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Non-Enlargeable Operators and Self-Cancelling Operators

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The ε-enlargement of a maximal monotone operator is a construct similar to the Brøndsted and Rockafellar ε-subdifferential enlargement of the subdifferential. Like the ε-subdifferential, the ε-enlargement of a maximal monotone operator has practical and theoretical applications.

In a recent paper in Journal of Convex Analysis, Burachik and Iusem studied conditions under which a maximal monotone operator is non-enlargeable, that is, its ε-enlargement coincides with the operator. Burachik and Iusem studied these non-enlargeable operators in reflexive Banach spaces, assuming the interior of the domain of the operator to be nonempty. In the present work, we remove the assumption on the domain of non-enlargeable operators and also present partial results for the non-reflexive case.

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1. Introduction

Let X be a real Banach space. We use the notation X^* for the topological dual of X and $\langle \cdot, \cdot \rangle$ for the duality product in $X \times X^*$:

$$\langle x, x^* \rangle = x^*(x).$$

Whenever necessary, we will identify X with its image under the canonical injection of X into X^{**}. A point-to-set operator $T : X \rightrightarrows X^*$ is a relation on $X \times X^*$:

$$T \subset X \times X^*$$

and $x^* \in T(x)$ means $(x, x^*) \in T$. From now on, for $T : X \rightrightarrows X^*$, we define

$$-T = \{(x, -x^*) \mid (x, x^*) \in T\},$$

so that $-T : X \rightrightarrows X^*$, $(-T)(x) = -(T(x))$. An operator $T : X \rightrightarrows X^*$ is *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in T.$$

and it is *maximal monotone* if it is monotone and maximal (with respect to the inclusion) in the family of monotone operators of X to X^*. Maximal monotone operators

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in Banach spaces arise, for example, in the study of PDE's, equilibrium problems and calculus of variations.

The *conjugate* of $f : X \to \overline{\mathbb{R}}$ is $f^* : X^* \to \overline{\mathbb{R}}$,

$$f^*(x^*) = \sup_{x \in X} \langle x, x^* \rangle - f(x).$$

and the *effective domain* of f is

$$\operatorname{dom}(f) = \{x \in X \mid f(x) < \infty\}.$$

The subdifferential of f is the point to set operator $\partial f : X \rightrightarrows X^*$,

$$\partial f(x) = \{x^* \mid f(y) \geq f(x) + \langle y - x, x^* \rangle, \ \forall y\}.$$

In a paper where many fundamental techniques were introduced, Rockafellar proved that the subdifferential of a proper, convex, lower semicontinuous function in a Banach space is maximal monotone [27]. Rockafellar's proof relied on the ε-*subdifferential*, a concept introduced previously by Brøndsted and Rockafellar [4], which is defined as follows, for $f : X \to \overline{\mathbb{R}}$:

$$\partial_\varepsilon f(x) = \{x^* \mid f(y) \geq f(x) + \langle y - x, x^* \rangle - \varepsilon, \ \forall y\}.$$

Note that $\partial f \subset \partial_\varepsilon f$, for any $\varepsilon \geq 0$, and the inclusion may be proper if $\varepsilon > 0$. Hence, the ε-subdifferential is an "enlargement" of the subdifferential. It is easy to check that for any $\varepsilon > 0$, the ε-subdifferential of a proper, convex, lower semicontinuous function f is non-empty at any point where f is finite. One of the key properties of the ε-subdifferential of a proper, convex, lower semicontinuous function f, used on Rockafellar proof is the fact, proved by Brøndsted and Rockafellar [4], that points of $\partial_\varepsilon f$ are close to ∂f, and this distance can be estimated. This property is known as Brøndsted-Rockafellar property of the ε-subdifferential. Although created by Brøndsted and Rockafellar for theoretical purposes, the ε-subdifferential has extensive practical applications in convex optimization [33, 34, 11, 22, 19].

