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On Closed Starshaped Sets

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We present several new results on closed starshaped sets in $\mathbb{R}^d$ which mainly refer to the kernel and the boundary structure of such sets. In particular, we introduce so-called dispensable points which, in a certain sense, play the same role for closed starshaped sets that extreme points play for convex sets. Based on this notion we prove an analogue of the famous Krein-Milman theorem.

Keywords: Asymptotic structure, cone, dispensable point, extreme point, kernel, Krein-Milman theorem, Minkowski's theorem, starshaped set, support cone

1991 Mathematics Subject Classification: 52-99, 52A01, 52A30

1. Introduction

This paper contains various new results on closed starshaped sets in d-dimensional linear space $\mathbb{R}^d$, $d \geq 2$. The main tools are introduced in Sections 2 and 3, such as basic notions from set theory, topology, linear algebra, convexity and, particularly important, properties and special types of cones. A general reference for cones is [3]. In Section 4 we derive some asymptotic properties of unbounded closed starshaped sets, including also classification results, and Section 5 contains theorems on the existence of support cones of closed starshaped sets and on the representation of such sets via support cones. Section 6 contains some separation theorems on (not only) starshaped sets, and in the final sections the main result is prepared and proved. Namely, we introduce so-called dispensable points for closed starshaped sets which somehow play the same role for closed starshaped sets that extreme points play for closed convex sets. Geometric properties of such points are described, also using the kernels of the starshaped sets under consideration, and finally an analogue of the famous Krein-Milman theorem (replacing "convex set" and "extreme points" by "starshaped set with compact kernel" and "dispensable points", respectively) is given.

2. Basic notation and definitions

Unless otherwise stated, we shall work in $\mathbb{R}^d$ and use standard concepts from set theory, topology, linear algebra, and convexity. If $A \subset \mathbb{R}^d$ is any set, its *complement* will be denoted by A'. If a, b are different points, by $[a, b]$, $[a, b)$ and (a, b) we shall denote the *segment with endpoints a and b*, the *half-line* or *ray with origin a through b*, and

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the *line through* a *and* b, respectively. The replacement of [by] in $[a,b]$ or $[a,b)$ simply means that the endpoint or origin a does not belong to the segment or ray; analogously for the replacement of] by [in $[a,b]$. We agree that $[a,a]=\{a\}$, and we shall say that a segment is *non-degenerate* if its endpoints are different. Similarly, open and closed intervals in $\mathbb{R}$ will be denoted by $]\alpha,\beta[$, $[\alpha,\beta]$, $[\alpha,+\infty[$, etc. We shall denote half-lines from the origin and lines through the origin by Δ and Γ, respectively, and sometimes we shall refer to Δ as a *direction*. When necessary, we shall write Δ_u instead of Δ for the half-line $[0,u)$, where $u \neq 0$. If $A \subset \mathbb{R}^d$, $B \subset \mathbb{R}^d$, and $\Lambda \subset \mathbb{R}$, then $A+B=\{a+b : a \in A, b \in B\}$ and $\Lambda A = \{\lambda a : \lambda \in \Lambda, a \in A\}$. If A or Λ are singletons, we shall simply write $a+B$ and λA.

If $A \subset \mathbb{R}^d$ is any set, its *affine hull* will be denoted by $\operatorname{aff} A$, its *convex hull* by $\operatorname{conv} A$, and its *relative interior* (with respect to $\operatorname{aff} A$) and its *interior* will be denoted by $\operatorname{relint} A$ and $\operatorname{int} A$, respectively. The *closure* and the *boundary* of A will be denoted by $\operatorname{cl} A$ and $\operatorname{bd} A$, respectively. By $U(x,\varepsilon)$ and $B(x,\varepsilon)$ we shall denote the open and closed balls with center x and radius ε, respectively. We say that a set A is a *hunk* if $\operatorname{int} A$ is connected and $A = \operatorname{cl}(\operatorname{int} A)$. If A is a set and $p \notin A$, the *join* of p and A, denoted by $[p,A]$, is the union of all segments $[p,a]$ for $a \in A$. More generally, the *join* of two disjoint sets A and B is the union $[A,B]$ of all segments $[a,b]$ with $a \in A$, $b \in B$. If A is convex, $[p,A]$ coincides with the set $\operatorname{conv}(\{p\} \cup A)$, which we shall denote by $\operatorname{conv}(p,A)$. Similarly, if x is a point and X is a set, we shall write $x \cup X$ instead of the more clumsy $\{x\} \cup X$. If A is a closed convex set, by $\operatorname{ext} A$ we shall denote the set of *extreme points* of A. As usual, *flats* are the translates of subspaces of any dimension, and we say that a set S is *line-free* if there is no line included in S. For the sake of completeness, we recall that if S is a set and $x \in S$, the *star of* x *in* S is the set $\operatorname{st}(x,S) = \{y \in S : [x,y] \subset S\}$. From this we get that a set S is said to be *starshaped* if there exists some $x \in S$ such that $\operatorname{st}(x,S) = S$. The *kernel of* S, denoted by $\ker S$, is the set of all $x \in S$ such that $\operatorname{st}(x,S)=S$. Without explicit mention we shall assume that all starshaped sets are non-empty. Obviously, a set S is convex if and only if $S = \operatorname{conv} S$. If S is a (starshaped) set, a *convex component* of S is a maximal (with respect to inclusion) convex subset of S.

