Journal of Convex Analysis
Volume 17 (2010), No. 1, 111–130

Points of Upper Local Uniform Monotonicity in Calderón-Lozanovskiĭ Spaces

Paweł Kolwicz
*Institute of Mathematics, Poznań University of Technology,
Piotrowo 3A, 60-965 Poznań, Poland*
pawel.kolwicz@put.poznan.pl

Ryszard Płuciennik
*Institute of Mathematics, Poznań University of Technology,
Piotrowo 3A, 60-965 Poznań, Poland*
ryszard.pluciennik@put.poznan.pl

Received: June 22, 2008
Revised manuscript received: January 27, 2009

We start with some results concerning $ULUM$ and UM points in general Köthe sequence spaces. Next we present full characterization of $ULUM$ points in Calderón-Lozanovskiĭ spaces. In particular, we obtain the respective criteria for $ULUM$ points in Orlicz-Lorentz spaces and Orlicz spaces.

Keywords: Köthe spaces, Calderón-Lozanovskiĭ spaces, points of upper local uniform monotonicity, local monotonicity properties

2000 Mathematics Subject Classification: 46E30, 46B20, 46B42, 46B45

1. Introduction

Recall that monotonicity properties (strict and uniform monotonicity) play analogous role in the best dominated approximation problems in Banach lattices as do the respective rotundity properties (strict and uniform rotundity) in the best approximation problems in Banach spaces (see [26]). Moreover, they are crucial in many problems since they provide a tool for estimating a norm. It is worth noticing that monotonicity properties are applicable in the ergodic theory ([1]). Recall also that they are restrictions of appropriate rotundity properties to the set of couples of comparable elements in the positive cone of a Köthe space E (see [14]). Clearly, the points of lower (upper) monotonicity of a Banach lattice E play an analogous role as the extreme points in a Banach space X. Similarly, the role of points of upper (lower) local uniform monotonicity in Banach lattices is analogous to that of points of local uniform rotundity in Banach spaces. The monotonicity properties in Calderón-Lozanovskiĭ spaces have been studied in several papers (see [5], [10], [22]). The local monotonicity structure of Calderón-Lozanovskiĭ spaces has been considered in [18]. However, the precise full criteria have been presented only for points of lower and upper monotonicity. Considering $LLUM$ and $ULUM$-points, the authors of [18] gave only some sufficient and some necessary conditions, basing often on too strong assumptions. The $LLUM$-points of Calderón-Lozanovskiĭ spaces have been characterized in [24]. In the present paper we

ISSN 0944-6532 / \$ 2.50

shall give full criteria for $ULUM$-points of Calderón-Lozanovskiĭ spaces. It is natural to study $LLUM$ and $ULUM$-points separately, because it appears that the structure of $LLUM$ and $ULUM$-points is quite different in E_φ. Namely, considering the $LLUM$-point x, the local $\Delta_2^E(x)$ condition is crucial, while in the case of an $ULUM$-point x the respective global condition Δ_2^E is essential which is indeed really stronger than the local $\Delta_2^E(x)$ in general.

2. Preliminaries

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ be the sets of reals, nonnegative reals and positive integers, respectively. Set $\{k \leq\} := \{n \in \mathbb{N} : k \leq n\}$ and $\{< k\} = \mathbb{N} \setminus \{k \leq\}$ for any $k \in \mathbb{N}$. As usual $S(X)$ (resp. $B(X)$) stands for the unit sphere (resp. the closed unit ball) of a real Banach space $(X, \|\cdot\|_X)$.

Let (T, Σ, μ) be a $\sigma-$finite and complete measure space. By $L^0 = L^0(T)$ we mean the set of all μ-equivalence classes of real valued measurable functions defined on T.

A Banach space $E = (E, \|\cdot\|_E)$ is said to be a *Köthe space* if E is a linear subspace of L^0 and:

(i) if $x \in E, y \in L^0$ and $|y| \leq |x|$ μ-a.e., then $y \in E$ and $\|y\|_E \leq \|x\|_E$;
(ii) there exists a function x in E that is positive on the whole T (see [21] and [27]).

Every *Köthe space* is a Banach lattice under the obvious partial order ($x \geq 0$ if $x(t) \geq 0$ for μ-a.e. $t \in T$). In particular, if we consider the space E over a nonatomic measure, then we shall say that E is a *Köthe function space.* If we replace the measure space (T, Σ, μ) by the counting measure space $\left(\mathbb{N}, 2^{\mathbb{N}}, m\right)$, then we will say that E is a *Köthe sequence space* and we denote it by e. In the last case the i-th unit vector is defined as $e_i = (0, ..., 0, 1, 0, ...)$, where "1" is an i-th coordinate of e_i.

The set $E_+ = \{x \in E : x \geq 0\}$ is called the *positive cone of* E. For any subset $A \subset E$ define $A_+ = A \cap E_+$.

A Köthe space is called a *symmetric space* if for any $x \in E$ and $y \in L^0$ with $x^* = y^*$ we have that $y \in E$ and $\|y\|_E = \|x\|_E$, where x^* denotes the *nonincreasing rearrangement of* x given by

$$x^*(t) = \inf\{s \geq 0 : \mu\{t \in T : |x(t)| > s\} \leq t\}, \quad t > 0.$$

For basic properties of symmetric spaces and rearrangements we refer to [29] and to the monographs [2], [25].

A point $x \in E$ is said to have an *order continuous norm* if for any sequence (x_m) in E such that $0 \leq x_m \leq |x|$ and $x_m \to 0$ μ-a.e. we have $\|x_m\|_E \to 0$. A Köthe space E is called *order continuous* ($E \in (OC)$) if every element of E has an order continuous norm (see [21], [27] and [30]). As usual E_a stands for the subspace of order continuous elements of E. It is known that $x \in E_a$ iff $\left\|x\chi_{A_n}\right\|_E \downarrow 0$ for any sequence $\{A_n\}$ satisfying $A_n \downarrow \varnothing$ (that is $A_n \supset A_{n+1}$ and $\mu\left(\bigcap_{n=1}^\infty A_n\right) = 0$). Clearly, $\mu\left(\bigcap_{n=1}^\infty A_n\right) = 0$ iff $\chi_{A_n} \to 0$ μ-a.e. in T. Moreover, for a Köthe sequence space e, $x \in e_a$ iff $\left\|x\chi_{\{n,n+1,...\}}\right\|_e \to 0$ as $n \to \infty$.

A point $x \in E_+ \setminus \{0\}$ is said to be a *point of lower monotonicity* (*upper monotonicity*) if for any $y \in E_+$ such that $y \leq x$ and $y \neq x$ ($x \leq y$ and $y \neq x$), we have $\|y\|_E < \|x\|_E$

($\|x\|_E < \|y\|_E$). A point $x \in E_+$ is called a *point of lower local uniform monotonicity* (*upper local uniform monotonicity*) if $\|x_n - x\|_E \to 0$ for any sequence $x_n \in E$ such that $0 \le x_n \le x$ and $\|x_n\|_E \to \|x\|_E$ ($x \le x_n$ and $\|x_n\|_E \to \|x\|_E$). We will write shortly that x is an *LM-point, UM-point, LLUM-point* and *ULUM-point,* respectively. Recall that if each point of $E_+ \setminus \{0\}$ is an UM-point (equivalently LM-point), then we say that E is *strictly monotone* ($E \in (SM)$) (see [3], [14]). Similarly, if each point of $E_+ \setminus \{0\}$ is an $LLUM$-point [$ULUM$-point], then we say that E is *lower locally uniformly monotone* ($E \in (LLUM)$) [*upper locally uniformly monotone* ($E \in (ULUM)$)]. Notice that global properties $LLUM$ and $ULUM$ are different in general (see [16]). We say that E is *uniformly monotone* ($E \in (UM)$) provided for every $q \in (0,1)$ there exists $p \in (0,1)$ such that for all $0 \le y \le x$ satisfying $\|x\|_E = 1$ and $\|y\|_E \ge q$, we have $\|x-y\|_E \le 1-p$ (see [3], [14]).

In the whole paper φ denotes an *Orlicz function*, i.e. $\varphi : \mathbb{R} \to [0,\infty]$, φ is convex, even, vanishing and continuous at zero, left continuous on $(0,\infty)$ and not identically equal to zero. Denote

$$a_\varphi = \sup\{u \ge 0 : \varphi(u) = 0\} \quad \text{and} \quad b_\varphi = \sup\{u \ge 0 : \varphi(u) < \infty\}.$$

We write $\varphi > 0$ when $a_\varphi = 0$ and $\varphi < \infty$ if $b_\varphi = \infty$. Let $\varphi_r = \varphi\chi_{G_\varphi}$, where

$$G_\varphi = \begin{cases} \{0\} \cup (a_\varphi, b_\varphi] & \text{if } \varphi(b_\varphi) < \infty, \\ \{0\} \cup (a_\varphi, b_\varphi) & \text{otherwise.} \end{cases} \tag{1}$$

Define on L^0 a convex semimodular I_φ by

$$I_\varphi(x) = \begin{cases} \|\varphi \circ x\|_E & \text{if } \varphi \circ x \in E, \\ \infty & \text{otherwise,} \end{cases}$$

where $(\varphi \circ x)(t) = \varphi(x(t))$, $t \in T$. By the Calderón-Lozanovskiĭ space E_φ we mean

$$E_\varphi = \{x \in L^0 : I_\varphi(cx) < \infty \text{ for some } c > 0\}$$

equipped with so called *Luxemburg norm* defined by

$$\|x\|_\varphi = \inf\{\lambda > 0 : I_\varphi(x/\lambda) \le 1\}.$$

If $E = L^1$ ($e = l^1$), then E_φ (e_φ) is the Orlicz function (sequence) space equipped with the Luxemburg norm. If $E = \Lambda_\omega$ -the Lorentz function space ($e = \lambda_\omega$), then E_φ (e_φ) is the corresponding Orlicz-Lorentz function (sequence) space denoted by $(\Lambda_\omega)_\varphi$ ($(\lambda_\omega)_\varphi$) and equipped with the Luxemburg norm (see [12], [14], [22]).

We will assume in the whole paper that E has the *Fatou property*, that is, if $0 \le x_n \uparrow x \in L^0$ with $(x_n)_{n=1}^\infty$ in E and $\sup_n \|x_n\|_E < \infty$, then $x \in E$ and $\|x\|_E = \lim_n \|x_n\|_E$. Since E has the Fatou property, E_φ has also this property, whence E_φ is a Banach space (see [28]). For arbitrary $x \in L^0$ we define

$$\theta(x) := \sup\{\lambda > 0 : I_\varphi(\lambda x) < \infty\},$$

where $\sup \emptyset = 0$.