If $T : X \rightrightarrows X^*$ is maximal monotone, then inclusion on T may be characterized by a family of inequalities:

$$(x, x^*) \in T \iff (\langle x - y, x^* - y^* \rangle \geq 0, \ \forall (y, y^*) \in T).$$

Veselý [35] defined an ε-monotone operator as an operator which satisfies this condition with $-\varepsilon$ replacing 0 in the above inequality. Martínez-Legaz and Thera [23] also noted that the above inequality could be relaxed. Burachik, Iusem and Svaiter proposed the T^ε enlargement [7] of a maximal monotone operator $T : X \rightrightarrows X^*$ as follows: for $\varepsilon \geq 0$,

$$(x, x^*) \in T^\varepsilon \iff (\langle x - y, x^* - y^* \rangle \geq -\varepsilon, \ \forall (y, y^*) \in T). \tag{1}$$

The T^ε enlargement has many similarities to the ε-subdifferential proposed by Brøndsted and Rockafellar [4]. For example, in the interior of the domain of T, for ε bounded away from 0, the mapping

$$(x, \varepsilon) \mapsto T^\varepsilon(x) = \{x^* \mid (x, x^*) \in T^\varepsilon\}$$

is locally Lipschitz continuous, with respect to the Hausdorff metric. This enlargement also satisfies (in reflexive spaces) a property similar to the Brøndsted-Rockafellar property of the ε-subdifferential. Beside that, the T^ε enlargement has also theoretical [26, 25, 21, 3] and algorithmic applications [28, 9, 8, 29, 20, 30, 24]. For a survey in the subject, see [6].

Our aim is to investigate those maximal monotone operators $T : X \rightrightarrows X^*$ which are "non-enlargeable", that is,

$$T^\varepsilon = T, \quad \forall \varepsilon \geq 0. \tag{2}$$

This question has been previously addressed by Burachik and Iusem [5] and the present work is inspired by that article of Burachik and Iusem.

The T^ε enlargement is one among a family of enlargements, defined and studied in [31]. These enlargements share some basic properties and T^ε is the biggest element in this family. Moreover, if T happens to be the subdifferential of some convex function f, then the ε-subdifferential of f also belongs to this family and the inclusion

$$\partial_\varepsilon f \subset (\partial f)^\varepsilon$$

is proper, in general.

The T^ε enlargement is closely tied to the Fitzpatrick function, which we discuss next. To honor Fitzpatrick, we shall use φ, the Greek "f", to denote Fitzpatrick function [12] associated with a maximal monotone operator $T : X \rightrightarrows X^*$:

$$\varphi_T(x, x^*) = \sup_{(y,y^*)\in T} \langle x, y^*\rangle + \langle y, x^*\rangle - \langle y, y^*\rangle . \tag{3}$$

Observe that φ_T is convex, lower semicontinuous on the $w \times w^*$ topology of $X \times X^*$ and

$$\varphi_T(x, x^*) \geq \langle x, x^*\rangle, \qquad T = \{(x, x^*) \mid \varphi_T(x, x^*) = \langle x, x^*\rangle\} . \tag{4}$$

The above inequality is a generalization of Fenchel-Young inequality. Indeed if f is a proper convex lower semicontinuous function on X, then

$$f(x) + f^*(x^*) \geq \langle x, x^*\rangle, \qquad \partial f = \{(x, x^*) \mid f(x) + f^*(x^*) = \langle x, x^*\rangle\}.$$

So given f, defining the Fenchel-Young function associated with f,

$$h_{FY} : X \times X^* \to \overline{\mathbb{R}}, \qquad h_{FY}(x, x^*) = f(x) + f(x^*), \tag{5}$$

we have a convex function bounded bellow by the duality product and equal to it on ∂f. Fitzpatrick proved that associated with each maximal monotone operator T there is a family of functions with these properties and that φ_T is the minimal element of this family. Brøndsted and Rockafellar observed that the ε-subdifferential can be characterized by the function h_{FY}:

$$\begin{aligned} \partial_\varepsilon f &= \{(x, x^*) \mid f(x) + f^*(x^*) \leq \langle x, x^*\rangle + \varepsilon\} \\ &= \{(x, x^*) \mid h_{FY}(x, x^*) \leq \langle x, x^*\rangle + \varepsilon\}. \end{aligned} \tag{6}$$