We shall say that a point $z \in [x,y)$ is *subsequent to* y *in* $[x,y)$ if $z \notin [x,y]$. If S is a starshaped set and $m \in \ker S$, we shall say that a point $l \in S$ is *the last point of a ray* $[m,x)$ *in* S if $l \in [m,x)$ and there are no points $y \in [m,x) \cap S$ subsequent to l in $[m,x)$. Obviously, these points may not exist if S is not closed.

3. Cones

A set $C \subset \mathbb{R}^d$ is a *cone* if there exists a point $a \in \mathbb{R}^d$ such that

$$]0,\infty[\cdot(C-a) \subset C-a\,.$$

The point a is called *apex of the cone*, and it does not necessarily belong to the cone. A cone may have not only one apex; but if one of its apices belongs to the cone, all of them belong to it. The set of all apices of C is a flat called *summit* of C and denoted by ${}^{\gamma}C$[1]. Therefore

$${}^{\gamma}C = \{y :]0,\infty[\cdot(C-y) \subset C-y\}\,.$$

If ${}^{\gamma}C \subset C$, the cone is called *sharp;* otherwise it is *dull.* A sharp cone C is *salient* if no line through any of its apices is included in C, and in that case it has a unique apex. If C is a salient cone with apex a, the *opposite cone of* C is the cone $\operatorname{opp} C = 2a - C$.

For any cone C with apex a, its translate to the origin $C_0 = C - A$ is called the *centralized cone of* C.

If $A \subset \mathbb{R}^d$ and $a \in \mathbb{R}^d$, the *conic hull of* A *from* a is the set

$$[a, A) = \{a\} + [0,\infty[\cdot(A-a)\,.$$

It is the smallest (with respect to inclusion) sharp cone with apex a including A. If $a = 0$, it will be simply called the *conic hull of* A and denoted by $C(A)$. If A is a convex set, then $[a, A)$ is a convex cone for any a. The *witness cone* $[a, A)_0$ *of* A is the centralized conic hull of A from a:

$$[a, A)_0 = [a, A) - a\,.$$

It is well known (see [1]) that if A is convex and compact, then $[a, A)$ is a closed convex cone, but this may not be the case if A is merely closed. If $a \notin A$ and A is convex, the convex cone $[a, 2a - A) = \bigcup_{x\in A} [a, 2a - x)$ is called *opposite to* $[a, A)$.

Unless explicitly stated, all cones considered in this paper are supposed to be non-trivial, that is, they are not reduced to their apices.

4. On the asymptotic structure of starshaped sets

The following statements describe "asymptotic properties" that closed starshaped sets may have. For the first one, the reader is reminded that for convex sets unboundedness implies the existence of half-lines included in these sets, and that this property applies here (since $\ker S$ is convex).

Theorem 4.1. *Let S be an unbounded closed starshaped set. Then there exists a half-line Δ such that $\ker S + \Delta \subset S$. Moreover, if $\ker S$ is unbounded, then $S + \Delta \subset S$, where Δ is now a half-line Δ such that $m + \Delta \subset \ker S$ for some $m \in \ker S$.*

Proof. Let $m \in \ker S$. If S is unbounded, for every $n \in \mathbb{N}$ there exists an $x_n \in S$ such that $\|x_n - m\| \geq n$. Obviously, $[m, x_n] \subset S$ for every $n \in \mathbb{N}$. Let

$$S_m = \left\{x \in \mathbb{R}^d : \|x - m\| = 1\right\}$$

and, for every $n \in \mathbb{N}$, $s_n \in S_m \cap [m, x_n) \subset S_m \cap S$. Also it is clear that $S_m \cap [m, x_n) = \{s_n\}$ for every $n \in \mathbb{N}$. By compactness of S_m there exists a subsequence of the sequence (s_n) which is convergent to a point $s \in S_m \cap S$. Eventually renaming the elements of the subsequence, we may assume that $(s_n) \to s$, and it is clear that the corresponding x_n verify $\|x_n - m\| \geq n$ for every $n \in \mathbb{N}$, because they form a subsequence of the

[1] In the same way we can define the summit of A for any subset A of a linear space. It is always an affine subvariety of $\operatorname{aff} A$ (see [3], pp. 14–15).