We say an Orlicz function φ satisfies *condition* $\Delta_2(0)$ (resp. $\Delta_2(\infty)$) if there exist $K > 0$ and $u_0 > 0$ such that $\varphi(u_0) > 0$ (resp. $\varphi(u_0) < \infty$) and the inequality $\varphi(2u) \leqslant K\varphi(u)$ holds for all $u \in [0, u_0]$ (resp. $u \in [u_0, \infty))$. If there exists $K > 0$ such that $\varphi(2u) \leqslant K\varphi(u)$ for all $u \geqslant 0$, then we say that φ satisfies *condition* $\Delta_2(\mathbb{R}_+)$. We write for short $\varphi \in \Delta_2(0)$, $\varphi \in \Delta_2(\infty)$, $\varphi \in \Delta_2(\mathbb{R}_+)$, respectively. Obviously, $\varphi \in \Delta_2(\mathbb{R}_+)$ if and only if $\varphi \in \Delta_2(0)$ and $\varphi \in \Delta_2(\infty)$.

For a Köthe space E and an Orlicz function φ we say that φ satisfies *condition* Δ_2^E ($\varphi \in \Delta_2^E$ for short) if:

1) $\varphi \in \Delta_2(0)$ whenever $E \hookrightarrow L^\infty$;
2) $\varphi \in \Delta_2(\infty)$ whenever $L^\infty \hookrightarrow E$;
3) $\varphi \in \Delta_2(\mathbb{R}_+)$ whenever neither $L^\infty \hookrightarrow E$ nor $E \hookrightarrow L^\infty$ (see [12]),

where the symbol $E \hookrightarrow F$ stands for the continuous embedding of the space E into the space F.

Relationships between the modular I_φ and the norm $\|\cdot\|_\varphi$ are collected in [22].

3. $ULUM$-points in Köthe sequence spaces

Proposition 3.1. *Let e be a Köthe sequence space. A point $x \in e_+$ is an $LLUM$-point of e if and only if x is an LM-point and x has an absolutely continuous norm.*

Proof. Since for any Köthe space E every $LLUM$-point of E is an LM-point of E and, by Lemma 6 in [18], any $LLUM$-point $x \in S(E)_+$ has absolutely continuous norm, the necessity of the theorem is obvious. We need to prove the sufficiency only. Let $x \in e_+$ and (x_n) be a sequence such that $0 \leq x_n \leq x$ and $\|x_n\|_e \to \|x\|_e$. Notice that the sequences $(x_n(i))_{n=1}^\infty$ are bounded for any $i \in \mathbb{N}$. By the diagonal method, we conclude that there is $y \in l^0$ and a subsequence (x_{n_k}) of (x_n) such that $x_{n_k}(i) \to y(i)$ for all $i \in \mathbb{N}$. Obviously, $0 \leq y \leq x$ and, by the Fatou property of e, $\|x_{n_k}\|_e \to \|y\|_e$. Hence, by the assumptions, we conclude $\|y\|_e = \|x\|_e$. Since x is an LM-point, we have $y = x$. Moreover, $x - x_{n_k} \to 0$ coordinatewise and $0 \leq x - x_{n_k} \leq x$. By the fact that x has absolutely continuous norm, it follows that $\|x_{n_k} - x\|_e \to 0$ as $k \to \infty$. Notice that (x_{n_k}) with the above properties can be extracted from arbitrary subsequence of (x_n), so by virtue of the double extract convergence theorem, we also get that $\|x_n - x\|_e \to 0$ as $n \to \infty$, which finishes the proof of the theorem. □

We conclude immediately that a *Köthe sequence space e is $LLUM$* iff $e \in (OC)$ and $e \in (SM)$ (see [10]).

The natural question is whether the similar characterization as in Proposition 3.1 is valid for $ULUM$-points. The following example gives the negative answer.

Example 3.2. Consider the space c_0 equipped with the norm

$$\|x\|_S = \sup_{k\in\mathbb{N}} |x(k)| + \sum_{k=1}^{\infty} \frac{|x(k)|}{2^{k-1}}$$

for any $x \in c_0$. Take $x = \frac{1}{2}e_1$. It is easy to see that x is an UM-point and x has an absolutely continuous norm. We will show that x is not an $ULUM$-point. Really, define

$x_n = \frac{1}{2}(e_1 + e_n)$. Then $x \le x_n$. Moreover, $\|x\|_S = 1$, $\|x_n\|_S = 1 + \frac{1}{2^n}$ and $\|x_n - x\|_S = \frac{1}{2} + \frac{1}{2^n} > \frac{1}{2}$ for any $n \in \mathbb{N}$. Consequently, $\|x_n\|_S \to \|x\|_S$ and $\|x_n - x\|_S \nrightarrow 0$, so x is not an $ULUM$-point.

Proposition 3.3. *Let e be a Köthe sequence space.*

a) *If a point $x \in e_+$ is an $ULUM$-point of e, then x is an UM-point and for any sequence $x_n \in e$ such that $x \le x_n$ and $\|x_n\|_e \to \|x\|_e$ there holds $\|x_n\chi_{\{k\le\}}\|_e \to \|x\chi_{\{k\le\}}\|_e$ for any $k \in \mathbb{N}$, where $\{k \le\} = \{k, k+1, k+2, ...\}$.*

b) *Under the assumption that x has absolutely continuous norm, the converse is also true.*

Proof. *a)* Let $x \in e_+$. It is clear that if x is an $ULUM$-point of e, then x is an UM-point. Suppose that (x_n) is a sequence such that $0 \le x \le x_n$ and $\|x_n\|_e \to \|x\|_e$. Since x is an $ULUM$-point of e, we have $\|x_n - x\|_e \to 0$. Then

$$\left| \|x_n\chi_{\{k\le\}}\|_e - \|x\chi_{\{k\le\}}\|_e \right| \le \|(x_n - x)\chi_{\{k\le\}}\|_e \le \|x_n - x\|_e \to 0$$

for any $k \in \mathbb{N}$.

b) Let $x \in e_+$ be an UM-point and (x_n) be a sequence such that $0 \le x \le x_n$ and $\|x_n\|_e \to \|x\|_e$. Notice that the sequence x_n is convergent coordinatewise to x. If not, then there is i_0 such that $x_n(i_0) \nrightarrow x(i_0)$. Without loss of generality we can assume that there exists a positive number ε_0 such that $x_n(i_0) > x(i_0) + \varepsilon_0$ for any $n \in \mathbb{N}$. Define $y = x\chi_{\mathbb{N}\setminus\{i_0\}} + (x(i_0) + \varepsilon_0)\chi_{\{i_0\}}$. By the fact that x is an UM-point, $\|y\|_e > \|x\|_e$. On the other hand, since $0 \le x \le y \le x_n$, we have $\|y\|_e \le \|x_n\|_e$ for any $n \in \mathbb{N}$. Consequently, $\lim_{n\to\infty} \|x_n\|_e \ge \|y\|_e > \|x\|_e$. A contradiction.

By our assumption, $\|x_n\chi_{\{k\le\}}\|_e \to \|x\chi_{\{k\le\}}\|_e$ for any $k \in \mathbb{N}$. Now suppose that x has absolutely continuous norm. Take $\varepsilon > 0$ and suppose that k_0 is so large that $\|x\chi_{\{k_0\le\}}\|_e < \varepsilon/4$. Then there is $n_1 \in \mathbb{N}$ such that

$$\|x_n\chi_{\{k_0\le\}}\|_e - \|x\chi_{\{k_0\le\}}\|_e < \frac{\varepsilon}{4}$$

for every $n \ge n_1$. Hence $\|x_n\chi_{\{k_0\le\}}\|_e < \varepsilon/2$ for any $n \ge n_1$. Since $x_n(i) \to x(i)$ for any $i \in \mathbb{N}$, we conclude that $x_n\chi_{\{<k_0\}} - x\chi_{\{<k_0\}} \to 0$ in norm, whence there is n_2 such that

$$\|x_n\chi_{\{<k_0\}} - x\chi_{\{<k_0\}}\|_e < \frac{\varepsilon}{4}$$

for any $n \ge n_2$. Therefore

$$\begin{aligned} \|x_n - x\|_e &= \left\|\left(x_n\chi_{\{<k_0\}} - x\chi_{\{<k_0\}}\right) + x_n\chi_{\{k_0\le\}} - x\chi_{\{k_0\le\}}\right\|_e \\ &\le \|x_n\chi_{\{<k_0\}} - x\chi_{\{<k_0\}}\|_e + \|x_n\chi_{\{k_0\le\}}\|_e + \|x\chi_{\{k_0\le\}}\|_e < \varepsilon \end{aligned}$$

for any $n \ge \max\{n_1, n_2\}$, which finishes the proof. □

Proposition 3.3 *b)* is not true without assumption that x has absolutely continuous norm, which is ilustrated by the following example.

Example 3.4. Consider the space l^∞ equipped with the norm $\|\cdot\|_S$ defined as in Example 3.2. Take $x = \left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{2}{5}, ...\right)$. Obviously, x is an UM-point. Since $\left\|x\chi_{\{m\leq\}}\right\|_S > \frac{2}{5}$ for any $m \geq 2$, x has not absolutely continuous norm. Moreover, taking $x_n = x + \frac{1}{5}e_{2n}$ for any $n \in \mathbb{N}$, we get $x \leq x_n$, and $\|x_n\|_S \to \|x\|_S$. Since $\|x_n - x\|_S > \frac{1}{5}$, x is not an $ULUM$-point. Finally, we show that for any sequence (x_n) in l^∞ such that $x \leq x_n$ and $\|x_n\|_S \to \|x\|_S$ there holds $\left\|x_n\chi_{\{k\leq\}}\right\|_S \to \left\|x\chi_{\{k\leq\}}\right\|_S$ for any $k \in \mathbb{N}$. Suppose for the contrary that $x \leq x_n$, $\|x_n\|_S \to \|x\|_S$ and there are $\delta > 0$ and $k_0 \in \mathbb{N}$ such that $\left\|x_n\chi_{\{k_0\leq\}}\right\|_S - \left\|x\chi_{\{k_0\leq\}}\right\|_S \geq \delta$ for infinitely many n (say for all n). Since $x \leq x_n$ and $\|x_n\|_S \to \|x\|_S$, we conclude that $\sup_{i\geq k_0} |x_n(i)| \to \sup_{i\geq k_0} |x(i)|$ as $n \to \infty$. Hence

$$\sum_{i=k_0}^{\infty} \frac{|x_n(i)|}{2^{i-1}} - \sum_{i=k_0}^{\infty} \frac{|x(i)|}{2^{i-1}} > \frac{\delta}{2}$$

for $n \geq N_1$. Moreover, there is $k_1 > k_0$ with $\sum_{i=k_1}^{\infty} \frac{|x_n(i)|}{2^{i-1}} < \frac{\delta}{8}$ for each n. Thus

$$\sum_{i=k_0}^{k_1} \frac{|x_n(i)|}{2^{i-1}} - \sum_{i=k_0}^{k_1} \frac{|x(i)|}{2^{i-1}} > \frac{\delta}{8}. \tag{2}$$

On other hand x is an UM-point and consequently, by the proof of Proposition 3.3 b) and our assumptions, $x_n \to x$ coordinatewise. This is a contradiction with (2).