Likewise, it is trivial to check that φ_T characterizes the T^ε enlargement of a maximal monotone $T : X \rightrightarrows X^*$:

$$T^\varepsilon = \{(x, x^*) \mid \varphi_T(x, x^*) \leq \langle x, x^*\rangle + \varepsilon\}. \tag{7}$$

Given a maximal monotone operator $T : X \rightrightarrows X^*$, Fitzpatrick defined [12] the family $\mathcal{F}_T$ as those convex, lower semicontinuous functions in $X \times X^*$ which are bounded below by the duality product and coincide with it on T:

$$\mathcal{F}_T = \left\{ h \in \overline{\mathbb{R}}^{X\times X^*} \;\middle|\; \begin{array}{l} h \text{ is convex and lower semicontinuous} \\ \langle x, x^*\rangle \le h(x,x^*), \quad \forall (x,x^*) \in X \times X^* \\ (x,x^*) \in T \Rightarrow h(x,x^*) = \langle x, x^*\rangle \end{array} \right\}. \tag{8}$$

Fitzpatrick proved that φ_T belongs to this family and it is its minimal element. Moreover, he also proved that if $h \in \mathcal{F}_T$ then h represents T in the following sense:

$$(x,x^*) \in T \iff h(x,x^*) = \langle x, x^*\rangle.$$

For the case of the subdifferential of a proper convex lower semicontinuous function f, defining h_{FY} as (5),

$$h_{FY} \in \mathcal{F}_{\partial f}.$$

Moreover, h_{FY} is separable. It would be most desirable to find separable elements in $\mathcal{F}_T$. Unfortunately, this family has a separable element if and only if T is a subdifferential [10]. Another interesting property of h_{FY} is that this function is a fixed point of the mapping

$$\mathcal{J} : \overline{\mathbb{R}}^{X\times X^*} \to \overline{\mathbb{R}}^{X\times X^*}, \qquad \mathcal{J}g(x,x^*) = g^*(x^*,x).$$

Burachik and Svaiter observed that $\mathcal{F}_T$ is invariant under $\mathcal{J}$ [10] and Svaiter proved that there always exists a fixed point of $\mathcal{J}$ in $\mathcal{F}_T$[32]. These fixed points have meet some applications in the study of PDE'S under the attractive name "self-dual" [14, 13, 18, 15, 17, 16] in the pioneering works of Ghoussoub.

2. Non-enlargeable operators

Direct use of (7) shows that problem (2) is equivalent to finding those maximal monotone operators T such that

$$\operatorname{dom}(\varphi_T) = T. \tag{9}$$

It has been recently proved in [1] and in [2], independently, that if a maximal monotone $T \subset X \times X^*$ is convex, then it is an affine subspace of $X \times X^*$. As φ_T is convex, the above condition implies that T is convex. Therefore, we can reduce our problem to finding those maximal monotone operators which are affine subspaces and satisfy (2).

If $T \subset X \times X^*$ and $(x_0, x_0^*) \in X \times X^*$, defining

$$\begin{aligned} T_0 &= T - \{(x_0, x_0^*)\} \\ &= \{(x - x_0, x^* - x_0^*) \mid (x,x^*) \in T\}. \end{aligned}$$

We have

$$(T_0)^\varepsilon = T^\varepsilon - \{(x_0, x_0^*)\}. \tag{10}$$

So, we can restrict our attention tho those maximal monotone operators which are subspaces of $X \times X^*$ and satisfy (2), and the general case will be obtained by translations of these subspaces.

Define, for $B \subset X \times X^*$

$$B^{\vdash} = \{(y, y^*) \mid \langle x, y^* \rangle + \langle y, x^* \rangle = 0, \ \forall (x, x^*) \in B\}. \tag{11}$$

Note that $B^{\vdash}$ can be written in terms of the annihilator of a family in $(X \times X^*)^*$:

$$B^{\vdash} = {}^a\{(x^*, x) \mid (x, x^*) \in B\}$$

Lemma 2.1. *If $T \subset X \times X^*$ is maximal monotone and a subspace, then*

1. $T^{\vdash} \subset \{(x, x^*) \mid \varphi_T(x, x^*) = 0\}$,
2. $T \cap T^{\vdash} = T \cap \{(x, x^*) \mid \langle x, x^* \rangle = 0\}$.