original sequence. Let $\Delta = [m, s_n) - m$. We have $[m, s) = m + \Delta \subset S$, since S is closed. Then

$$[\ker S : [m, s)] = \operatorname{conv}(\ker S \cup [m, s))$$

is a convex cylinder included in S, and therefore its closure satisfies $\ker S + \Delta \subset S$, again since S is closed. Assume now that there exists an $m \in \ker S$ and a half-line Δ such that $m + \Delta \subset \ker S$, and let $x \in S$. Then $[x, y] \subset S$ for every $y \in m + \Delta$, and so $x + \Delta \subset S$, since S is closed. □

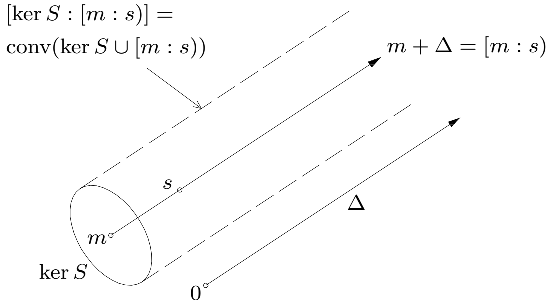

Figure 4.1

Corollary 4.2. *Let S be a closed starshaped set and Δ a direction. If $m + \Delta \subset S$ for some $m \in \ker S$, then $\ker S + \Delta \subset S$. If $m + \Delta \subset \ker S$ for some $m \in \ker S$, then $\ker S + \Delta \subset \ker S$ and $S + \Delta \subset S$. Conversely, if $S + \Delta \subset S$ for some direction Δ, then $\ker S + \Delta \subset \ker S$.*

Proof. Let $t \in \ker S$. Then $[t, p] \subset S$ for any $p \in m + \Delta$. Hence $t + \Delta \subset S$, since S is closed. The second assertion is obvious, since $\ker S$ is closed and convex. As for the converse, if there exists some $m \in \ker S$ such that $m + \Delta \subsetneqq \ker S$, let $t \in (m+\Delta) \cap (\ker S)'$ and let p be any point in S. We shall prove that $[t, p] \subset S$, contradicting the assumption that $t \notin \ker S$. If $x \in [t, p]$, then $x \in [m, y] \subset S$, since $m \in \ker S$ and $y \in p + \Delta \subset S$ (see Figure 4.1).

This concludes the proof. □

Corollary 4.3. *Let S be a closed starshaped set. If there exists a flat $F \subset S$, then $\ker S + (F - F) \subset S$.*

The kernel of a starshaped set S may include lines or, more generally, flats. But the structure of such sets is relatively simple.

Theorem 4.4. *Let S be a closed starshaped set and assume that there exists a flat $F \subset \ker S$. Then $S \cap (F - F)^{\perp}$ is starshaped and*

$$S = S \cap (F - F)^{\perp} + (F - F) .$$

Moreover, if F is a flat included in $\ker S$ and there is no flat F' such that $F \subsetneqq F' \subset \ker S$, then $\ker \left[S \cap (F - F)^{\perp}\right]$ is line-free.

Proof. This follows from the above results. □

Corollary 4.5. *Let S be a closed starshaped set. If a set $H \subset \ker S$ is a hyperplane, then S is a convex set of one of the following types:* (*a*) *the whole space;* (*b*) *a closed half-space;* (*c*) *a layer between two parallel hyperplanes. In cases* (*b*) *and* (*c*) *the bounding hyperplanes of S are parallel to H.*

Proof. If $H \subset \ker S$ is a hyperplane, then $(H - H)^{\perp}$ is one-dimensional such that $S \cap (H-H)^{\perp}$ could be: (a) a line, (b) a half-line, and (c) a segment. The result follows from the above theorem. $\square$

This generalizes known results on convex sets; see [12] and [2], p. 26 (Exercise 12).

Definition 4.6. Let S be a closed starshaped set and let Δ be a direction. We say that Δ is a *recession direction of* S if $\ker S + \Delta \subset \ker S$, and it is called an *infinity direction of* S if $\ker S + \Delta \subset S$. The set of all recession directions of S will be called the *recession cone of* S, denoted by $\operatorname{rc} S$, and the set of all infinity directions of S will be called the *infinity cone of* S and denoted by $\operatorname{ic} S$.

Note that a recession direction for a starshaped set S is a recession direction for the conex set $\ker S$.