We will prove more facts about UM and $ULUM$-points, whenever e is a symmetric Köthe sequence space.

Lemma 3.5. *Let e be a symmetric Köthe sequence space. If $x \in e_+$ is an UM-point, then $x \in c$ and $\lim_{j\to\infty} x(j) = \inf_{j\in\mathbb{N}} x(j)$.*

Proof. It is enough to show that there are a subset $A \subset \mathbb{N}$ and a bijection $\pi : \mathbb{N} \to \mathbb{N}\backslash A$ such that $x^*(i) = x(\pi(i))$ for any $i \in \mathbb{N}$ and $x(i) = \lim_{j\to\infty} x^*(j)$ for any $i \in A$. First we assume that

(+) there is i_0 such that $x(i_0) < x^*(j)$ for any $j \in \mathbb{N}$.

Since x^* is a nonincreasing sequence and $x(i_0)$ is its lower bound, there is the limit x_0 of the sequence x^*, i.e. $x_0 = \lim_{j\to\infty} x^*(j)$. If $x(i_0) < x_0$, then, taking into account that x is an UM-point and setting $\lambda = \frac{1}{2}(x_0 - x(i_0))$, we have that $x^* = (x + \lambda e_{i_0})^*$ and

$$\|x^*\|_e \geq \|x + \lambda e_{i_0}\|_e > \|x\|_e = \|x^*\|_e,$$

a contradiction. Hence $x(i_0) = \lim_{j\to\infty} x^*(j)$. In this case we define

$$A = \left\{i \in \mathbb{N} : x(i) = \lim_{j\to\infty} x^*(j)\right\}.$$

If condition (+) does not hold, then we put $A = \varnothing$. $\square$

Remark 3.6. It is obvious that if e is a symmetric Köthe sequence space and x is an $ULUM$-point, then x^* is an $ULUM$-point. The converse is not true. Really, take

$e = l^\infty$. Define

$$x = \sum_{i=1}^{\infty} \left(\frac{3}{4} + \frac{1}{4} (-1)^{i+1} \right) e_i \quad \text{and} \quad y = \sum_{i=1}^{\infty} \left(\frac{1}{4} + \frac{1}{4} (-1)^{i} \right) e_i.$$

Then $x \in S(l^\infty)$, $y \geq 0$ and $y \neq 0$. Moreover

$$\|x+y\|_{l^\infty} = \left\| \sum\nolimits_{i=1}^{\infty} e_i \right\|_{l^\infty} = 1 = \|x\|_{l^\infty},$$

whence x in not an UM-point. On the other hand $x^* = \sum_{i=1}^{\infty} e_i$ is an $ULUM$-point.

Corollary 3.7. *Suppose that e is a symmetric Köthe sequence space. Let $x \in e_+$. The following statements are equivalent:*

a) *A point x is an $ULUM$-point.*

b) *A point x is an UM-point and x^* is an $ULUM$-point.*

Proof. The implication $a) \Rightarrow b)$ is obvious by Remark 3.6, so we will show $b) \Rightarrow a)$. Applying the proof of Lemma 3.5, if $A = \varnothing$, then there is a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $x^*(i) = x(\pi(i))$ for any $i \in \mathbb{N}$. Let (x_n) be a sequence of elements of e such that $x \leq x_n$ and $\|x_n\|_e \to \|x\|_e$. Define $\widetilde{x}_n(i) = x_n(\pi(i))$. Then $x^* \leq \widetilde{x}_n$ and

$$\|\widetilde{x}_n\|_e = \|x_n\|_e \to \|x\|_e = \|x^*\|_e.$$

Since x^* is an $ULUM$-point, we have $\|x_n - x\|_e = \|\widetilde{x}_n - x^*\|_e \to 0$. Thus x is an $ULUM$-point.

Now, suppose that $A \neq \varnothing$. By Lemma 3.5, $x(i) = x_0$ for any $i \in A$. Take a sequence (x_n) satisfying the same properties as in the previous part of the proof. Again define $\widetilde{x}_n(i) = x_n(\pi(i))$, where π is the same as in the proof of Lemma 3.5. Then $x^* \leq \widetilde{x}_n$ and

$$\|x^*\|_e \leq \|\widetilde{x}_n\|_e = \left\|x_n \chi_{\mathbb{N}\setminus A}\right\|_e \leq \|x_n\|_e \to \|x\|_e = \|x^*\|_e,$$

whence $\|\widetilde{x}_n\|_e \to \|x^*\|_e$. Hence, by the fact that x^* is an $ULUM$-point, we have $\|\widetilde{x}_n - x^*\|_e \to 0$. Further, by the triangle inequality, we have

$$\|x_n - x\|_e \leq \|\widetilde{x}_n - x^*\|_e + \|(x_n - x)\chi_A\|_e.$$

To finish the proof, it is enough to show that $\|(x_n - x)\chi_A\|_e \to 0$.

We claim that for any $\varepsilon > 0$ there is n_ε such that $|x_n(i) - x_0| < \varepsilon$ for every $i \in A$ and $n > n_\varepsilon$, where $x_0 = \lim_{j\to\infty} x^*(j) = x(i)$ for each $i \in A$. It is clear in the case if $\operatorname{card}(A) < \aleph_0$, because $x_n \to x$ coordinatewise (see the proof of Proposition 3.3). Suppose that $\operatorname{card}(A) = \aleph_0$ and that our claim is not true. Then there are $\varepsilon_0 > 0$, sequences (n_k) of positive integers and (i_k) of elements of A such that $|x_{n_k}(i_k) - x_0| > \varepsilon_0$ for any $k \in \mathbb{N}$ (we have $x_0 = x(i_k)$). Hence $x_{n_k}\chi_A \geq x_0 + \varepsilon_0 e_{i_k}$ for any $k \in \mathbb{N}$. Consequently, $x_{n_k} \geq x + \varepsilon_0 e_{i_k}$ for any $k \in \mathbb{N}$. Since $x(i) = x(j)$ for each $i, j \in A$, by the symmetry of the space e and the fact that x is an UM-point, we have

$$\|x + \varepsilon_0 e_{i_k}\|_e = \|x + \varepsilon_0 e_{i_1}\|_e = a > \|x\|_e.$$

Consequently, in view of our assumptions, we get

$$\|x\|_e = \lim_{n\to\infty} \|x_n\|_e = \lim_{k\to\infty} \|x_{n_k}\|_e \geq a > \|x\|_e ,$$

a contradiction. Hence

$$\|(x_n - x)\,\chi_A\|_e < \varepsilon$$

for any $n > n_\varepsilon$ and consequently $\|(x_n - x)\,\chi_A\|_e \to 0$, which finishes the proof. □

The implication $b) \Rightarrow a)$ is not true in general in nonsymmetric Köthe sequence spaces (see Example 3.4, where $x^* = \frac{2}{5}\sum_{i=1}^{\infty} e_i$ is an $ULUM$-point).

4. Points of upper local uniform monotonicity of E_φ

Remark 4.1. In the next theorem we will assume that the Köthe space E is order continuous and symmetric. Under these assumptions in the case when E is function space, we have $E \not\hookrightarrow L^\infty$. Therefore $E \hookrightarrow L^\infty$ is possible only for Köthe sequence spaces, i.e. $E = e$. Moreover, for the symmetric and nontrivial Köthe sequence spaces we always have that $e \hookrightarrow l^\infty$. Hence the Δ_2^E-condition means the $\Delta_2(\infty)$-condition whenever $L^\infty \hookrightarrow E$, the $\Delta_2(\mathbb{R})$-condition (i.e. both $\Delta_2(\infty)$ and $\Delta_2(0)$ conditions) if $L^\infty \not\hookrightarrow E$ and $\Delta_2(0)$ condition when $E = e$.

Theorem 4.2. *Let E be an order continuous symmetric Köthe space. Suppose additionally that if $L^\infty \not\hookrightarrow E$ in the function case or $E = e$, then $a_\varphi = 0$. Moreover, for a sequence case assume that $\varphi(b_\varphi)\,\|e_1\|_e > 1$. A point $x \in S(E_\varphi)_+$ is an $ULUM$-point if and only if $x = b_\varphi \chi_T$ or the following conditions are satisfied:*

a) $x \geq a_\varphi \chi_T$,
b) $\varphi \in \Delta_2^E$,
c) $\varphi \circ x$ *is an $ULUM$-point in E.*

Proof. *Necessity.* Let $x \in S(E_\varphi)_+$ be an $ULUM$-point. Since x is also an UM-point, by Theorem 1 from [18], $x = b_\varphi \chi_T$ or ($x \geq a_\varphi \chi_T$ and $I_\varphi(x) = 1$).

By Remark 4.1, to prove $b)$ we will show first that if E is a Köthe function space and $\varphi \notin \Delta_2(\infty)$, then there are $D_n \in \Sigma$ with $\mu(D_n) \to 0$ and a sequence (y_n) in $S(E_\varphi)_+$ such that $\operatorname{supp} y_n = D_n$ and $I_\varphi(y_n) \to 0$. Consider three possible cases in which the condition $\Delta_2(\infty)$ is not satisfied.

Case 1. If $b_\varphi < \infty$ and $\varphi(b_\varphi) < \infty$, then, taking an arbitrary sequence of sets $D_n \in \Sigma$ with $\mu(D_n) \to 0$ and $\left\|\varphi(b_\varphi)\,\chi_{D_n}\right\|_E \leq 1$ for each $n \in \mathbb{N}$, and defining $y_n = b_\varphi \chi_{D_n}$, we have

$$\|y_n\|_\varphi = \inf\{\lambda > 0 : I_\varphi(y_n/\lambda) \leq 1\} = \inf\left\{\lambda > 0 : \left\|\varphi(b_\varphi/\lambda)\,\chi_{D_n}\right\|_E \leq 1\right\} = 1$$

for any $n \in \mathbb{N}$. Moreover, $I_\varphi(y_n) = \left\|\varphi(b_\varphi)\,\chi_{D_n}\right\|_E \to 0$, by order continuity and symmetry of E.