Proof. To prove item *1.*, take $(x, x^*) \in T^{\vdash}$. As $(0, 0) \in T$, for any $(y, y^*) \in T$, $\langle y, y^* \rangle \geq 0$. Therefore

$$\begin{aligned} \varphi_T(x, x^*) &= \sup_{(y,y^*) \in T} \langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle \\ &= \sup_{(y,y^*)} -\langle y, y^* \rangle = 0. \end{aligned}$$

To prove item *2.*, first use item *1.* to obtain

$$T \cap T^{\vdash} \subset T \cap \{(x, x^*) \mid \varphi_T(x, x^*) = 0\}.$$

As $\varphi_T(x, x^*) = \langle x, x^* \rangle$ for any $(x, x^*) \in T$, we conclude

$$T \cap T^{\vdash} \subset T \cap \{(x, x^*) \mid \langle x, x^* \rangle = 0\}.$$

To prove the other inclusion, take $(x, x^*) \in T \cap \{(x, x^*) \mid \langle x, x^* \rangle = 0\}$. As T is a subspace, if $(y, y^*) \in T$, then, for any $\lambda \in \mathbb{R}$

$$\lambda(x, x^*) + (y, y^*) \in T.$$

As $(0, 0) \in T$,

$$\langle \lambda x + y, \lambda x^* + y^* \rangle = \lambda[\langle x, y^* \rangle + \langle y, x^* \rangle] + \langle y, y \rangle \geq 0.$$

As λ is arbitrary, we conclude that the expression inside the brackets must be 0. $\square$

It is interesting to observe that $\langle x, x^* \rangle$ is non-linear and non-convex in (x, x^*). Even though, the points on T where this expression vanish is a subspace, which may be trivial.

We will be concerned with a special type of linear point-to set operators

Definition 2.2. An operator $A : X \rightrightarrows X^*$ is **self-cancelling** if A is a subspace and

$$\langle x, x^* \rangle = 0, \quad \forall (x, x^*) \in A.$$

This definition is an extension of the definition of skew-symmetric operators of Burachik and Iusem [5] and of the definition of skew linear of Bauschke, Wang and Yau [2]. The relations between these classes will be discussed in the Section 3.

Lemma 2.3. *If $A \subset X \times X^*$ is self-cancelling, then $A \subset A^{\vdash}$.*

Proof. Take $(x, x^*), (y, y^*) \in A$. Then, $(x + y, x^* + y^*) \in A$ and so

$$\langle x + y, x^* + y^* \rangle = \langle x, x^* \rangle + \langle x, y^* \rangle + \langle y, x^* \rangle + \langle y, y^* \rangle = 0.$$

To end the proof, note that $\langle x, x^* \rangle = \langle y, y^* \rangle = 0$. □

Lemma 2.4. *If $A \subset X \times X^*$ is self-cancelling and $A^{\vdash}$ is maximal monotone then, for any $(x_0, x_0^*) \in X \times X^*$, the operator*

$$T = A^{\vdash} + \{(x_0, x_0^*)\}$$

is non-enlargeable, or equivalently, $\mathrm{dom}(\varphi_T) = T$.

Proof. In view of (10), it suffices to prove this lemma for $(x_0, x_0^*) = 0$. In that case, if $(x, x^*) \notin A^{\vdash}$, there exists $(y, y^*) \in A$ such that

$$\langle x, y^* \rangle + \langle y, x^* \rangle \neq 0.$$

As A is a subspace and $A \subset A^{\vdash}$,

$$\varphi_{A^{\vdash}}(x, x^*) \geq \sup_{\lambda \in \mathbb{R}} \langle x, \lambda y^* \rangle + \langle \lambda y, x^* \rangle - \langle \lambda y, \lambda y^* \rangle = \sup_{\lambda \in \mathbb{R}} \langle x, \lambda y^* \rangle + \langle \lambda y, x^* \rangle.$$

Combining the above equation we obtain $\varphi_{A^{\vdash}}(x, x^*) \geq \infty$. □