5. Linear accessibility

Theorem 5.1. *Let S be a starshaped set such that* $\operatorname{int}\ker S \neq \varnothing$. *If $x \in \operatorname{cl} S$ and $m \in \operatorname{int}\ker S$, then $[m, x[\ \subset \operatorname{int} S$.*

Proof. Let us first prove that $]m, x[\ \subset S$. Let $y = \lambda m + (1-\lambda)x$, with $0 < \lambda < 1$, and $\varepsilon > 0$ be such that $U(m,\varepsilon) \subset \ker S$. If $\delta = \frac{\lambda\varepsilon}{1-\lambda}$, then $U(x,\delta) \cap S \neq \varnothing$. Let $z \in U(x,\delta) \cap S$. Then $[z,y) \cap U(m,\varepsilon) \neq \varnothing$ and, if $t \in [z,y) \cap U(m,\varepsilon)$, then $y \in [t,z] \subset S$. Let now C be any convex component of S such that $[m,x[\ \subset C$. Then $m \in \operatorname{int} C$ and $x \in \operatorname{cl} C$. By the linear accessibility theorem for convex sets then $[m,x[\ \subset \operatorname{int} C \subset \operatorname{int} S$. $\square$

Corollary 5.2. *If S is a closed starshaped set such that* $\operatorname{int}\ker S \neq \varnothing$, *then S is a hunk.*

Proof. The property that $\operatorname{int} S$ is (arcwise) connected is immediate. Let $x \in S = \operatorname{cl} S$ and $m \in \operatorname{int}\ker S$. Then $[m,x[\ \subset \operatorname{int} S$ whence $x \in \operatorname{cl}\operatorname{int} S$. The converse inclusion is obvious. $\square$

Conjecture 5.3. *If S is a starshaped set such that* $\dim\ker S \geq 2$ *and* $\ker S$ *is rotund, then* $\dim S > \dim\ker S$.

6. Support cones

An important notion for our investigations is that of a support cone of a starshaped set.

Definition 6.1. Let $S \subset \mathbb{R}^d$ be any set. A convex cone C with apex a and non-empty interior is a *support cone of S at a* if $a \in S$, $S \subset (\operatorname{int} C)'$ and C is a maximal (with respect to inclusion) convex cone with these properties.

Theorem 6.2. *Let $S \subset \mathbb{R}^d$ be a closed starshaped set with* $\operatorname{int}\ker S \neq \varnothing$. *Then for each $x \in \operatorname{bd} S$ there exists a support cone C of S at x.*

Proof. Let $m \in \operatorname{int}\ker S$. We shall prove that the ray $[m,x)\backslash[m,x] \subset S'$. If there exists a point $p \in [m,x)\backslash[m,x] \cap S$, then $x \in [m,p[\subset \operatorname{int} S$ by Theorem 5.1, which contradicts the assumption that $x \in \operatorname{bd} S$. By a similar argument, if $U \subset \ker S$ is an open ball centered at m, then $(\operatorname{opp}[x,U]) \setminus \{x\} \subset S'$. Therefore $\operatorname{opp}[x,U]$ is a pointed convex cone with apex x and non-empty interior such that $S \subset (\operatorname{int}\operatorname{opp}[x,U])'$. Therefore, the family $\mathbb{F}$ of cones with these properties is non-empty. We consider the partial order relation $\subset$ in $\mathbb{F}$. Then, if $\mathbb{C}$ is a linearly ordered subfamily of $\mathbb{F}$, $C = \bigcup \mathbb{C}$ is clearly an element of $\mathbb{F}$ which is an upper bound of $\mathbb{C}$. By Zorn's lemma there exists a maximal element of $\mathbb{F}$, which by definition is a support cone of S. □

Remark 6.3. Indeed, the condition $\operatorname{int}\ker S \neq \varnothing$ cannot be dropped in Theorem 6.2, but we mention that it is not necessary. Consider the following examples. The first one be

$$S_1 = \left\{(\xi,\nu) : -1 \leq \xi \leq 1, -1 \leq \nu \leq \sqrt{|\xi|}\right\} \cup \{(0,\nu) : \nu \geq 0\}$$

in $\mathbb{R}^2$. Then $x = (0,0) \in \operatorname{bd} S_1$, and any cone with apex x has non-empty intersection with S_1. And the second one be

$$S_2 = \{(\xi,\nu) : -1 \leq \xi \leq 1, \nu = 0\} \cup \{-1 \leq \nu \leq 1, \xi = 0\},$$

also in $\mathbb{R}^2$. Then $\operatorname{int}\ker S_2 = \varnothing$, but for each $x \in \operatorname{bd} S_2$ there exists a support cone C of S_2 at x.