Case 2. Suppose that $b_\varphi < \infty$ and $\varphi(b_\varphi) = \infty$. Take an arbitrary set A of finite measure and a sequence of reals (u_n) such that $u_n \to b_\varphi$. By order continuity of E, there is a sequence (A_n) of measurable and disjoint sets such that $\mu(A_n) \leq \mu(A)/2^n$

and $\left\|\varphi\left(u_n\right)\chi_{A_n}\right\|_E \leq 1/2^n$ for any $n \in \mathbb{N}$. Define $y_n = \sum_{k=n}^{\infty} u_k \chi_{A_k}$. Then for any $\lambda < 1$ there is $n_0 \in \mathbb{N}$ such that $u_{n_0}/\lambda > b_\varphi$, so $\varphi\left(u_{n_0}/\lambda\right) = \infty$. Consequently, $I_\varphi\left(y_n/\lambda\right) = \infty$ for any $\lambda < 1$. But for $\lambda = 1$, we have

$$I_\varphi\left(y_n\right) = \left\|\varphi \circ \left(\sum_{k=n}^{\infty} u_k \chi_{A_k}\right)\right\|_E \leq \sum_{k=n}^{\infty} \left\|\varphi\left(u_k\right)\chi_{A_k}\right\|_E \leq \sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}} < 1.$$

Hence $\left\|y_n\right\|_\varphi = 1$ for any $n \in \mathbb{N}$ and $I_\varphi\left(y_n\right) \to 0$. Define $D_n = \bigcup_{k=n}^{\infty} A_k$. Then $\operatorname{supp} y_n = D_n$ and $\mu\left(D_n\right) \to 0$.

In the proof of the following case we apply similar methods as in [12], but we present the whole proof for the sake of convenience.

Case 3. Suppose that $b_\varphi = \infty$. If $\varphi \notin \Delta_2(\infty)$, then there is an increasing sequence (u_n) such that $u_n \to \infty$ and

$$\varphi\left(\left(1+\frac{1}{n}\right)u_n\right) > \frac{2^n}{\left\|\chi_A\right\|_E}\varphi\left(u_n\right) \quad \text{and} \quad \varphi\left(u_n\right) \geq 1,$$

where $A \in \Sigma$ is an arbitrary set of positive and finite measure such that $\left\|\chi_A\right\|_E < 1$. Notice that if $A = A_1 \cup A_2$, $A_1 \cap A_2 = \varnothing$ and $\mu\left(A_1\right) = \mu\left(A_1\right) = \mu(A)/2$, then

$$\left\|\chi_A\right\|_E = \left\|\chi_{A_1} + \chi_{A_2}\right\|_E \leq \left\|\chi_{A_1}\right\|_E + \left\|\chi_{A_2}\right\|_E = 2\left\|\chi_{A_1}\right\|_E,$$

whence $\left\|\chi_{A_1}\right\|_E \geq \left\|\chi_A\right\|_E/2$. Let (A_n) be a sequence of disjoint measurable sets such that $\mu\left(A_n\right) = \mu(A)/2^n$ for any $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} A_n = A$. Then, by the above observation, $\left\|\chi_{A_n}\right\|_E \geq \left\|\chi_A\right\|_E/2^n$ for any $n \in \mathbb{N}$. By order continuity of the space E, the function defined $\nu(B) = \left\|\chi_B\right\|_E$ for any $B \in \Sigma \cap A$ is an absolutely continuous submeasure with respect to measure μ. Hence it has the Darboux property (see [6]) and consequently for any $n \in \mathbb{N}$ there exists a set $B_n \subset A_n$ such that

$$\nu\left(B_n\right) = \left\|\chi_{B_n}\right\|_E = \frac{1}{2^n\varphi\left(u_n\right)}\left\|\chi_A\right\|_E.$$

Obviously, $(B_n)_{n=1}^{\infty}$ is a sequence of disjoint sets. Define

$$y_n = \sum_{k=n}^{\infty} u_k \chi_{B_k} \quad \text{and} \quad D_n = \bigcup_{k=n}^{\infty} B_k$$

for any $n \in \mathbb{N}$. Obviously, $\operatorname{supp} y_n = D_n$ and $\mu\left(D_n\right) \to 0$. Moreover, we have

$$\begin{aligned} I_\varphi\left(y_n\right) &= \left\|\sum_{k=n}^{\infty} \varphi\left(u_k\right)\chi_{B_k}\right\|_E \leq \sum_{k=n}^{\infty} \varphi\left(u_k\right)\left\|\chi_{B_k}\right\|_E \\ &= \sum_{k=n}^{\infty} \frac{1}{2^k}\left\|\chi_A\right\|_E = \frac{1}{2^{n-1}}\left\|\chi_A\right\|_E < \frac{1}{2^{n-1}} \end{aligned}$$

for any $n \in \mathbb{N}$. Hence $I_\varphi(y_n) \to 0$. Moreover, for every $\lambda \in (0,1)$ and $n \in \mathbb{N}$ there is $n_0 \geq n$ such that $1 + 1/n_0 < 1/\lambda$. Then

$$\begin{aligned} I_\varphi\left(\frac{y_n}{\lambda}\right) &= \left\| \sum_{k=n}^{\infty} \varphi\left(\lambda^{-1} u_k\right) \chi_{B_k} \right\|_E \geq \left\| \varphi\left(\lambda^{-1} u_{n_0}\right) \chi_{B_{n_0}} \right\|_E \\ &> \left\| \varphi\left(\left(1 + \frac{1}{n_0}\right) u_{n_0}\right) \chi_{B_{n_0}} \right\|_E > \frac{2^{n_0}}{\|\chi_A\|_E} \varphi\left(u_{n_0}\right) \left\| \chi_{B_{n_0}} \right\|_E = 1, \end{aligned}$$

whence $\|y_n\|_\varphi = 1$. This finishes the proof of *Case 3*.

Consider the situation when $\varphi \notin \Delta_2(0)$.

Case 4. If $L^\infty \not\hookrightarrow E$ (resp. $E = e$), $a_\varphi = 0$ and $\varphi \notin \Delta_2(0)$, then for any set $A \in \Sigma$ (resp. $A \subset \mathbb{N}$) of infinite measure there are a decreasing sequence of sets (C_n) in $\Sigma \cap A$ (resp. $C_n \subset A$) of infinite measure and a sequence of functions (y_n) in $S(E_\varphi)_+$ (resp. $S(e_\varphi)_+$) such that $\operatorname{supp} y_n = C_n$ for any $n \in \mathbb{N}$, $C_n \downarrow \varnothing$ and $I_\varphi(y_n) \to 0$.

To prove this claim, suppose that $L^\infty \not\hookrightarrow E$ (resp. $E = e$), $a_\varphi = 0$ and $\varphi \notin \Delta_2(0)$. Take a set $A \in \Sigma$ (resp. $A \subset \mathbb{N}$) such that $\chi_A \notin E$. In view of symmetry of E, $\mu(A) = \infty$. The condition $\varphi \notin \Delta_2(0)$ implies that there is an decreasing sequence (u_n) such that $u_n \to 0$ and

$$\varphi\left(\left(1 + \frac{1}{n}\right) u_n\right) > 2^n \varphi(u_n) \quad \text{and} \quad \varphi(u_n) \leq 1.$$

Without loss of generality, passing to a subsequence if necessary, we can assume that $\varphi(u_n) \leq 1/2^n$. Really, since $\varphi(u_n) \to 0$, there is an increasing sequence (n_k) of positive integers such that $\varphi(u_{n_k}) \leq 1/2^k$ for any $k \in \mathbb{N}$. Noticing that $n_k \geq k$ for any $k \in \mathbb{N}$, we have

$$\varphi\left(\left(1 + \frac{1}{k}\right) u_{n_k}\right) \geq \varphi\left(\left(1 + \frac{1}{n_k}\right) u_{n_k}\right) > 2^{n_k} \varphi(u_{n_k}) \geq 2^k \varphi(u_{n_k}).$$

To get the desired subsequence it is enough to put $v_k = u_{n_k}$ for any $k \in \mathbb{N}$.

In the case when $L^\infty \not\hookrightarrow E$ we take a sequence of disjoint sets (A_n) of finite measure such that $\varphi(u_n) \left\|\chi_{A_n}\right\|_E = 1/2^n$ and $A_n \subset A$ for any $n \in \mathbb{N}$. Moreover, by our assumption, $\left\|\chi_{A_n}\right\|_E \geq 1$. By symmetry of the space E, there is a positive number $\beta > 0$ such that $\mu(A_n) \geq \beta$ for any $n \in \mathbb{N}$.

If $E = e$, then the sequence (u_n) can be taken such that $\varphi(u_n) \left\|\chi_{\{e_1\}}\right\|_e \leq 1/2^n$ for any $n \in \mathbb{N}$. Hence we can construct a finite set $A_n \subset A$ such that $\frac{1}{2^n} \leq \varphi(u_n) \left\|\chi_{A_n}\right\|_e \leq \frac{1}{2^{n-1}}$ for any $n \in \mathbb{N}$. To this end, note that if $A = \{a_i : i \in \mathbb{N}\}$, then, by the Fatou property, the sequence of the norms $\left\|\chi_{\{a_1, a_2, \ldots, a_k\}}\right\|_e \to \infty$. Hence we can take $A_1 = \{a_1, a_2, \ldots, a_{k_1}\}$ as the smallest set such that $\frac{1}{2} \leq \varphi(u_1) \left\|\chi_{A_1}\right\|_e$. Then for any element $a \in A_1$ we have $\varphi(u_1) \left\|\chi_{A_1 \setminus \{a\}}\right\|_e < \frac{1}{2}$ and

$$\varphi(u_1) \left\|\chi_{A_1}\right\|_e = \varphi(u_1) \left\|\chi_{A_1 \setminus \{a\}} + \chi_{\{a\}}\right\|_e \leq \varphi(u_1) \left\|\chi_{A_1 \setminus \{a\}}\right\|_e + \varphi(u_1) \left\|\chi_{\{a\}}\right\|_e < 1.$$

Define $A_n = \left\{a_{k_{n-1}}, \ldots, a_{k_n}\right\}$, where k_n is the smallest positive integer such that $\frac{1}{2^n} \leq \varphi(u_n) \left\|\chi_{A_n}\right\|_e$. By the same argumentation as for the set A_1, we conclude that $\frac{1}{2^n} \leq$

$\varphi(u_n)\left\|\chi_{A_n}\right\|_e \le \frac{1}{2^{n-1}}$. Define for both cases

$$y_n = \sum_{k=n}^{\infty} u_k \chi_{A_k} \quad \text{and} \quad C_n = \bigcup_{k=n}^{\infty} A_k$$

for any $n \in \mathbb{N}$. Obviously, $C_n \downarrow \varnothing$, $\operatorname{supp} y_n = C_n$ and $\mu(C_n) = \infty$ for any $n \in \mathbb{N}$. Moreover,

$$I_\varphi(y_n) = \left\| \sum_{k=n}^{\infty} \varphi(u_k)\chi_{A_k} \right\|_E \le \sum_{k=n}^{\infty} \varphi(u_k)\left\|\chi_{A_k}\right\|_E \le \sum_{k=n}^{\infty} \frac{1}{2^{k-1}} = \frac{1}{2^{n-2}}$$

for any $n \in \mathbb{N}$. Consequently, $I_\varphi(y_n) \to 0$. Similarly as in *Case 3*, for any $\lambda \in (0,1)$ and $n \in \mathbb{N}$ a positive integer $n_0 \ge n$ can be found such that $1 + 1/n_0 < 1/\lambda$. Hence, by an analogous way as in *Case 3*, we get $I_\varphi(y_n/\lambda) > 1$, which implies that $\|y_n\|_\varphi = 1$ for any $n \in \mathbb{N}$, as we claimed in *Case 4*.