Theorem 2.5. *If X is reflexive, then a maximal monotone T is non-enlargeable, if and only if there exists an self-canceling A and $(x, x^*) \in X \times X^*$ such that $A^{\vdash}$ is maximal monotone and*

$$T = A^{\vdash} + \{(x, x^*)\}.$$

Moreover, if T is non-enlargeable and $(x, x^*) \in T$, then the maximal A satisfying the above condition is

$$(T - (x, x^*))^{\vdash}.$$

Proof. First assume that T is non-enlargeable and $(0, 0) \in T$. Define

$$A = T^{\vdash}.$$

Using Lemma 2.1, item 1, we conclude that $\varphi_T(x, x^*) = 0$ for all $(x, x^*) \in A$. As $\mathrm{dom}(\varphi_T) = T$, we conclude that $A \subset T$. Therefore,

$$A = T \cap A = T \cap T^{\vdash}.$$

Combining the above equation with the definition of A and Lemma 2.1, item 2, we conclude that A is self-cancelling. Moreover A is the maximal self-cancelling operator contained in T. As T is a closed subspace and X is reflexive, direct use of Hahn-Banach theorem yields

$$T = (T^{\vdash})^{\vdash} = A^{\vdash}.$$

Note also that the above defined A is maximal in the family

$$\{B \subset X \times X^* \mid T = B^{\vdash}\}$$

Conversely, if for some self-cancelling A, $T = A^{\vdash}$, then according to Lemma 2.4 T is non-enlargeable.

The general case follows now using (10). □

3. On maximal monotone operators obtained from self-cancelling operators

Now we shall analyze those maximal monotone operators discussed in Theorem 2.5.

Proposition 3.1. *If $A \subset X \times X^*$ is self-cancelling and $A^\vdash$ is monotone, then $A^\vdash$ is maximal monotone.*

Proof. Take $(x_0, x_0^*) \notin A^\vdash$. Then there exists $(y, y^*) \in A$ such that

$$\langle x_0, y\rangle + \langle y, x_0^*\rangle \neq 0.$$

Using Lemma 2.3 we conclude that $(y, y^*) \in A^\vdash$. For any $\lambda \in \mathbb{R}$,

$$\langle x_0 + \lambda y, x_0^* + \lambda y^*\rangle = \langle x_0, x_0^*\rangle + \lambda[\langle x_0, y^*\rangle + \langle y, x_0^*\rangle].$$

Combining the two above equations with the fact that $A^\vdash$ is a subspace, we conclude that $\{(x_0, x_0^*)\} \cup A^\vdash$ is not monotone. $\square$

Observe that in Lemma 2.1, the maximal monotone operator T may not be in the family of $A^\vdash$ with A self-cancelling.

In the next proposition we will use the following result, which follows from the Hahn-Banach theorem in locally convex linear spaces.

Lemma 3.2. *Let Z be a locally convex topological linear space. If A is a subspace of Z, then*

$$\mathrm{cl}(A) = \{z \in Z \mid f(z) = 0,\ \forall f \in A^a\}$$

where A^a is the family of continuous linear functionals which vanish at A.

Proposition 3.3. *If A is self-cancelling and $A^\vdash$ is monotone, then*

$$\tilde{A} := \mathrm{cl}_{w\times w*} A = \mathrm{cl}_{s\times w*} A$$

is a maximal element of the family of self-cancelling operators, where $\mathrm{cl}_{w\times w*}$ and $\mathrm{cl}_{s\times w*}$ denotes the closure in the weak$\times$weak-$*$ and strong$\times$weak-$*$ topologies, respectively.

Proof. The dual of $X \times X^*$ in the weak$\times$weak-$*$ and strong$\times$weak-$*$ topologies is $X^* \times X$. Combining this observation with Lemma 3.2 we conclude that the closure of A in these topologies is the same and that

$$\tilde{A} = (A^\vdash)^\vdash.$$

Using Lemma 2.3 we obtain $A \subset A^\vdash$. Direct inspection of definition (11) shows that $A^\vdash$ is closed in the weak$\times$weak-$*$ topology. Therefore

$$A \subset \tilde{A} \subset A^\vdash.$$

Combining the two above equations we obtain $(A^\vdash)^\vdash \subset A^\vdash$. If $(x, x^*) \in \tilde{A} = (A^\vdash)^\vdash$, then $(x, x^*) \in A^\vdash$ and $2\langle x, x^*\rangle = 0$. Therefore $\tilde{A}$ is self-cancelling.