Let S be a closed starshaped set such that $\operatorname{int}\ker S \neq \varnothing$ and $p \notin S$. Then, given any $m \in \operatorname{int}\ker S$, there exists a unique $x \in]m,p[\ \cap \operatorname{bd} S$. Clearly $]x,p) \subset S'$, whence p is included in all support cones C_x of S at x. Moreover, since $m \in \operatorname{int}\ker S$, p is an interior point of C_x. Therefore

$$S' \subset \bigcup\nolimits_{x \in \operatorname{bd} S} \operatorname{int} C_x .$$

The reverse inclusion is obvious, whence

$$S' = \bigcup\nolimits_{x \in \operatorname{bd} S} \operatorname{int} C_x .$$

Taking complements we get

$$S = \bigcap\nolimits_{x \in \operatorname{bd} S} (\operatorname{int} C_x)' .$$

Theorem 6.4. *If $S \subset \mathbb{R}^d$ is a closed starshaped set with* $\operatorname{int}\ker S \neq \varnothing$, *then*

$$S = \bigcap\nolimits_{x \in \operatorname{bd} S} (\operatorname{int} C_x)' ,$$

where C_x is any support cone of S at x.

This theorem generalizes the corresponding theorem for convex sets. Namely, in that case the support cones are half-spaces, and the complements of the interiors of such half-spaces are the usual support half-spaces.

7. Separation of starshaped sets

Disjoint convex sets are usually separated by means of hyperplanes and their associated half-spaces. As we saw in the last section, for starshaped sets cones and their complements play a role analogous to the role that half-spaces play for convex sets.

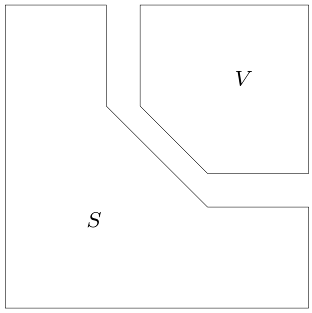

Figure 7.1

Now we want to derive separation theorems which, in some way, generalize the idea of separation of convex sets. In general, by a single cone we cannot separate a starshaped set from a disjoint convex set, even not in the case that both are compact; consider the simple example in $\mathbb{R}^2$ shown in Figure 7.1.

But we have the following

Theorem 7.1. *Let S and V be two disjoint sets, where S is starshaped and closed with* $\operatorname{int}\ker S \neq \varnothing$ *and V is compact. Then there exists a finite family of convex cones K_i, $i = 1, \cdots, n$, such that*

$$V \subset \bigcup\nolimits_{i=1}^{n} K_i,$$

$$\bigcup\nolimits_{i=1}^{n} K_i \cap S = \varnothing.$$

Proof. Let $p \in V$ and $m \in \operatorname{int}\ker S$. Let $x \in S$ be the last point of $[m, p)$ in S, and $q \in]x, p[\subset S'$. Let $\varepsilon > 0$ be such that $U(m, \varepsilon) \subset \ker S$ and $\rho = \varepsilon \cdot \frac{\|p-q\|}{\|m-q\|}$. Then clearly $[q, U(p, \rho)) \subset S'$ because, if $z \in]q, U(p, \rho)) \cap S$, then $[z, q] \cap U(m, \varepsilon) \neq \varnothing$ whence $q \in S$ (see Figure 7.2).

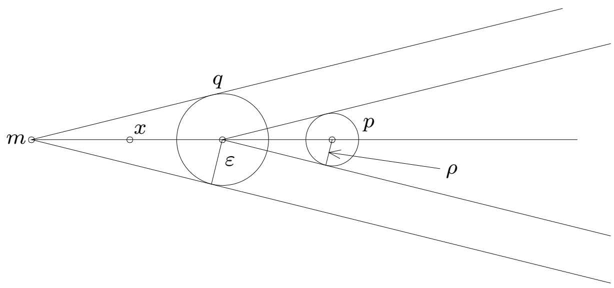

Figure 7.2

Let $K(p) =]q, U(p, \rho))$. Then $K(p)$ is an open convex cone such that $p \in K(p) \subset S'$, and the family of such cones $K(p)$, for $p \in V$, is an open covering of V. Let

$K_1, \cdots, K_n$ be a finite subcovering of V. Then $K_1, \cdots, K_n$ verifies the conclusion of the theorem. □

As seen before, in general there is not a convex cone K disjoint from S that includes V, even if V is a compact, convex set. But there exist convex *tails* of cones with these properties, where a convex conic tail is any set $T = V + K$, with V compact and convex and K a convex cone with apex 0. Moreover, by means of such tails we can strictly separate a closed starshaped set S from a disjoint compact, convex set V, in the sense that there exists a convex conic tail T such that $V \subset \operatorname{int} T$ and $S \subset \operatorname{int} T'$.

