rx_0) \geq r(\Psi(x_0) - c), \tag{5}$$

and for $r < 0$:

$$I_g(rx_0) = \Psi(rx_0) - I_h(rx_0) \leq r\Psi(x_0) - cr = r(\Psi(x_0) - c). \tag{6}$$

Clearly since I_g is Lipschitzian, the numerical function $r \mapsto I_g(rx_0)$ is Lipschitzian and (5) and (6) prove, since X is a vector space, that $I\!R$ is the range of I_g. Hence, using (3): $I\!R = I_g(X) = rir_{I_f}(I_g)$, and the assumption (H_1) of Theorem 3.1 is satisfied when the set $C \subset I\!R$ is equal to any single point.
On the other hand, X being a vector space, we remark that for every $y^* \neq 0$, we have:

$$\inf_{x \in \operatorname{Dom} I_g} I_f(x) + \langle y^*, I_g(x)\rangle \leq \inf_{r \in I\!R} I_f(rx_0) + \langle y^*, I_g(rx_0)\rangle. \tag{7}$$

Let us suppose that $y^* > 0$; then since $\gamma_i \in]0, 1[$, with (2), (4) and (6) we obtain:

$$\inf_{r<-1} I_f(rx_0) + \langle y^*, I_g(rx_0)\rangle$$
$$\leq \inf_{r<-1} \sum_{i=1}^{k} |r|^{\gamma_i} \|\beta_i\|_{L_{p/(p-\gamma_i)}}.\|x_0\|_{L_p}^{\gamma_i} + \int \alpha d\mu + ry^*(\Psi(x_0) - c) = -\infty.$$

Similarly, if $y^* < 0$, using (2), (4) and (5):

$$\inf_{r>1} I_f(rx_0) + \langle y^*, I_g(rx_0)\rangle = -\infty,$$

therefore with (7) we have proved the second assumption of Theorem 3.1:

$$y^* \neq 0 \Rightarrow \inf_{x \in \operatorname{Dom} I_g} I_f(x) + \langle y^*, I_g(x)\rangle = -\infty.$$

Applying Theorem 3.1, when $C \subset I\!R$ is equal to any singleton r_0, if $\mathcal{X} = \{x \in X,\ I_g(x) = r_0\}$, since $\operatorname{Dom} I_g = X$, we obtain:

$$\inf_{x \in \mathcal{X}} I_f(x) = \inf_{x \in X} I_f(x) \ \Big(= I_{\operatorname{ess\,inf}_X f(x)}\Big).$$

Now for the proof without the assumption $h(\omega, 0) = 0,$ a.e., since $g(0)$ is integrable, remark that $I_g(x) = 0 \Leftrightarrow I_{g(x)-g(0)} = -I_g(0)$ and we apply our proved result with $r_0 = -I_g(0) = -I_h(0)$, and the integrand $g' = g - g(0) = \langle x^*, .\rangle - h'$, where $h' = h - h(0)$. This is possible since $h'(\omega, 0) = 0$, and clearly $I_{h'}$ has the same Lipschitz constant as that of I_h. The proof of Corollary 3.2 is complete. Remark that in [13], [14], [15], the measurability of the integrands f and h is taken in a apparently weaker sense.
We conclude by a simple criterion with "disintegrated" assumptions.

Corollary 3.3. *Assume that the tribe $\mathcal{T}$ is μ-complete and the Banach space E is separable. Let X be a decomposable subset of $L_0(\Omega, E)$ rich in $L_0(M)$ for a suitable multifunction M with measurable graph, and let C be a convex subset of $\mathbb{R}^d$. Let f and g be two integrands as in Section 2, define $u_{y^*}(\omega) = \inf\{f(\omega, e) + \langle y^*, g(\omega, e)\rangle,\ e \in M(\omega) \cap \operatorname{Dom} g(\omega, .)\}$. Suppose that the following assumptions are fulfilled:*

(H_1) $0 \in ri(C - r_{I_f}(I_g))$,

(H_3) $y^* \in \mathbb{R}^d,\ y^* \neq 0,\ \sup_{c\in C}\langle y^*, c\rangle < +\infty \Rightarrow \mu(\{u_{y^*} = -\infty\}) > 0.$

Then, setting $\mathcal{X} = \{x \in X : \ I_g(x) \in C\}$, we have:

$$\inf_{x\in\mathcal{X}} I_f(x) = \inf_{x\in\operatorname{Dom} I_g} I_f(x) = I_{u_0},$$

provided the above left hand side is finite.