Suppose that $B = [\tilde{A} \cup \{x_0, x_0^*\}]$ is self-cancelling. In that case,

$$B^\vdash \subset A^\vdash.$$

In particular, $B^{\vdash}$ is monotone. Using Proposition 3.1 we conclude that $B^{\vdash}$ is maximal monotone. Hence the above inclusion holds as an equality. Therefore,

$$B \subset (B^{\vdash})^{\vdash} = (A^{\vdash})^{\vdash} = \tilde{A},$$

and $(x_0, x_0^*) \in \tilde{A}$. $\square$

Now, we will discuss maximal self-cancelling operators.

Lemma 3.4. *Let $A : X \rightrightarrows X^*$ self-cancelling. Then A is maximal in the family of self-cancelling operators if and only if*

$$A = A^{\vdash} \cap \{(x, x^*) \mid \langle x, x^* \rangle = 0\}.$$

Proof. As A is self-cancelling, using Definition 2.2 and Lemma 2.3 we have

$$A \subset A^{\vdash} \cap \{(x, x^*) \mid \langle x, x^* \rangle = 0\}. \tag{12}$$

If the above inclusion holds as an equality, $A \subset B$ and B is self-cancelling, then $B^{\vdash} \subset A^{\vdash}$ and by the same token

$$B \subset B^{\vdash} \cap \{(x, x^*) \mid \langle x, x^* \rangle = 0\} \subset A^{\vdash} \cap \{(x, x^*) \mid \langle x, x^* \rangle = 0\} = A.$$

Therefore, in this case A is maximal self-cancelling.

Conversely, if A is maximal self-cancelling and

$$(z, z^*) \in A^{\vdash}, \qquad \langle z, z^* \rangle = 0,$$

then, for any $(x, x^*) \in A$ and $\lambda \in \mathbb{R}$,

$$\langle x + \lambda z, x^* + \lambda z^* \rangle = 0.$$

This means that the subspace generated by $A \cup \{(z, z^*)\}$ is self-cancelling. Therefore, $(z, z^*) \in A$. Altogether we have

$$A^{\vdash} \cap \{(x, x^*) \mid \langle x, x^* \rangle = 0\} \subset A$$

which combined with the (12) shows that the above inclusion holds as an equality. $\square$

A natural question is whether $A^{\vdash}$ is maximal monotone whenever A is maximal self-cancelling. Up to now we have a partial answer to this question.

Proposition 3.5. *If $A : X \rightrightarrows X^*$ is maximal self-cancelling, the $A^{\vdash}$ or $-A^{\vdash}$ is maximal monotone.*

Proof. Recall that $-A = \{(x, -x^*) \mid (x, x^*) \in A\}$ so that

$$(-A)^{\vdash} = -(A^{\vdash}) \ ,$$

Take $(x, x^*), (y, y^*) \in A^{\vdash}$. Suppose that

$$\langle x, x^* \rangle > 0, \qquad \langle y, y^* \rangle < 0. \tag{13}$$

Then, for some $\theta \in (0,1)$

$$z_\theta = \theta x + (1-\theta)y, \qquad z^*_\theta = \theta x^* + (1-\theta)y^*.$$

satisfy

$$\langle z_\theta, z^*_\theta \rangle = 0.$$

Hence, using Lemma 3.4 we have $(z_\theta, z^*_\theta) \in A$. Therefore $\langle z_\theta, y^* \rangle + \langle y, z^*_\theta \rangle = 0$. Direct use of the definitions of z_θ, z^*_θ gives

$$\theta(x, x^*) = (z_\theta, z^*_\theta) - (1-\theta)(y, y^*),$$

which readily implies

$$\begin{aligned} \theta^2 \langle x, x^* \rangle &= \langle z_\theta - (1-\theta)y, z^*_\theta - (1-\theta)y^* \rangle \\ &= (1-\theta)^2 \langle y, y^* \rangle \end{aligned}$$

in contradiction with (13). Therefore (13) can not hold for $(x, x^*), (y, y^*) \in A^\vdash$ and $A^\vdash$ or $-A^\vdash$ is monotone. Maximal monotonicity of $A^\vdash$ or $-A^\vdash$ now follows from Proposition 3.1. □