Theorem 7.2. *Let S and V be two disjoint sets, where S is starshaped and closed, with* $\operatorname{int}\ker S \neq \varnothing$*, and V is compact and convex. Then*

(a) *there exists a convex cone K with apex* 0 *such that $V \subset V+K$ and $(V+K) \cap S = \varnothing$,*

(b) *there exist a compact, convex set $W \supsetneq V$ and a convex cone K with apex* 0 *such that $V \subset W + K$, $(W + K) \cap S = \varnothing$, $V \subset \operatorname{int}(W + K)$, and $S \subset \operatorname{int}(W + K)'$.*

Proof. (a) Since $\ker S$ is closed and V is compact, there exist $m \in \ker S$, $p \in V$ that realize the minimal distance between $\ker S$ and V. Since V is a compact, convex set, $[m, V)$ is a closed convex cone [1], and for every $p \in V$ the half-line $[m, p)\backslash[m, p] \subset S'$. Let $[m, V)_0$ be the witness cone of V from m. Then $V \subset V + [m, V)_0 \subset S'$, so that $K = [m, V)_0$ verifies the assertion.

(b) Let $\varepsilon > 0$ be the minimal distance between S and V. Then $W = V + B(0, \varepsilon/2)$ is a compact, convex set disjoint from S. From (a) there exists a closed convex cone K with apex 0 such that $W \subset W + K$ and $(W + K) \cap S = \varnothing$. It is clear that $W + K$ verifies the assertion. □

8. Dispensable points

A basic theorem of classical convexity theory, usually referred to as Minkowski's theorem, states that a compact, convex subset of $\mathbb{R}^d$ may be "recovered" from the set of its extreme points, in the sense that the set itself is the convex hull of its extreme points. This theorem was generalized in several directions, e.g. with respect to unbounded sets by Klee (see [4] and [5]), subsets of spaces of infinite dimension by Krein and Milman (see [6]), etc., and today it is common to call it in general the Krein-Milman theorem for convex sets; see, e.g., §4 in [7], §1.4 in [11], and §2.6 in [13]. We intend to get a similar result for starshaped subsets of $\mathbb{R}^d$. For related (but different) results we refer to [8], [9], and [10].

We begin by asking for a useful analogue to extreme points of convex sets for the starshaped case. A suitable property is well known: a point x is an extreme point of a closed, convex set A if and only if the set $A \backslash \{x\}$ is also convex. In this sense extreme points are *dispensable points* of the set although, paradoxically, in another sense they are the essential points of the set (in view of Minkowski's theorem).

In the case of a closed starshaped set S we shall say that a point d is *dispensable* if and only if $\ker S \backslash \{d\} = \ker (S \backslash \{d\})$. This is a direct generalization of the above property of extreme points of closed, convex sets. The points of S that are not dispensable will be called *indispensable*, and the set of dispensable points of a given starshaped set S will be denoted by $\operatorname{disp} S$.

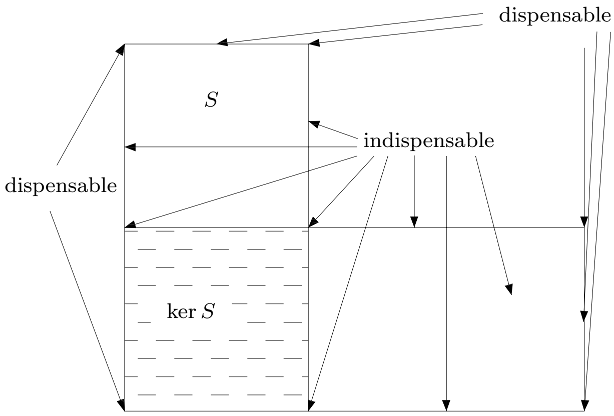

Figure 8.1

Roughly speaking, a point $p \in S$ is indispensable if there exist points $m \in \ker S$ and $q \in S \setminus \{p\}$ such that $p \in [m, q]$ (see Figure 8.2).

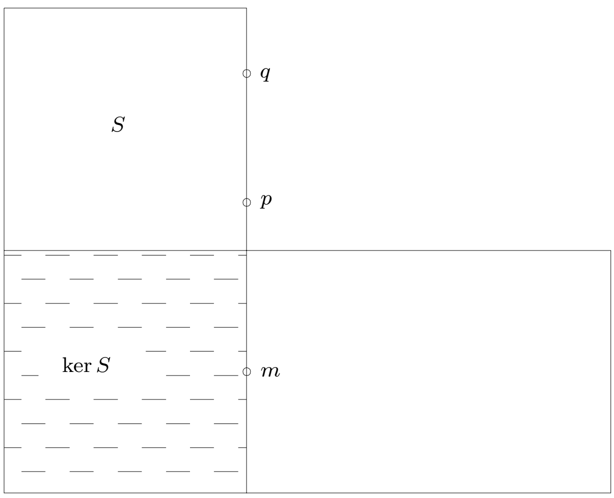

Figure 8.2

Remark 8.1. Being an extreme point of a convex set is a local property; that is, if $p \in C$ is an extreme point of a closed, convex set C, then there exists a neighborhood U of p such that, no matter how C is altered without losing convexity outside of U, p still is an extreme point of the (new) set. This is not true for dispensable points. They strongly depend on the convex kernel of the set, and this convex kernel can be altered completely by changing the set far from the point. The examples in Figure 8.3 illustrate this: the "upper part" of the set remains unaltered. Nevertheless, some points of this upper part change their character from one set to the other; see Figure 8.3.