Proof of Corollary 3.3. Since $\inf_{x\in\mathcal{X}} I_f(x)$ is finite, for every $y^* \in \mathbb{R}^d$ the function u_{y^*} is bounded above by an integrable function, hence due to (H_3):

$$y^* \neq 0,\ \sup_{c\in C}\langle y^*, c\rangle < +\infty \Rightarrow I_{u_{y^*}} = -\infty.$$

But as remarked in Theorem 2.4: $I_{u_{y^*}} = \inf_{x\in\operatorname{Dom} I_g} I_f(x) + \langle y^*, I_g(x)\rangle$.
This proves that the assumption (H_2) of Theorem 3.1 is satisfied, and therefore its conclusion holds:

$$\inf_{x\in\mathcal{X}} I_f(x) = \inf_{x\in\operatorname{Dom} I_g} I_f(x) = I_{u_0},$$

The proof of Corollary 3.3 is complete.

Acknowledgements. The author thanks the anonymous referee for his useful comments.

References

[1] J. P. Aubin, I. Ekeland: Minimisation de critères intégraux, C. R. Acad. Sci., Paris, Sér. A 281 (1975) 285–288.

[2] R. J. Aumann, M. Perles: A variational problem arising in economics, J. Math. Anal. Appl. 11 (1965) 488–503.

[3] R. J. Aumann: Integrals of set-valued functions, J. Math. Anal. Appl. 12 (1965) 1–12.

[4] A. Bourass: Comparaison de Fonctionnelles Intégrales. Conditions de Croissance Caractéristiques et Application à la Semi Continuité dans les Espaces Intégraux de Type Orlicz, Thèse d'État, Perpignan (1983).

[5] A. Bourass, E. Giner: Kuhn-Tucker conditions and integral functionals, J. Convex Analysis 8(2) (2001) 533–553.

[6] C. Castaing, M. Valadier: Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics 580, Springer, Berlin (1977).

[7] F. Hiai, H. Umegaki: Integrals, conditional expectations, and martingales of multivalued functions, J. Multivariate Anal. 7 (1977) 149–182.

[8] A. D. Ioffe, V. M. Tikhomirov: On minimization of integral functionals, Funct. Anal. Appl. 3 (1969) 218–227.

[9] A. D. Ioffe, V. M. Tikhomirov: Theory of Extremal Problems, Nauka, Moscow (1974), (in Russian); North Holland, Amsterdam (1979) (in English).

[10] A. Ionescu Tulcea, C. Ionescu Tulcea: Topics in the Theory of Lifting, Springer, New York (1969).

[11] J. Neveu: Bases Mathématiques du Calcul des Probabilités, Masson, Paris (1970).

[12] J. J. Moreau: Fonctionnelles Convexes, Collège de France, Paris (1966–67).

[13] B. Ricceri: A variational property of integral functionals on L^p spaces of vector-valued functions, C. R. Acad. Sci., Paris, Sér. I (1994) 337–342.

[14] B. Ricceri: More on a variational property of integral functionals, J. Optimization Theory Appl. 94(3) (1997) 757–763.

[15] B. Ricceri: Further considerations on a variational property of integral functionals, J. Optimization Theory Appl. 106(3) (2000) 677–681.

[16] R. T. Rockafellar: Integrals which are convex functionals, Pac. J. Math. 24(3) (1968) 525–539.

[17] R. T. Rockafellar: Integrals which are convex functionals. II, Pac. J. Math. 39(2) (1971) 439–469.

[18] R. T. Rockafellar: Integral functionals, normal integrands and measurable selections, in: Nonlinear Operators and the Calculus of Variations (Bruxelles, 1975), J. P. Gossez et al. (ed.), Lecture Notes in Mathematics 543, Springer, Berlin (1976) 157–206.

[19] R. T. Rockafellar, R. J.-B. Wets: Variational Analysis, Springer, Berlin (1998).