Now suppose that E is a Köthe function space, $x \in S(E_\varphi)_+$ is an $ULUM$-point and $x \neq b_\varphi \chi_T$. Then there are a set A of positive and finite measure and a number $a \in (0,1)$ such that $x(t) < b_\varphi$ for μ-*a.e.* $t \in A$ and $\theta(x\chi_A) > \frac{1}{1-a}$. If $\varphi \notin \Delta_2(\infty)$, then, by *Cases 1–3*, we can find a sequence (D_n) of measurable subsets of A with $\mu(D_n) \to 0$ and a sequence (y_n) in $S(E_\varphi)_+$ such that $\|y_n\|_\varphi = 1$, $\operatorname{supp} y_n = D_n$ for any $n \in \mathbb{N}$ and $I_\varphi(y_n) \to 0$. Define $x_n = x + a y_n$ for any $n \in \mathbb{N}$. Obviously, $x_n \ge x$. Since x is also an UM-point, by Theorem 1 in [18], $I_\varphi(x) = 1$. Hence

$$\begin{aligned} 1 = I_\varphi(x) \le I_\varphi(x_n) &\le I_\varphi\left(x\chi_{T\setminus D_n}\right) + I_\varphi\left(x\chi_{D_n} + a y_n\right) \\ &= I_\varphi\left(x\chi_{T\setminus D_n}\right) + I_\varphi\left((1-a)\frac{x}{1-a}\chi_{D_n} + a y_n\right) \\ &\le I_\varphi(x) + (1-a) I_\varphi\left(\frac{x}{1-a}\chi_{D_n}\right) + a I_\varphi(y_n) \to 1 \end{aligned} \tag{3}$$

because $I_\varphi\left(\frac{x}{1-a}\chi_A\right) < \infty$. Consequently, $I_\varphi\left(\frac{x}{1-a}\chi_{D_n}\right) \to 0$, by order continuity of E. Thus $\|x_n\|_\varphi \to 1 = \|x\|_\varphi$. But

$$\|x_n - x\|_\varphi = a\|y_n\|_\varphi = a > 0$$

for any $n \in \mathbb{N}$, which means that x is not an $ULUM$-point. The obtained contradiction shows that if E is a Köthe function space, then φ always satisfies the $\Delta_2(\infty)$ condition. In particular, if $L^\infty \hookrightarrow E$, then $\varphi \in \Delta_2^E$.

Suppose that $L^\infty \not\hookrightarrow E$ or $E = e$, $a_\varphi = 0$, $x \in S(E_\varphi)_+$ is an $ULUM$-point and $\varphi \notin \Delta_2(0)$. Since $L^\infty \not\hookrightarrow E$ or e is symmetric order continuous Köthe sequence space, then either $\chi_{T\setminus \operatorname{supp} x} \notin E$ or $\chi_{\operatorname{supp} x} \notin E$ (if $E = e$, then $T = \mathbb{N}$). If $\chi_{T\setminus \operatorname{supp} x} \notin E$, then, taking $A = T\setminus \operatorname{supp} x$, by the claim in *Case 4*, we can construct a decreasing sequence of sets (C_n) in $\Sigma \cap A$ of infinite measure and a sequence of functions (y_n) in $S(E_\varphi)_+$ such that $\operatorname{supp} y_n = C_n \subset A$ and $I_\varphi(y_n) \to 0$. Now, putting $x_n = x + y_n$ for any $n \in \mathbb{N}$, we have that $x_n \ge x$ and $1 = I_\varphi(x) \le I_\varphi(x_n) \le I_\varphi(x) + I_\varphi(y_n) \to 1$. Hence $\|x_n\|_\varphi \to \|x\|_\varphi$ and $\|x_n - x\|_\varphi = 1$, which contradicts the fact that x is an $ULUM$-point. Thus $\varphi \in \Delta_2(0)$.

Now suppose that $\chi_{\operatorname{supp} x} \notin E$. It implies that $\mu(\operatorname{supp} x) = \infty$. Let (u_n) be a decreasing sequence from *Case 4* such that $\varphi(u_n) \leq \frac{1}{2^n \|\chi_B\|_E}$, where $\mu(B) = 1$ and let $a \in (0,1)$ be a real number. Then, by order continuity and symmetry of E, a sequence (A_n) of measurable disjoint sets of finite measure can be found such that $\mu(A_n) \geq 1$, $A_n \subset \operatorname{supp} x$, $x\chi_{A_n} \leq (1-a)\, u_n \chi_T$ and $\varphi(u_n) \left\|\chi_{A_n}\right\|_E = 1/2^n$ for any $n \in \mathbb{N}$. Denote $A = \bigcup_{k=1}^{\infty} A_k$ and $C_n = \bigcup_{k=n}^{\infty} A_k$. Defining as in *Case 4* the sequence $y_n = \sum_{k=n}^{\infty} u_k \chi_{A_k}$ in $S(E_\varphi)_+$, we have that $I_\varphi(y_n) \to 0$. Since $C_n \downarrow \varnothing$ and

$$I_\varphi\left(\frac{x}{1-a}\chi_A\right) \leq I_\varphi\left(\sum_{n=1}^{\infty} u_n \chi_{A_n}\right) \leq \sum_{n=1}^{\infty} \varphi(u_n) \left\|\chi_{A_n}\right\|_E = 1,$$

by order continuity of E, we have

$$I_\varphi\left(\frac{x}{1-a}\chi_{C_n}\right) = \left\|\varphi \circ \left(\frac{x}{1-a}\right)\chi_{C_n}\right\|_E \to 0.$$

Hence, defining $x_n = x + a y_n$ for any $n \in \mathbb{N}$ and repeating the sequence of inequalities (3) (replacing D_n by C_n), we obtain that $I_\varphi(x_n) \searrow 1$, whence $\|x_n\|_\varphi \to \|x\|_\varphi$. Moreover, $\|x_n - x\|_\varphi = a > 0$ for any $n \in \mathbb{N}$. The same contradiction as above shows that $\varphi \in \Delta_2(0)$. For the case $E = e$ a finite sequence (A_n) of disjoint subsets of $\mathbb{N}$ can be found such that $\mu(A_n) \geq 1$, $A_n \subset \operatorname{supp} x$, $x\chi_{A_n} \leq (1-a)\, u_n \chi_T$ and $1/2^{n+1} \leq \varphi(u_n) \left\|\chi_{A_n}\right\|_E \leq 1/2^n$ for any $n \in \mathbb{N}$.

Combining the above, if $L^\infty \hookrightarrow E$, then $\varphi \in \Delta_2(\infty)$. If $L^\infty \not\hookrightarrow E$, then $\varphi \in \Delta_2(0)$ and $\varphi \in \Delta_2(\infty)$, i.e. $\varphi \in \Delta_2(\mathbb{R}_+)$. If $E = e$, then $\varphi \in \Delta_2(0)$. Consequently, in all four cases $\varphi \in \Delta_2^E$, which finishes the proof of b).

If E is a Köthe function space, the condition c) follows immediately from Proposition 4 in [18]. We remind to the reader that $\theta(x)$ defined in [18] is smaller than one if and only if $\theta(x)$ defined by us is bigger than one. It is enough to notice that if the measure μ is nonatomic, then the condition $\Delta_2(\infty)$ implies that $\theta(x) > 1$ and $[0,\infty) \subset \varphi([0,\infty))$. If e is a Köthe sequence space and $\varphi(b_\varphi) = \infty$, then also the assumptions of Proposition 4 in [18] are satisfied.

It remains to prove condition c) in the case when $E = e$, $b_\varphi < \infty$ and $\varphi(b_\varphi) < \infty$. We have that $\varphi(b_\varphi)\|e_1\|_e > 1$. Let $x \in S(e_\varphi)_+$ be an $ULUM$-point. Then $x(i) < b_\varphi$ for any $i \in \mathbb{N}$. Let i_0 be the smallest number for which

$$x(i_0) = b = \sup\{x(i) : i \in \mathbb{N}\} < b_\varphi.$$

Such i_0 always exists, because, by order continuity and symmetry of e, $x(i) \to 0$. Take $u_0 = \frac{b+b_\varphi}{2}$ and $c = \frac{b+b_\varphi}{2b}$. Then, by the $\Delta_2(0)$-condition for φ, there is $K > 1$ such that $\varphi(cu) \leq K\varphi(u)$ for any $|u| \leq \frac{u_0}{c}$. Hence

$$I_\varphi(cx\chi_{\mathbb{N}}) \leq KI_\varphi(x\chi_{\mathbb{N}}) \leq KI_\varphi(x) = K < \infty. \tag{4}$$

Let $(y_n) \subset e_+$ be a sequence such that $y_n \geq \varphi \circ x$ for any $n \in \mathbb{N}$ and $\|y_n\|_e \to \|\varphi \circ x\|_e = 1$. Without loss of generality, excluding a finite number of elements, if necessary, we can assume that $\|y_n\|_e < \varphi(b_\varphi)\|e_1\|_e$ for any $n \in \mathbb{N}$. Hence $y_n(i) < \varphi(b_\varphi)$ for any $i, n \in \mathbb{N}$.