If S is convex, the condition "$d \in S$ is indispensable" (i.e., $S \setminus \{d\}$ is not convex) is clearly equivalent to the condition that there exist points $x, y \in S$ such that $d \in]x, y[$. For starshaped sets this translates to

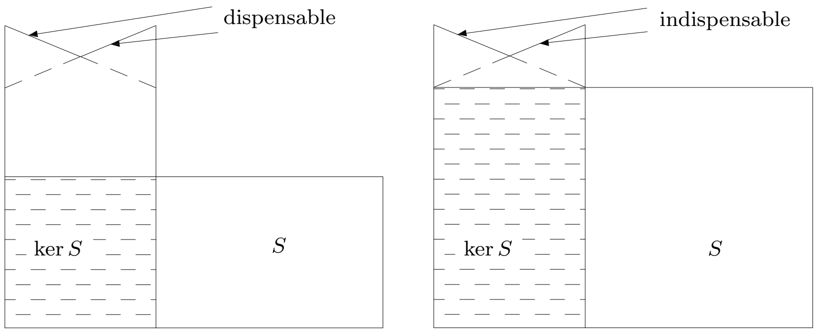

Figure 8.3

Proposition 8.2. *Let S be a closed starshaped set. Then $d \in S$ is indispensable if and only if there exist points $x \in \ker S$ and $y \in S$ such that $d \in]x, y[$.*

Proof. In any case it is clear that if $d \in S$, then

$$\ker (S \setminus \{d\}) \subset (\ker S) \setminus \{d\}.$$

($\Rightarrow$) d is indispensable if and only if the above inclusion is strict. Therefore there exists a point $x \in (\ker S) \setminus \{d\}$ such that $x \notin \ker (S \setminus \{d\})$, whence there exists a point $y \in S$ such that $d \in]x, y[$.

($\Leftarrow$) If there exist points $x \in \ker S$, $y \in S$ such that $d \in]x, y[$, then $x \in (\ker S) \setminus \{d\}$ and $x \notin \ker (S \setminus \{d\})$. $\square$

Proposition 8.3. *Let S be a closed starshaped set. Then $p \in S$ is dispensable if for every $m \in \ker S$ the point p is the last point of $[m, p)$ in S. As an easy consequence,* $\operatorname{disp} S \subset \operatorname{bd} S$.

This proposition follows directly from the defnitions, like also the next one.

Proposition 8.4. *Let S be a closed starshaped set. If $p \in \operatorname{bd} S$ is indispensable, then there exist different points $a, b \in \operatorname{bd} S$ such that $p \in [a, b] \subset \operatorname{bd} S$.*

9. Structure theorems

As was stated before (and despite their name), dispensable points of a starshaped set are the essential ones for recovering the set, as extreme points are the essential points for convex sets.

Lemma 9.1. *Let S be a closed starshaped set with* $\ker S$ *compact. Then*

$$S = (\ker S + \operatorname{rc} S) \cup [\ker S, \operatorname{disp} S]$$

Proof. The $\supset$ inclusion is obvious. Let us prove the converse. Let $p \in S \setminus \ker S$. Then there exists a hyperplane H which strictly separates p from $\ker S$. Parametrize the translates H_λ of H so that $H = H_0$ and $p \in H_\lambda$ for some $\lambda > 0$. The cone $[p, 2p - \ker S)$ opposite to $[p, \ker S)$ is closed, and so is

$$S \cap [p, 2p - \ker S).$$

Moreover, $[p, 2p - \ker S)\setminus\{p\} \subset H_\lambda^+$, where H_λ^+ is the open half-space determined by H_λ towards the side of increasing λ. For $\rho > \lambda$ consider the intersections

$$H_\rho \cap S \cap [p, 2p - \ker S).$$

There are two possibilities: $H_\rho \cap S \cap [p, 2p - \ker S) \neq \varnothing$ for every $\rho > \lambda$, or there exists a largest $\rho_0 > \lambda$ such that $H_{\rho_0} \cap S \cap [p, 2p - \ker S) \neq \varnothing$ but $H_\rho \cap S \cap [p, 2p - \ker S) = \varnothing$ for every $\rho > \rho_0$. In the first case there exists a half-line $[p, 2p - m)$, for some $m \in \ker S$, such that $]p, 2p) \subset S \cap [p, 2p - \ker S)$. Then its translate $[0, p - m) \subset \operatorname{rc} S$, and therefore $p \in \ker S + \operatorname{rc} S$. In the other case H_{ρ_0} is a support hyperplane of the convex set $S \cap [p, 2p - \ker S)$ at a point

$$d \in S \cap [p, 2p - \ker S) \cap H_{\rho_0}.$$

We shall prove that in this case d is a dispensable point of S. If d would be indispensable, then there would exist points $n \in \ker S$, $q \in S$ such that $q \neq d$ and $d \in [n, q]$ (Figure 9.1).

