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A Characterization of Injective Linear Transformations

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We prove a characterization of the injective linear transformations on real vector spaces: Let X and Y be an m-dimensional and an n-dimensional real vector spaces ($n \geq m \geq 2$), respectively. Assume that a mapping $f : X \to Y$ satisfies $\dim f(X) \geq 2$ and $f(o) = o$. Then, f is an injective linear transformation if and only if f maps every line in X onto a (corresponding) line in Y and preserves the ordering on line.

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1. Introduction

Throughout this paper, we will denote by X and Y an m-dimensional and an n-dimensional real vector spaces, respectively. We use the common notation o to denote the origin of X or Y because there is no danger of misunderstanding. For distinct points p_1 and p_2 of X, we mean by an open segment (open side) p_1p_2 the line segment between p_1 and p_2 without its endpoints, i.e., it is the set of all points represented by $\lambda p_1 + (1-\lambda)p_2$ for all $0 < \lambda < 1$. A subset A of X is called a k-flat (k-dimensional affine space) if it is a translate of a k-dimensional subspace of X (cf. [1, 4]).

A mapping $f : X \to Y$ is said to be a linear transformation (linear mapping) if and only if it is additive and homogeneous. The linear transformation plays an important rôle in studying the theory of vector spaces.

The following theorem is known as the 'Fundamental Theorem of Affine Geometry', which provides us with a useful criterion for the linear transformations (see [2, 2.F]):

Theorem 1.1. *If a bijective mapping $f : X \to Y$ $(m = n \geq 2)$, $f(o) = o$, maps lines in X into lines in Y, then f is linear.*

We remark that the above criterion can be applied only for the bijective mappings. Indeed, the surjectivity of f plays an important rôle in the proof of Theorem 1.1.

We are now interested in some criteria of the linear transformations when the relevant mappings are not assumed to be *a priori* surjective. The following theorem presents a

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criterion of linear mappings without the surjectivity assumption (ref. [5, pp. 167–169]):

Theorem 1.2. *Assume that* m *and* n *are integers given with* $n \geq m \geq 2$. *If a mapping* $f : X \to Y$, *with* $\dim f(X) \geq 2$ *and* $f(o) = o$, *satisfies both the following conditions*

(L') *f maps every line in X into a line in Y;*

(P) *f maps any two parallel flats in X onto two parallel flats in Y,*

then f is a linear transformation.

In [5, p. 169], it has been remarked that the hypotheses (L') and (P) imply that f preserves the ratio of the segment pr to the segment qr if $p, q, r \in X$ are collinear and q lies between p and r with $q \neq r$ and moreover if p, q, r are mapped into a line. Conversely, denoting by (R) the last statement, it was also remarked that (L') and (R) imply (P). Therefore, it is easy to see that Theorem 1.2 is equivalent to [5, Satz 11.6].

The purpose of this paper is to prove a new characterization of the injective linear transformations when the relevant mapping f is not supposed to be surjective.

A mapping $f : X \to Y$ will be said to preserve the ordering on line (or the betweenness) if $f(p)$, $f(q)$, $f(r)$ are collinear and $f(q)$ lies between $f(p)$ and $f(r)$ whenever $p, q, r \in X$ are collinear and q is between p and r. (Here, q will be said to be between p and r if q can be expressed as $\lambda p + (1 - \lambda)r$ for some $0 \leq \lambda \leq 1$. We observe that this definition of betweenness is slightly different from that defined in [3, p. 100].)

In this paper, we will prove that a mapping $f : X \to Y$ ($n \geq m \geq 2$), with $\dim f(X) \geq 2$ and $f(o) = o$, is an injective linear transformation if and only if f satisfies both the following conditions

(L) *f maps every line in X onto a line in Y*;

(O) *f preserves the ordering on line.*

We here remark that the condition (L) implies the condition (L').

2.. Preliminaries

For $i = 1, 2$, let A_i be a k_i-flat in X which is a translate of a k_i-dimensional subspace V_i, where $0 < k_i \leq m$. We say that A_1 and A_2 are parallel if $V_1 \subset V_2$ or $V_2 \subset V_1$.