It follows from the proof of Proposition 3.3 *b*) that $y_n \to \varphi \circ x$ coordinatewise. Denoting $z_n = \varphi_r^{-1} \circ y_n$, we have $b_\varphi > z_n(i) \geq x(i)$ for any $i, n \in \mathbb{N}$ and $I_\varphi(z_n) = \|y_n\|_e \to 1$, whence $\|z_n\|_\varphi \to 1$. Since x is an $ULUM$-point, we obtain $\|z_n - x\|_\varphi \to 0$. Let (z_{n_k}) be a subsequence of (z_n) such that

$$\sum_{k=1}^{\infty} \|z_{n_k} - x\|_\varphi < \frac{(b_\varphi - b)}{2b_\varphi}.$$

Define $h := \sum_{k=1}^{\infty}(z_{n_k} - x)$. Obviously, $z_{n_k} - x \leq h$ for each $k \in \mathbb{N}$. Since

$$\|h\chi_{\{i\}}\|_\varphi \leq \|h\|_\varphi < \frac{b_\varphi - b}{2b_\varphi}$$

for any $i \in \mathbb{N}$, we have

$$1 \geq I_\varphi\left(\frac{2b_\varphi h\chi_{\{i\}}}{b_\varphi - b}\right) = \varphi\left(\frac{2b_\varphi h(i)}{b_\varphi - b}\right)\|e_1\|_e,$$

whence

$$h(i) \leq \varphi_r^{-1}\left(\frac{1}{\|e_1\|_e}\right)\frac{b_\varphi - b}{2b_\varphi} \leq \frac{b_\varphi - b}{2}.$$

Therefore, for any $i \in \mathbb{N}$, we have

$$x(i) + h(i) \leq b + \frac{(b_\varphi - b)}{2} \leq \frac{(b_\varphi + b)}{2} \leq b_\varphi$$

and

$$\frac{c}{c-1}h(i) \leq \frac{\frac{b+b_\varphi}{2b}}{\frac{b+b_\varphi}{2b} - 1}\frac{b_\varphi - b}{2} = \frac{(b_\varphi + b)}{2}.$$

Hence, again, by the $\Delta_2(0)$-condition for $u_1 = \frac{b+b_\varphi}{2}$ and $c_1 = \frac{c}{c-1}$ there is K_1 such that $\varphi(c_1 u) \leq K_1\varphi(u)$ for any $|u| \leq \frac{u_1}{c_1}$. Thus

$$I_\varphi\left(\frac{c}{c-1}h\chi_{\mathbb{N}}\right) \leq K_1 I_\varphi(h) \leq K_1\|h\|_\varphi < \infty. \tag{5}$$

Therefore, by (4), (5) and convexity of φ, we have

$$I_\varphi(x+h) = \|\varphi \circ (x+h)\|_e \leq \frac{1}{c}I_\varphi(cx) + \frac{c-1}{c}I_\varphi\left(\frac{c}{c-1}h\right) < \infty,$$

whence $\varphi \circ (x+h) \in e$. Since $\varphi \circ z_{n_k} \to \varphi \circ x$ coordinatewise and

$$0 \leq y_{n_k} - \varphi \circ x = \varphi \circ z_{n_k} - \varphi \circ x \leq \varphi \circ (x+h)$$

for every $k \in \mathbb{N}$, by order continuity of e, we obtain

$$\|y_{n_k} - \varphi \circ x\|_e \to 0.$$

Applying the double extract subsequence theorem, we get $\|y_n - \varphi \circ x\|_e \to 0$. Consequently, $\varphi \circ x$ is an $ULUM$-point, which finishes the proof of necessity.

Sufficiency. Suppose that $x := b_\varphi \chi_T$ is not an $ULUM$-point. Then there are a sequence (y_n) in $(E_\varphi)_+$ and a positive real number ε such that $\|y_n\|_\varphi \geq \varepsilon$ for any $n \in \mathbb{N}$ and $\|x + y_n\|_\varphi \to 1$. Define

$$B_n = \left\{ t \in T : y_n(t) \geq \frac{\varepsilon}{2} x(t) \right\}.$$

Then

$$\varepsilon \leq \left\|y_n \chi_{T \setminus B_n}\right\|_\varphi + \left\|y_n \chi_{B_n}\right\|_\varphi < \frac{\varepsilon}{2} + \left\|y_n \chi_{B_n}\right\|_\varphi .$$

Consequently, $\left\|y_n \chi_{B_n}\right\|_\varphi > \frac{\varepsilon}{2}$ and $\mu(B_n) > 0$ for any $n \in \mathbb{N}$. Moreover

$$\begin{aligned} I_\varphi\left(\frac{x + y_n}{1 + \varepsilon^2/2}\right) &= \left\|\varphi \circ \left(\frac{x + y_n}{1 + \varepsilon^2/2}\right)\right\|_E \\ &\geq \left\|\varphi \circ \left(\frac{x + y_n}{1 + \varepsilon^2/2}\right) \chi_{B_n}\right\|_E > \left\|\varphi\left(\frac{1 + \varepsilon/2}{1 + \varepsilon^2/2} b_\varphi\right) \chi_{B_n}\right\|_E = \infty \end{aligned}$$

because $1 + \varepsilon^2/2 < 1 + \varepsilon/2$. Therefore $\|x + y_n\|_\varphi \geq 1 + \varepsilon^2/2$, a contradiction. In consequence $x = b_\varphi \chi_T$ is an $ULUM$-point.

Suppose that E is a Köthe function space and $x \in S(E_\varphi)_+$ and conditions $a)$, $b)$, $c)$ are satisfied. Let (x_n) be a sequence of elements of $(E_\varphi)_+$ such that $x_n \geq x$ for all $n \in \mathbb{N}$ and $\|x_n\|_\varphi \to \|x\|_\varphi = 1$. We will show that $\|x_n - x\|_\varphi \to 0$. Note that $E \not\hookrightarrow L^\infty$ because $E \in (OC)$. Then $\varphi \in \Delta_2^E$ means $\varphi \in \Delta_2(\infty)$ whenever $L^\infty \hookrightarrow E$ or $\varphi \in \Delta_2(\mathbb{R}_+)$ if $L^\infty \not\hookrightarrow E$, whence, in any case, $\varphi < \infty$. Thus, we conclude that $\|\varphi \circ x\|_E = I_\varphi(x) = 1$ and

$$\|\varphi \circ x_n\|_E = I_\varphi(x_n) \to 1 = \|\varphi \circ x\|_E$$

(see [5]). Moreover, $\varphi \circ x \leq \varphi \circ x_n$ for all $n \in \mathbb{N}$ because φ is an increasing function. Therefore, by assumption $c)$,

$$\|\varphi \circ x_n - \varphi \circ x\|_E \to 0. \tag{6}$$

Applying superadditivity of φ on $\mathbb{R}_+$, we obtain

$$I_\varphi(x_n - x) = \|\varphi \circ (x_n - x)\|_E \leq \|\varphi \circ x_n - \varphi \circ x\|_E , \tag{7}$$

whence $I_\varphi(x_n - x) \to 0$. As above, by the Δ_2^E-condition, we conclude that $\varphi \in \Delta_2(\infty)$ whenever $L^\infty \hookrightarrow E$ or $\varphi \in \Delta_2(\mathbb{R}_+)$ whenever $L^\infty \not\hookrightarrow E$.

If $\varphi \in \Delta_2(\mathbb{R}_+)$, then $\varphi > 0$ and consequently $\|x_n - x\|_\varphi \to 0$, which means that x is an $ULUM$-point of E_φ. Suppose that $L^\infty \hookrightarrow E$ and $\varphi \in \Delta_2(\infty)$. Condition (6) and assumption $a)$ imply that $x_n - x \to 0$ μ-a.e.. Consequently, by Lemma 8 in [8], $\|x_n - x\|_\varphi \to 0$ as desired.

Suppose that e is a symmetric and order continuous Köthe sequence space and $\varphi(b_\varphi) = \infty$. Then, repeating the same argumentation as for function spaces, we conclude that under assumptions $a)$, $b)$, $c)$, $x \in S(e_\varphi)_+$ is an $ULUM$-point (see [9], [22] for the required results to apply).

It remains to proof the hypothesis in the case when $b_\varphi < \infty$ and $\varphi(b_\varphi) < \infty$. Let (x_n) be a sequence in $(e_\varphi)_+$ such that $x_n \geq x$ for all $n \in \mathbb{N}$ and $\|x_n\|_\varphi \to \|x\|_\varphi = 1$. Since, $\varphi(b_\varphi)\|e_1\|_e > 1$, there is $\alpha \in (0,1)$ such that $\|x_n\|_\varphi < \varphi(\alpha b_\varphi)\|e_1\|_e$ for n large enough. Consequently, $x(i) \leq x_n(i) < \alpha b_\varphi$ for any $i \in \mathbb{N}$ and $n \in \mathbb{N}$ large enough. Then conditions (6) and (7) are satisfied, whence $I_\varphi(x_n - x) \to 0$. By the Δ_2^e-condition, we conclude that $\|x_n - x\|_\varphi \to 0$, whence $x \in S(e_\varphi)_+$ is an $ULUM$-point, which finishes the proof. □

Remark 4.3. By Lemma 1.4 in [22], the assumption $\varphi(b_\varphi)\|e_1\|_e \geq 1$ under the $\Delta_2(0)$-condition is necessary and sufficient to the equivalence $I_\varphi(x) = 1 \Leftrightarrow \|x\|_\varphi = 1$ for any $x \in e_\varphi$. Our assumption with the sharp inequality $\varphi(b_\varphi)\|e_1\|_e > 1$ is slightly stronger.

The case $L^\infty \not\hookrightarrow E$ with $a_\varphi > 0$ deviates from the pattern set by Theorem 4.2. Since in such a case $\varphi \notin \Delta_2^E = \Delta_2(\mathbb{R}_+)$, one can expect that $S(E_\varphi)_+$ has no $ULUM$-points. On the other hand the next theorem shows that if $L^\infty \not\hookrightarrow E$ and $a_\varphi > 0$, then Theorem 4.2 (with a small restriction) holds true, whenever we replace the condition $\varphi \in \Delta_2^E$ by the conditions $\varphi \in \Delta_2(\infty)$ and $I_\varphi(x) = 1$.

Lemma 4.4. *For any Orlicz function φ and any $\alpha \in (0,1]$ the inequality*

$$\varphi(u+v) \geq \varphi(u) + \varphi(\alpha a_\varphi + v)$$

holds for any $u \geq \alpha a_\varphi$ and $v \geq 0$.

Proof. By Lemma 1 in [17], the inequality holds for $\alpha = 1$. Hence, by monotonicity of φ, the inequality is true for any $\alpha \in (0,1]$. □

Theorem 4.5. *Suppose that E is an order continuous symmetric Köthe function space such that $L^\infty \not\hookrightarrow E$, $a_\varphi > 0$. Let $x \in S(E_\varphi)_+$ be such that $\varphi \circ x$ is an $ULUM$-point in E. A point x is an $ULUM$-point in E_φ if and only if the following conditions are satisfied:*

a) $x \geq a_\varphi \chi_T$,
b) $\varphi \in \Delta_2(\infty)$,
c) $I_\varphi(x) = 1$.

Proof. *Necessity.* Let $x \in S(E_\varphi)_+$ be an $ULUM$-point. Since x is also an UM-point, by Theorem 1 in [18], $x \geq a_\varphi \chi_T$ and $I_\varphi(x) = 1$, i.e. *a*) and *c*) are satisfied. Similarly as in the proof of Theorem 4.2, we conclude *b*).