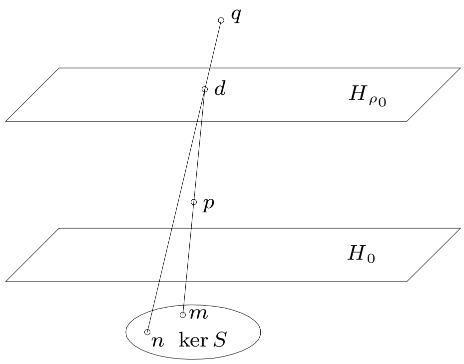

Figure 9.1

Note that m, n, d, p, and q belong to the plane determined by m, n and d. We repeat the drawing, now in that plane (Figure 9.2).

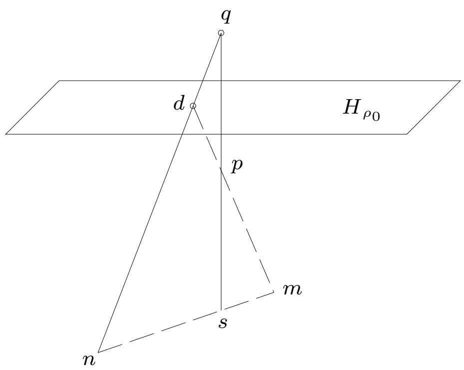

Figure 9.2

The half-line $[q, p)$ intersects the segment $[m, n]$ at a point s which belongs to $\ker S$, by convexity. Then $q \in S \cap [p, 2p - \ker S)$. This contradicts the fact that H_{ρ_0} is a support hyperplane of $S \cap [p, 2p - \ker S)$ at d. $\square$

Theorem 9.2. *Let S be a closed starshaped set with* $\ker S$ *compact. Then*

$$S = (\ker S + \operatorname{rc} S) \cup [\operatorname{conv}(\operatorname{ext}\ker S), \operatorname{disp} S].$$

Theorem 9.3. *Let S be a compact starshaped set. Then*

(a) $S = [\operatorname{conv}(\operatorname{ext}\ker S), \operatorname{disp} S]$.

(b) *If f is a real linear functional, then there exists a point $x \in \operatorname{ext}\ker S \cup \operatorname{disp} S$ such that*

$$f(x) = \sup_{y\in S} f(y).$$

Proof. (a) This is a consequence of the last theorem.

(b) Since S is compact, there exists a point $m \in S$ such that $f(m) = \sup_{x\in S} f(x)$. Assume that $f(m) > f(x)$ for every $x \in \operatorname{ext}\ker S \cup \operatorname{disp} S$. By ($a$) there exists a point $p \in \operatorname{disp} S$ such that $m \in [\operatorname{conv}(\operatorname{ext}\ker S), p]$, whence there exist $x_1, \cdots, x_r \in \operatorname{ext}\ker S$ and $\lambda_0, \cdots, \lambda_r \geq 0$ such that

$$m = \lambda_0 p + \sum_{i=1}^{r} \lambda_i x_i, \qquad \sum_{i=0}^{r} \lambda_i = 1.$$

Therefore

$$\begin{aligned} f(m) &= \lambda_0 f(p) + \sum_{i=1}^{r} \lambda_i f(x_i) < \lambda_0 f(m) + \sum_{i=1}^{r} \lambda_i f(m) \\ &= f(m) \left(\sum_{i=0}^{r} \lambda_i \right) = f(m), \end{aligned}$$

a contradiction. □

10. The final cut

Our final theorem says that from the set of stars of the dispensable points of a compact starshaped set one can recover this set and, moreover, even its kernel.

Theorem 10.1. *Let S be a compact starshaped set. Then*

$$S = \bigcup_{x\in \operatorname{disp} S} \operatorname{st}(x, S),$$

$$\ker S = \bigcap_{x\in \operatorname{disp} S} \operatorname{st}(x, S).$$

Proof. The first assertion follows from the remark that if $x \in S$, then

$$\operatorname{conv}(x \cup \ker S) \subset \operatorname{st}(x, S) \subset S.$$

Let $y \in S$ be an arbitrary point of S. By Theorem 9.3 there exist points $p \in \operatorname{disp} S$ and $m \in \ker S$ such that $y \in [m, p]$. Let $z \in \bigcap_{x\in \operatorname{disp} S} \operatorname{st}(x, S)$. Then $z \in \operatorname{st}(p, S)$ whence $[z, p] \subset S$. Therefore $[z, y] \subset [m, [z, p]] \subset S$ whence $z \in \ker S$. The reverse inclusion is obvious. □

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