Working in the setting of reflexive Banach spaces, Burachik and Iusem [5] defined a skew-symmetric operator as a linear continuous operator $L : X \to X^*$ such that

$$L = -L^*$$

where L^* is the adjoint of L. As $L^* : X^{**} \to X^*$, it is natural to consider, in a reflexive Banach space, $L^* : X \to X^*$. In that case, L^* is defined as

$$\langle Lx, y \rangle = \langle x, L^* y \rangle, \quad \forall x, y \in X.$$

Note that $L^* = (-L)^\vdash$. Bauschke, Wang and Yao [2], still working in reflexive Banach spaces, extended this definition of adjoint to an arbitrary linear point-to-set operator $L : X \rightrightarrows X^*$ as $L^* = (-L)^\vdash$. For these authors, a point to set operator $L : X \rightrightarrows X^*$ is skew if it is linear and $L = -L^*$. It is trivial to verify that a skew-symmetric operator (in the sense of [5]) is always a skew operator (in the sense on [2]). We will use the the notation $D(M)$ and $R(M)$ for the domain and the range of $M : X \rightrightarrows X^*$,

$$D(M) := \{x \in X \mid M(x) \neq \emptyset\}, \qquad R(M) := \{x^* \in X \mid M^{-1}(x^*) \neq \emptyset\}.$$

Lemma 3.6. *Let $M : X \rightrightarrows X^*$ be a linear point to set operator.*

1. *If M is a skew operator, then it is maximal self-cancelling.*
2. *If M is maximal self-cancelling and $D(M)$ is closed then it is skew.*
3. *If M is maximal self-cancelling, $R(M)$ is closed and X is reflexive, then it is skew.*

Proof. To prove item *1.*, suppose that M is skew. Then, M is self-cancelling. If A is self-cancelling and $M \subset A$, then using Lemma 2.3 and (11) we have

$$A \subset A^\vdash \subset M.$$

Therefore, $A = M$.

To prove item *2.*, suppose that M is maximal self-cancelling. Take

$$(x_0, x_0^*) \in M^{\vdash}$$

If $x_0 \notin D(M)$, then there exists y^* such that

$$\langle x, y^* \rangle = 0, \quad \forall x \in D(M), \qquad \langle x_0, y^* \rangle \neq 0.$$

In that case, $(0, y^*) \in M$ and $\langle x_0, y^* \rangle + \langle 0, x_0^* \rangle \neq 0$, in contradiction with the assumption $(x_0, x_0^*) \in M^{\vdash}$. Hence, $x_0 \in D(A)$ and there exists z^* such that $(x_0, z^*) \in M$. Therefore,

$$(x_0, x_0^*) - (x_0, z^*) \in M^{\vdash}.$$

To simplify the notation, let $u^* = x_0^* - z^*$. We have just proved that $(0, u^*) \in M^{\vdash}$. Let

$$V = \text{span}(M \cup \{(0, u^*)\}).$$

If $(x, x^*) \in M$ and $\lambda \in \mathbb{R}$, then

$$\langle x, x^* + \lambda u^* \rangle = \langle x, x^* \rangle + \lambda \langle x, u^* \rangle = 0.$$

Hence, V is self-cancelling. As M is maximal self-cancelling,

$$(0, u^*) = (x_0, x_0^*) - (x_0, z^*) \in M,$$

and $(x_0, x_0^*) \in M$. Altogether, using also Lemma 2.3 we have

$$M \subset M^{\vdash} \subset M$$

and so M is skew.

Item *3.* follows from item *2.*, applied to $X' = X^*$ and

$$M' = \{(x^*, x) \mid (x, x^*) \in M\}.$$

$\square$

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