Lemma 2.1. *For $i = 1, 2$, let A_i be a k_i-flat in X, where $0 < k_1 \leq k_2 \leq m$. A_1 is parallel to A_2 if and only if for every line ℓ_1 in A_1 there exists a line ℓ_2 in A_2 which is parallel to ℓ_1.*

Proof. Set $A_i = p_i + V_i$ for $i = 1, 2$, where p_i is a point of X and V_i is a k_i-dimensional subspace of X. Assume that for each line ℓ_1 in A_1 there is a line ℓ_2 in A_2 which is parallel to ℓ_1. Let $u \in V_1 \backslash \{o\}$ be arbitrary. For the line ℓ_1 through $p_1 + u$ and p_1, there is a line ℓ_2 in A_2 which is parallel to ℓ_1. Suppose ℓ_2 goes through $p_2 + v_1$ and $p_2 + v_2$ for some $v_1, v_2 \in V_2$ with $v_1 \neq v_2$. Then, there exists a real number α such that $u = \alpha(v_1 - v_2) \in V_2$, which implies $V_1 \subset V_2$, i.e., A_1 is parallel to A_2.

Now, let A_1 and A_2 are parallel, i.e., we assume that $V_1 \subset V_2$ ($k_1 \leq k_2$). Consider a line ℓ_1 in A_1 which goes through $p_1 + u_1$ and $p_1 + u_2$, where $u_1, u_2 \in V_1$ are distinct.

By the assumption, we get $u_1, u_2 \in V_1 \subset V_2$. Hence, the line ℓ_2 through $p_2 + u_1$ and $p_2 + u_2$ lies in A_2 and is parallel to ℓ_1. $\square$

For affinely independent points $p_0, \ldots, p_k$ of X, we denote by S_k the k-simplex with these points as its vertices. (Throughout this paper, the vertices of any simplex will be affinely independent.) Indeed, S_k is the set of all points having the representation, $\lambda_0 p_0 + \cdots + \lambda_k p_k$, for all $0 \le \lambda_0, \ldots, \lambda_k \le 1$ with $\lambda_0 + \cdots + \lambda_k = 1$. The relative interior of the k-simplex S_k is the set of all points which have the expression of the form $\lambda_0 p_0 + \cdots + \lambda_k p_k$ for all $0 < \lambda_0, \ldots, \lambda_k < 1$ with $\lambda_0 + \cdots + \lambda_k = 1$ and it is denoted by S_k°. From now on, we will use the terminology 'interior' instead of 'relative interior' for the sake of simplicity.

The number of $(k-1)$-dimensional faces of the k-simplex S_k is $k+1$, and these faces are called the facets of the simplex. From now on, every facet (resp. simplex) will frequently be denoted by $p_{i_1} p_{i_2} \cdots p_{i_k}$ $(0 \le i_1 < \cdots < i_k \le k)$ when the points $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ comprise the vertices of the facet (resp. simplex).

The following lemma is well known. Hence, we omit the proof.

Lemma 2.2. *Let the points $p_0, \ldots, p_k$ comprise the vertices of a k-simplex S_k in X, where $2 \le k \le m$. Suppose p is an interior point of S_k. The line through p_0 and p intersects the facet F, which is opposite to p_0, in a unique interior point of F. Moreover, the open segment joining p_0 and an interior point of F is included in the interior of S_k. Conversely, every interior point of S_k lies on an open segment joining p_0 and an interior point of F.*

We can apply Lemma 2.2 to the proof of the following lemma because every triangle may be considered as a 2-simplex.

Lemma 2.3. *If an injective mapping $f : X \to Y$ $(m, n \ge 2)$ satisfies the conditions (L) and (O), then f maps the interior of any triangle $p_0 p_1 p_2$ in X onto the interior of a triangle $f(p_0) f(p_1) f(p_2)$ in Y.*

Proof. Let S_2 denote a triangle $p_0 p_1 p_2$ in X and let p be an arbitrary interior point of S_2. In view of Lemma 2.2, the line through p_0 and p intersects the open side $p_1 p_2$ in a unique point, say p_3. Let us define $q = f(p)$ and $q_i = f(p_i)$ for $i = 0, 1, 2, 3$.

The injectivity of f, together with (O), implies that q_3 lies on the open side $q_1 q_2$ and q lies on the open segment $q_0 q_3$, which together with Lemma 2.2 implies that $q = f(p)$ belongs to the interior of a triangle $q_0 q_1 q_2$.