Sufficiency. Assume that $x \in S(E_\varphi)_+$, $\varphi \circ x$ is an $ULUM$-point in E and that *a*), *b*) and *c*) are satisfied. Suppose that x is not an $ULUM$-point. Then there are a sequence (y_n) in $(E_\varphi)_+$ and a positive real number ε such that $\|y_n\|_\varphi \geq 4\varepsilon$ for any $n \in \mathbb{N}$ and $\|x + y_n\|_\varphi \to 1$. Define

$$B_n = \{t \in T : y_n(t) \geq 2\varepsilon a_\varphi\}.$$

Then

$$4\varepsilon \leq \left\|y_n \chi_{T \setminus B_n}\right\|_\varphi + \left\|y_n \chi_{B_n}\right\|_\varphi < 2\varepsilon + \left\|y_n \chi_{B_n}\right\|_\varphi.$$

Consequently, $\left\|y_n \chi_{B_n}\right\|_\varphi > 2\varepsilon$ and $\mu(B_n) > 0$ for any $n \in \mathbb{N}$. We will show, that there is $\beta > 0$ such that $I_\varphi\left((a_\varphi + y_n)\chi_{B_n}\right) > \beta$ for any $n \in \mathbb{N}$. To do this, denote

$a = \liminf_{n\to\infty} \mu(B_n)$. Passing to a subsequence if necessary, we can assume that $a = \lim_{n\to\infty} \mu(B_n)$. In the case when $a \in (0, \infty]$, we can take $\beta = \|\varphi \circ (a_\varphi (1+2\varepsilon)) \chi_B\|_E$, where B is an arbitrary set in Σ with $\mu(B) = \frac{1}{2}a$ if $a < \infty$ or $\mu(B) = 1$ if $a = \infty$.

Let now $a = 0$ and define

$$C_n = \{t \in B_n : y_n(t) \geq 2a_\varphi\}.$$

By order continuity of E, $\left\|\varphi\left(\frac{2a_\varphi}{\varepsilon}\right)\chi_{B_n}\right\|_E \leq 1$ for n large enough. Again, without loss of generality, we can assume that $\left\|2a_\varphi \chi_{B_n}\right\|_\varphi < \varepsilon$ for any $n \in \mathbb{N}$. Then

$$2\varepsilon \leq \left\|y_n \chi_{B_n \setminus C_n}\right\|_\varphi + \left\|y_n \chi_{C_n}\right\|_\varphi < \left\|2a_\varphi \chi_{B_n}\right\|_\varphi + \left\|y_n \chi_{C_n}\right\|_\varphi < \varepsilon + \left\|y_n \chi_{C_n}\right\|_\varphi,$$

whence $\left\|y_n \chi_{C_n}\right\|_\varphi > \varepsilon$ for any $n \in \mathbb{N}$. It implies, by the definition of the norm, that $I_\varphi\left(\frac{y_n}{\varepsilon}\chi_{C_n}\right) > 1$ for any $n \in \mathbb{N}$. In view of the $\Delta_2(\infty)$-condition, there is a constant $K(\varepsilon) > 0$ such that $I_\varphi\left(y_n \chi_{C_n}\right) > 1/K(\varepsilon) = \beta$, and consequently

$$I_\varphi\left((a_\varphi + y_n)\chi_{B_n}\right) \geq I_\varphi\left(y_n \chi_{C_n}\right) > \beta$$

for any $n \in \mathbb{N}$. Hence

$$\left\|\varphi \circ \left((a_\varphi + y_n)\chi_{B_n}\right)\right\|_E = I_\varphi\left((a_\varphi + y_n)\chi_{B_n}\right) > \beta$$

for any $n \in \mathbb{N}$. Since $\varphi \circ x$ is an $ULUM$-point in E, a real number $\gamma(\beta) > 0$ can be found such that

$$\begin{aligned} I_\varphi(x + y_n) &= \|\varphi \circ (x + y_n)\|_E \geq \|\varphi \circ x + \varphi \circ (a_\varphi \chi_T + y_n)\|_E \\ &\geq \left\|\varphi \circ x + \varphi \circ \left((a_\varphi + y_n)\chi_{B_n}\right)\right\|_E \geq 1 + 3\gamma(\beta) \end{aligned}$$

for any $n \in \mathbb{N}$. For any $\eta > 0$ define

$$D_\eta = \{t \in T : x(t) \leq a_\varphi(1+\eta)\}.$$

By order continuity of E there is η_0 such that $\left\|\varphi \circ \left(x \chi_{D_{\eta_0}}\right)\right\|_E < \gamma(\beta)$. It is easy to see that we can choose $\eta_0 \in (0, \min\{\varepsilon, a_\varphi\})$. We have

$$\begin{aligned} &\left\|\varphi \circ \left(x \chi_{T \setminus D_{\eta_0}}\right) + \varphi \circ \left((a_\varphi + y_n)\chi_{B_n}\right)\right\|_E \\ &= \left\|\varphi \circ x - \varphi \circ \left(x \chi_{D_{\eta_0}}\right) + \varphi \circ \left((a_\varphi + y_n)\chi_{B_n}\right)\right\|_E \\ &\geq \left\|\varphi \circ x + \varphi \circ \left((a_\varphi + y_n)\chi_{B_n}\right)\right\|_E - \left\|\varphi \circ \left(x \chi_{D_{\eta_0}}\right)\right\|_E \\ &\geq 1 + 3\gamma(\beta) - \gamma(\beta) = 1 + 2\gamma(\beta) \end{aligned}$$

for any $n \in \mathbb{N}$. By the condition $\Delta_2(\infty)$, for any $K > 1$ there is $\lambda_K \in (1, K)$ such that $\varphi(v) \leq K\varphi\left(\frac{v}{\lambda_K}\right)$ for any $\frac{v}{\lambda_K} \geq a_\varphi\left(1 + \frac{1}{2}\eta_0\right)$. Take

$$K = \min\left\{1 + \gamma(\beta),\ \frac{1+\eta_0}{1+\eta_0/2}\right\}.$$

Notice that by the fact that $\lambda_K < \frac{1+\eta_0}{1+\eta_0/2}$, we have

$$\frac{x(t)}{\lambda_K} > \frac{1+\eta_0/2}{1+\eta_0} a_\varphi (1+\eta_0) = a_\varphi \left(1+\frac{\eta_0}{2}\right)$$

for μ-a.e. $t \in T \setminus D_{\eta_0}$ and

$$\frac{(a_\varphi + y_n(t))}{\lambda_K} \geq \frac{1+\eta_0/2}{1+\eta_0} a_\varphi (1+2\varepsilon) \geq a_\varphi \left(1+\frac{\eta_0}{2}\right)$$

for μ-a.e. $t \in B_n$. Hence, by Lemma 4.4, we obtain

$$\begin{aligned} I_\varphi\left(\frac{x+y_n}{\lambda_K}\right) &= \left\|\varphi \circ \left(\frac{x+y_n}{\lambda_K}\right)\right\|_E \geq \left\|\varphi \circ \left(\frac{x}{\lambda_K}\right) + \varphi \circ \left(\frac{(a_\varphi + y_n)\chi_{B_n}}{\lambda_K}\right)\right\|_E \\ &\geq \left\|\varphi \circ \left(\frac{x}{\lambda_K}\right)\chi_{T\setminus D_{\eta_0}} + \varphi \circ \left(\frac{(a_\varphi + y_n)\chi_{B_n}}{\lambda_K}\chi_{B_n}\right)\right\|_E \\ &\geq \frac{1}{K}\left\|\varphi \circ (x)\chi_{T\setminus D_{\eta_0}} + \varphi \circ \left((a_\varphi + y_n)\chi_{B_n}\right)\right\|_E \geq \frac{1+2\gamma(\beta)}{1+\gamma(\beta)} > 1 \end{aligned}$$

for any $n \in \mathbb{N}$. Consequently, $\|x+y_n\|_\varphi > \lambda_K > 1$ for any $n \in \mathbb{N}$. This contradiction finishes the proof. $\square$

Theorem 4.6. *Suppose that e is an order continuous symmetric Köthe sequence space and φ is an Orlicz function such that $a_\varphi > 0$ and $\varphi(b_\varphi)\|e_1\|_e > 1$. Let $x \in S(e_\varphi)_+$ be such that $\varphi \circ x$ is an ULUM-point in e. The point x is an ULUM-point of e_φ if and only if the following conditions are satisfied:*

a) $x \geq a_\varphi \chi_{\mathbb{N}}$,
b) $I_\varphi(x) = 1$.

Proof. *Necessity.* Let $x \in S(E_\varphi)_+$ be an $ULUM$-point. Since x is also an UM-point, by Theorem 1 in [18], a) and b) are satisfied.

Sufficiency. Suppose that $x \in S(e_\varphi)_+$, $\varphi \circ x$ is an $ULUM$-point in e and that conditions a) and b) are satisfied. Assume that x is not an $ULUM$-point. Then there are a sequence (y_n) in $(e_\varphi)_+$ and a positive real number ε such that $\|y_n\|_\varphi \geq 4\varepsilon$ for any $n \in \mathbb{N}$ and $\|x+y_n\|_\varphi \to 1$. Since $\varphi(b_\varphi)\|e_1\|_e > 1$, there is $\alpha \in (0,1)$ such that $\|x+y_n\|_\varphi < \varphi(\alpha b_\varphi)\|e_1\|_e$ for n large enough. Consequently, without loss of generality, we can assume that $x(i) + y_n(i) < \alpha b_\varphi$ for all i, $n \in \mathbb{N}$. Define

$$B_n = \{i \in \mathbb{N} : y_n(i) \geq 2\varepsilon a_\varphi\}.$$

Similarly as in the proof of Theorem 4.5, we get $\|y_n \chi_{B_n}\|_\varphi > 2\varepsilon$, whence

$$a = \liminf_{n\to\infty} m(B_n) \geq 1.$$

Moreover,

$$I_\varphi\left((a_\varphi + y_n)\chi_{B_n}\right) \geq \left\|\varphi \circ (a_\varphi(1+2\varepsilon))\chi_{\{e_1\}}\right\|_e = \beta$$

for any $n \in \mathbb{N}$. Hence, by the fact that $\varphi \circ x$ is an $ULUM$-point in e, repeating the same argumentation as in the proof of Theorem 4.5, a real number $\gamma(\beta) > 0$ can be found such that $I_\varphi(x + y_n) \geq 1 + 3\gamma(\beta)$ for any $n \in \mathbb{N}$. Further, defining

$$D_\eta = \{i \in \mathbb{N} : x(t) \leq a_\varphi(1 + \eta)\}$$

for any $\eta > 0$, similarly as in the proof of Theorem 4.5, we get

$$\left\| \varphi \circ \left(x\chi_{T \setminus D_{\eta_0}} \right) + \varphi \circ \left((a_\varphi + y_n) \chi_{B_n} \right) \right\|_e \geq 1 + 2\gamma(\beta)$$

for a certain real number $\eta_0 \in (0, \min\{\varepsilon, a_\varphi\})$ and any $n \in \mathbb{N}$. Since the function φ is uniformly convex on the interval $(\eta_0, \alpha b_\varphi)$, for any $K > 1$ there is $\lambda_K \in (1, K)$ such that $\varphi(v) \leq K\varphi\left(\frac{v}{\lambda_K}\right)$ for any $\frac{v}{\lambda_K} \in \left[a_\varphi\left(1 + \frac{1}{2}\eta_0\right), \frac{\alpha b_\varphi}{\lambda_K}\right]$. Again, repeating the same argumentation as at the end of the proof of Theorem 4.5, we get

$$I_\varphi\left(\frac{x + y_n}{\lambda_K}\right) \geq 1$$

for any $n \in \mathbb{N}$, whence $\|x + y_n\|_\varphi > \lambda_K > 1$ for for any $n \in \mathbb{N}$. This contradiction finishes the proof. □

5. Application to Orlicz-Lorentz and Orlicz spaces

Now we consider Orlicz-Lorentz spaces as a special class of Calderón-Lozanovskiĭ spaces.