Let q be an arbitrary interior point of the triangle $q_0 q_1 q_2$ in Y and let q_4 be the unique intersection point of the line through q_0 and q with the open side $q_1 q_2$ (see Lemma 2.2). Then, (L) and (O) imply that there is a point p_4 on the open side $p_1 p_2$ which is the pre-image of q_4 under f. Considering the hypotheses again, we conclude that there exists a point p on the open segment $p_0 p_4$ such that $q = f(p)$. Moreover, we get by Lemma 2.2 that $p \in S_2^\circ$. $\square$

In the following lemma, we prove that if an injective mapping f satisfies the conditions (L) and (O), then f maps each 2-flat in X onto a 2-flat in Y.

Lemma 2.4. *If an injective mapping $f : X \to Y$ $(m, n \geq 2)$ satisfies both conditions (L) and (O), then f maps every 2-flat in X onto a 2-flat in Y.*

Proof. Let P be an arbitrary 2-flat in X. Consider a sequence $\{S_2^i\}$ of regular triangles in P with the property that the centers of 3 sides of S_2^{i+1} comprise the vertices of S_2^i for all $i \in \mathbb{N}$. Since $(S_2^i)^\circ \subset (S_2^j)^\circ$ for $i < j$, by considering Lemma 2.3, we may conclude that f maps each $(S_2^i)^\circ$ into a fixed 2-flat Q in Y. With the fact

$$P = \bigcup_{i=1}^{\infty} (S_2^i)^\circ,$$

we conclude that f maps the whole P into the 2-flat Q in Y.

Assume now that there would exist a point $q \in Q \backslash f(P)$. Let m be a line in Q through q. If there exist two distinct points $p_1, p_2 \in P$ with $f(p_1), f(p_2) \in m$, the condition (L) implies that $m = f(\ell)$ for the line ℓ in P through p_1 and p_2. This implies that $q \in m = f(\ell) \subset f(P)$, a contradiction. Therefore, every line m in Q through q could have at most one common point with $f(P)$, and we know that $f(P)$ contains at least two distinct parallel lines in Q, say m_1 and m_2, not going through q. (f maps two distinct parallel lines in P onto two distinct parallel lines in Q because f is injective and $f(P)$ is included in the 2-flat Q.) Then, we could select a line m_3 in Q which goes through q and intersects m_1 and m_2. This implies that there exists a line in Q which goes through q and has at least two common points with $f(P)$, a contradiction. We have proved that f maps the whole 2-flat P onto a 2-flat Q in Y. □

3. Main Results

As before, let X and Y be an m-dimensional and an n-dimensional real vector spaces, respectively.

We apply the mathematical induction for proving the higher dimensional version of Lemmas 2.3 and 2.4.

Lemma 3.1. *Assume that the integers m, n are given with $n \geq m \geq 2$. If an injective mapping $f : X \to Y$ satisfies both conditions (L) and (O), then f maps the interior of any k-simplex $p_0 p_1 \cdots p_k$ in X onto the interior of a k-simplex $q_0 q_1 \cdots q_k$ in Y for every $k \in \{2, \ldots, m\}$, where $q_i = f(p_i)$ for $i = 0, 1, \ldots, k$. Moreover, f maps every k-flat in X onto a k-flat in Y.*

Proof. In Lemmas 2.3 and 2.4, we have already proved our assertions for $k = 2$. Assume now that our assertions are true for some $k \in \{2, \ldots, m-1\}$.

Let $p_0, p_1, \ldots, p_{k+1}$ be vertices of a $(k+1)$-simplex S_{k+1} in X. Suppose p is an interior point of S_{k+1}. Then, Lemma 2.2 implies that the line through p and p_0 intersects the facet $p_1\, p_2 \cdots p_{k+1}$ in a unique interior point of the facet, say p_{k+2}.

Let us define $q_i := f(p_i)$ for $i = 0, 1, \ldots, k+2$. By the assumptions for induction, the interior of the facet (k-simplex) $p_1\, p_2 \cdots p_{k+1}$ is mapped by f onto the interior of a k-simplex $q_1\, q_2 \cdots q_{k+1}$ in Y. Furthermore, the hypotheses (L) and (O), together with the injectivity of f, imply that q_0 is not contained in the k-simplex $q_1 q_2 \cdots q_k$

because p_0 does not belong to the facet $p_1p_2\cdots p_k$. Hence, the points $q_0, q_1, \ldots, q_{k+1}$ comprise vertices of a $(k+1)$-simplex S'_{k+1} in Y. By the same reason, q_{k+2} belongs to the interior of the k-simplex $q_1\, q_2 \cdots q_{k+1}$. In view of (L) and (O), $f(p)$ lies on the open segment q_0q_{k+2}. Hence, Lemma 2.2 implies that $f(p)$ belongs to the interior of the $(k+1)$-simplex S'_{k+1} in Y.

Now, let q be an interior point of S'_{k+1}. Then, Lemma 2.2 implies that the line through q and q_0 intersects the facet $q_1\, q_2 \cdots q_{k+1}$ in a unique interior point, say q_{k+3}. Hence, the assumptions for induction imply that there is an interior point p_{k+3} of the k-simplex $p_1\, p_2 \cdots p_{k+1}$ and that there exists a point p on the open segment p_0p_{k+3} with $q = f(p)$. Furthermore, Lemma 2.2 implies that p is an interior point of S_{k+1}. This completes the proof for our first assertion in the case of $k+1$.

We slightly modify the proof of Lemma 2.4 to prove the second assertion for $k+1$. Let P be a $(k+1)$-flat in X and $\{S^i_{k+1}\}$ a sequence of regular $(k+1)$-simplexes in P with the property that the centroid of each facet of S^{i+1}_{k+1} is a corresponding vertex of S^i_{k+1}. Since $(S^i_{k+1})^\circ \subset (S^j_{k+1})^\circ$ for $i < j$, by the first part of our proof, we see that f maps each $(S^i_{k+1})^\circ$ into a fixed $(k+1)$-flat Q in Y. Considering the fact

$$P = \bigcup_{i=1}^{\infty} (S^i_{k+1})^\circ,$$

it is obvious that f maps the whole P into the $(k+1)$-flat Q in Y.

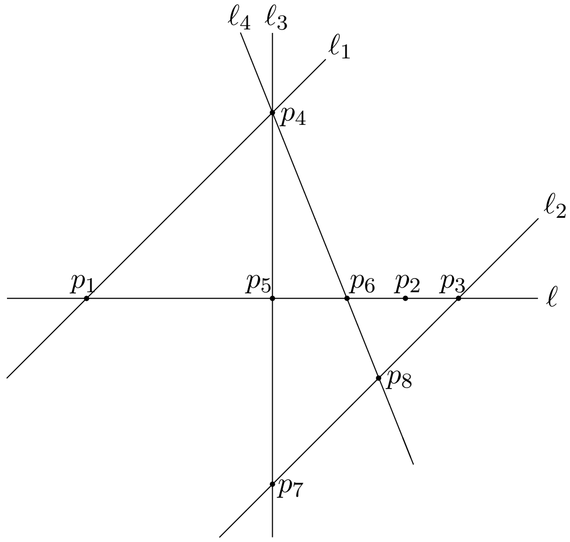
Assume that there would exist a $q \in Q \backslash f(P)$. Let m be a line in Q which goes through q. If there were two distinct points $p_1, p_2 \in P$ with $f(p_1), f(p_2) \in m$, then (L) implies that m coincides with $f(\ell)$ for the line ℓ through p_1 and p_2. Thus, $q \in m = f(\ell) \subset f(P)$, a contradiction. Hence, every line m in Q through q could have at most one intersection point with $f(P)$. However, the assumptions for induction, together with the injectivity of f, guarantee that f maps each pair of two parallel distinct k-flats in P (hyperplanes of P) onto parallel distinct k-flats in Q (hyperplanes of Q), and there exists a line m in Q, which goes through q and intersects both the parallel k-flats in Q (ref. [3, §2.6]), which leads to a contradiction. Consequently, we have also proved the second assertion for $k+1$, which completes our proof. □

In the following lemma, we will prove that the conditions (L) and (O), together with the additional condition that $\dim f(X) \geq 2$, guarantee the injectivity of f. This lemma seems to be of interest in its own right.