Recall that the function $\omega : [0, \gamma) \to R_+$ with $\gamma = \mu(T)$ is said to be the *weight function*, if it is nonnegative, nonincreasing and locally integrable function with the respect to the Lebesgue measure μ. Then the Lorentz function space Λ_ω consists of all functions $x \in L^0(T, \Sigma, \mu)$ such that $\|x\| = \int_0^\gamma x^*(t)\omega(t)dt < \infty$, where x^* is the nonincreasing rearrangement of x. Recall also that Lorentz sequence space λ_ω consists of all sequences $x = (x(i))$ such that $\sum_{i=1}^\infty x^*(i)\omega(i) < \infty$, where $\omega = (\omega(i))$ is a *weight sequence*, that is ω is a nonincreasing sequence of nonnegative real numbers. If $E = \Lambda_\omega$ or $e = \lambda_\omega$, then the Calderon-Lozanovskiĭ space E_φ (resp. e_φ) is the corresponding Orlicz-Lorentz function (resp. sequence) space $\Lambda_\varphi := (\Lambda_\omega)_\varphi$ (resp. $\lambda_\varphi := (\lambda_\omega)_\varphi$) (see [4], [12], [13], [19], [20], [22] and [23]).

Corollary 5.1. *Suppose that*

a) *the weight function ω is positive on $[0, \gamma)$ and $\int_0^\infty \omega(t)\,dt = \infty$ in the case when $\gamma = \infty$ and μ is nonatomic;*

b) *$\varphi(b_\varphi)\,\omega(1) > 1$ and $\sum_{n=1}^\infty \omega(i) = \infty$ whenever μ is the counting measure.*

A point $x \in S(\Lambda_\varphi)_+$ (resp. $x \in S(\lambda_\varphi)_+$) is an $ULUM$-point if and only if $x = b_\varphi\chi_T$ (resp. $x = b_\varphi\chi_\mathbb{N}$) or one of the following conditions is satisfied:

(i) *$\varphi \in \Delta_2(\infty)$, μ is nonatomic and either ($\mu(T) < \infty$ and $\alpha_\varphi = 0$) or ($a_\varphi > 0$, $x \geq a_\varphi\chi_T$, $I_\varphi(x) = 1$ and $\mu(T) \leq \infty$);*

(ii) *$\varphi \in \Delta_2(\mathbb{R}_+)$, μ is nonatomic and $\mu(T) = \infty$;*

(iii) $T = \mathbb{N}$, μ *is the counting measure and either* $\varphi \in \Delta_2(0)$ *or* ($a_\varphi > 0$, $x \geq a_\varphi \chi_\mathbb{N}$ *and* $I_\varphi(x) = 1$).

Proof. By Propositions 3 and 4 in [10], under the assumptions on ω the space Λ_ω as well as λ_ω is upper locally uniformly monotone. Hence $\varphi \circ x$ is an $ULUM$-point for any $x \in S(\Lambda_\varphi)_+$ (resp. $x \in S(\lambda_\varphi)_+$). Then the characterization follows immediately from Theorems 4.2, 4.5 and 4.6. □

Notice that if $\omega = \chi_T$ (resp. $\chi_\mathbb{N}$), then $\Lambda_\omega = L^1$ (resp. $\lambda_\omega = l^1$) and then the Orlicz-Lorentz space becomes an Orlicz space. Moreover, the assumption (a) from Corollary 5.1 is satisfied automatically, but (b) is reduced to the condition $\varphi(b_\varphi) > 1$. Hence the criterion for Orlicz spaces is the following:

Corollary 5.2. *Let* φ *be an Orlicz function such that* $\varphi(b_\varphi) > 1$. *A point* $x \in S(L_\varphi)_+$ *(resp.* $x \in S(l_\varphi)_+$*) is an* $ULUM$*-point if and only if* $x = b_\varphi \chi_T$ *(resp.* $x = b_\varphi \chi_\mathbb{N}$*) or one of the conditions* (*i*), (*ii*) *or* (*iii*) *from Corollary* 5.1 *is satisfied.*

6. Open problems.

We do not know answers to the following questions:

1. Is Theorem 4.2 true without requiring that E is an order continuous symmetric Köthe space?

2. Suppose that $a_\varphi > 0$, E (resp. e) is an order continuous symmetric Köthe function (resp. sequence) space such that $L^\infty \not\hookrightarrow E$. Is $\varphi \circ x$ always an $ULUM$-point in E (resp. in e) whenever $x \in S(E_\varphi)_+$ (resp. $x \in S(e_\varphi)_+$) is an $ULUM$-point? (see Theorem 4.5 and 4.6).

3. According to Lemma 1.4 from [22], the assumption $\varphi(b_\varphi)\|e_1\|_e \geq 1$ is natural. Theorems 4.2 and 4.6 are proved under the assumption that $\varphi(b_\varphi)\|e_1\|_e > 1$. Are Theorems 4.2 and 4.6 also true when $\varphi(b_\varphi)\|e_1\|_e = 1$?

References

[1] M. A. Akcoglu, L. Sucheston: On uniform monotonicity of norms and ergodic theorems in function spaces, Rend. Circ. Mat. Palermo, II. Ser., Suppl. 8 (1985) 325–335.

[2] C. Bennett, R. Sharpley: Interpolation of Operators, Academic Press, Boston (1988).

[3] G. Birkhoff: Lattice Theory, 3rd Ed., AMS, Providence (1967).

[4] J. Cerdá, H. Hudzik, A. Kamińska, M. Mastyło: Geometric properties of symmetric spaces with applications to Orlicz-Lorentz spaces, Positivity 2 (1998) 311–337.

[5] J. Cerdá, H. Hudzik, M. Mastyło: On the geometry of some Calderón-Lozanovskiĭ interpolation spaces, Indag. Math., New Ser. 6(1) (1995) 35–49.

[6] I. Dobrakov: On submeasures. I, Dissertationes Math., Warszawa (1974).

[7] P. Foralewski, H. Hudzik: On some geometric properties of generalized Calderón-Lozanovskiĭ sequence spaces, Acta Math. Hung. 80(1–2) (1998) 55–66.

[8] P. Foralewski, H. Hudzik: Some basic properties of generalized Calderón-Lozanovskiĭ spaces, Collect. Math. 48(4–6) (1997) 523–538.

[9] P. Foralewski, H. Hudzik: On some geometrical and topological properties of generalized Calderón-Lozanovskiĭ sequence spaces, Houston J. Math. 25(3) (1999) 523–542.

[10] P. Foralewski, P. Kolwicz: Local uniform rotundity in Calderón-Lozanovskiĭ spaces, J. Convex Analysis 14(2) (2007) 395–412.

[11] H. Hudzik, A. Kamińska: Monotonicity properties of Lorentz spaces, Proc. Amer. Math. Soc. 123(9) (1995) 2715–2721.

[12] H. Hudzik, A. Kamińska, M. Mastyło: Geometric properties of some Calderón-Lozanovskiĭ spaces and Orlicz-Lorentz spaces, Houston J. Math. 22 (1996) 639–663.

[13] H. Hudzik, A. Kamińska, M. Mastyło: On geometric properties of Orlicz-Lorentz spaces, Can. Math. Bull. 40(3) (1997) 316–329.

[14] H. Hudzik, A. Kamińska, M. Mastyło: Monotonicity and rotundity properties in Banach lattices, Rocky Mt. J. Math. 30(3) (2000) 933–949.

[15] H. Hudzik, P. Kolwicz, A. Narloch: Local rotundity structure of Calderón-Lozanovskiĭ spaces, Indag. Math., New Ser. 17(3) (2006) 373–395.

[16] H. Hudzik, W. Kurc: Monotonicity properties of Musielak-Orlicz spaces and dominated best approximation in Banach lattices, J. Approximation Theory 95 (1998) 353–368.

[17] H. Hudzik, X. Liu, T. Wang: Points of monotonicity in Musielak-Orlicz function spaces endowed with the Luxemburg norm, Arch. Math. 82(6) (2004) 534–545.

[18] H. Hudzik, A. Narloch: Local monotonicity structure of Calderón-Lozanovskiĭ spaces, Indag. Math., New Ser. 15(1) (2004) 1–12.

[19] A. Kamińska: Some remarks on Orlicz-Lorentz spaces, Math. Nachr. 147 (1990) 29–38.

[20] A. Kamińska: Uniform convexity of generalized Lorentz spaces, Arch. Math. 56 (1991) 181–188.

[21] L. V. Kantorovich, G. P. Akilov: Functional Analysis, 3rd Ed., Nauka, Moscow (1984), (in Russian).

[22] P. Kolwicz: Rotundity properties in Calderó n-Lozanovskiĭ spaces, Houston J. Math. 31(3) (2005) 883–912.

[23] P. Kolwicz, R. Płuciennik: On uniform rotundity in every direction in Calderón-Lozanovskiĭ sequence spaces, J. Convex Analysis 14(3) (2007) 621–645.

[24] P. Kolwicz, R. Płuciennik: Local $\Delta_2(x)$ condition as a crucial tool for local structure of Calderón-Lozanovskiĭ sequence spaces, J. Math. Anal. Appl. 356(2) (2009) 605–614.

[25] S. G. Krein, Yu. I. Petunin, E. M. Semenov: Interpolation of Linear Operators, Nauka, Moscow (1978), (in Russian).

[26] W. Kurc: Strictly and uniformly monotone Musielak-Orlicz spaces and applications to best approximation, J. Approximation Theory 69(2) (1992) 173–187.

[27] J. Lindenstrauss, L. Tzafriri: Classical Banach spaces. II: Function Spaces, Springer, Berlin (1979).

[28] W. A. J. Luxemburg: Banach Function Spaces, Van Gorcum, Assen (1955).

[29] W. A. J. Luxemburg: Rearrangement invariant Banach function spaces, in: Symposium in Analysis (Kingston / Canada, 1967), Queen's Papers in Pure and Applied Mathematics 10, Queen's University, Kingston / Canada (1967) 83–144.

[30] W. Wnuk: Banach Lattices with Order Continuous Norms, Polish Scientific Publishers PWN, Warszawa (1999).