Lemma 3.2. *Given integers $m, n \geq 2$, let $f : X \to Y$ satisfy the conditions (L) and (O). If $\dim f(X) \geq 2$, then f is injective.*

Proof. Assume that there exist distinct points $p_1, p_2 \in X$ with $f(p_1) = f(p_2)$. Let us denote by ℓ the line through p_1 and p_2. By (L) and (O), we can easily show that $f(p) = f(p_1)$ for all p between p_1 and p_2 on ℓ and that there exists a p_3 on ℓ with $f(p_3) \neq f(p_1)$. (Without loss of generality, suppose p_2 lies between p_1 and p_3.)

Since $\dim f(X) \geq 2$, we can choose a $p_4 \in X$ with $f(p_4) \notin f(\ell)$. Let ℓ_1 be the line through p_1 and p_4 and let ℓ_2 be the line through p_3 and parallel to ℓ_1. Choose points p_5 and p_6 between p_1 and p_2 on ℓ so close to p_1 that the line ℓ_3 (resp. ℓ_4) through p_4



and p_5 (resp. p_6) intersects ℓ_2 in a point p_7 (resp. p_8) and $f(p_1), f(p_3), f(p_7), f(p_8)$ are four distinct points in Y. (Notice that $f(\ell_2)$ is a line.)

Since $f(\ell_3)$ is a line through $f(p_5) = f(p_1)$ and $f(p_4) \neq f(p_1)$, we see that $f(\ell_3) = f(\ell_1)$. By a similar argument, we get $f(\ell_4) = f(\ell_1)$. Hence, $f(p_7)$ and $f(p_8)$ belong to $f(\ell_1)$ as well as $f(\ell_2)$. Because $f(p_7)$ and $f(p_8)$ are distinct, we have $f(\ell_1) = f(\ell_2)$. Thus, $f(p_3) \in f(\ell_2) = f(\ell_1)$. Moreover, the distinct points $f(p_1)$ and $f(p_3)$ belong to $f(\ell)$ as well as $f(\ell_1)$. Hence, we obtain $f(\ell_1) = f(\ell)$ and $f(p_4) \in f(\ell_1) = f(\ell)$, a contradiction. This proves the injectivity of f. $\square$

We are now ready to prove the main theorem of this paper, in which a new characterization of affine transformations on real vector spaces is introduced. As we see, it is not assumed that the mapping f is surjective.

Theorem 3.3. *Let X and Y be an m-dimensional and an n-dimensional real vector spaces, respectively, where $n \geq m \geq 2$. Assume that $f : X \to Y$ is a mapping with* $\dim f(X) \geq 2$. *Then, f is an injective affine transformation if and only if f satisfies both the conditions (L) and (O).*

Proof. Without loss of generality, assume that $f(o) = o$. If we assume that f is an injective linear transformation, then

$$f(\lambda p + (1-\lambda)q) = \lambda f(p) + (1-\lambda) f(q)$$

for any $p, q \in X$ and $\lambda \in \mathbb{R}$. Hence, f satisfies the conditions (L) and (O).

Now, assume that f satisfies the conditions (L) and (O). We assert that f satisfies the condition (P). Due to Lemma 3.2, f is injective. Let $A_i = p_i + V_i$ $(i = 1, 2)$ be two distinct k_i-flats in X with $V_1 \subset V_2$, i.e., A_1 is parallel to A_2. According to Lemma 3.1, the image B_i of A_i under f is also a k_i-flat in Y. Let m_1 be an arbitrary line in B_1. In view of (L), there exists a line ℓ_1 in A_1 with $m_1 = f(\ell_1)$. Since A_1 and A_2 are parallel, Lemma 2.1 implies that there is a line ℓ_2 in A_2 which is parallel to ℓ_1. Choose a 2-flat P

in X in which ℓ_1 and ℓ_2 lie. Then, Lemma 2.4 (or Lemma 3.1) implies that $Q = f(P)$ is a 2-flat in Y. Indeed, Q includes the lines $m_1 = f(\ell_1)$ and $f(\ell_2)$, $f(\ell_2) \subset B_2$, and the line $f(\ell_2)$ is parallel to m_1 because f is injective and satisfies (L). We have just proved that for each line m_1 in B_1 there exists a line in B_2 which is parallel to m_1, which implies by Lemma 2.1 that B_1 is parallel to B_2. That is, f satisfies the condition (P). Thus, we conclude by Theorem 1.2 that f is an injective linear transformation. □

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