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An Ill Posed Problem in $SBV_0^2(\Omega)$

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We study existence of solutions in $SBV_0^2(\Omega)$ of a variational problem involving bulk and interfacial energy terms, but with the bulk energy not attaining a minimum at the prescribed boundary value. We obtain an expression for the relaxed energy and give some properties satisfied by the solutions of the relaxed problem.

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Introduction

The question of finding necessary and sufficient conditions for existence of solutions of the problem

$$\inf\left\{\int_\Omega f(\nabla u(x))\,dx\ : u \in W_0^{1,\infty}(\Omega)\right\}, \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ and $f : \mathbb{R}^N \to \mathbb{R}_0^+$ is a lower semicontinuous function, has received considerable attention (cf. [7], [8], [10] and the references therein). Recently the same question was also considered in a generalized setting, allowing for other differential operators, namely the curl (cf. [4]) and general differential forms (cf. [3]).

Assuming f is convex, by Jensen's inequality and the divergence theorem, it follows that the infimum of (1) is given by $f(0)|\Omega|$, where $|\Omega|$ denotes the Lebesgue measure of the set Ω.

However, in trying to extend this problem to the space of functions of bounded variation we loose the insight about the infimum value of the energy which, in the Sobolev space setting, is obtained by the above argument.

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The case considering only a bulk energy term was treated in [16]. In this paper we consider the problem

$$(P_{\alpha\beta}) \qquad \inf\left\{\int_\Omega |\nabla u - \zeta_0|^2\,dx + \alpha|D^s u|(\Omega) + \beta H^{N-1}(S_u \cap \Omega)\;:\; u \in SBV_0^2(\Omega)\right\},$$

where $\alpha \geq 0$, $\beta > 0$, $\zeta_0 \in \mathbb{R}^N \setminus \{0\}$, ∇u is the density of the absolutely continuous part of the distributional derivative Du relative to the Lebesgue measure, and $D^s u$ is its singular part. The space $SBV_0^2(\Omega)$ consists of those functions $u \in SBV(\Omega)$ such that $\nabla u \in L^2(\Omega;\mathbb{R}^N)$, $H^{N-1}(S_u \cap \Omega) < +\infty$ and which have zero trace on $\partial\Omega$.

This is a prototype of a more general class of functionals, involving an interfacial energy term, which appear in the weak formulation of many problems, for example in image segmentation, fracture mechanics, minimal partitioning and so on (see [6] and the references therein).

More generally, one could also consider bulk energy terms with integrands of the form $f(\nabla u)$, for nonconvex functions $f : \mathbb{R}^N \to \mathbb{R}_0^+$ and with $0 \notin \{\zeta \in \mathbb{R}^N : f(\zeta) = f^{**}(\zeta)\}$, where f^{**} denotes the convex envelope of f (similarly to what was done in [10] and [3]), as well as other expressions for the surface energy term, but we will restrict our analysis to the case presented above.

For this functional one has weak compactness in $SBV^2(\Omega)$ (see [1]) but the imposition that admissible functions have zero trace creates an ill posed problem in the sense that $(P_{\alpha\beta})$ doesn't necessarily have solutions.

It is clear that

$$\inf(P_{\alpha\beta}) \leq |\zeta_0|^2|\Omega|$$

since $u = 0$ is an admissible function. However, it is not clear what the value of $\inf(P_{\alpha\beta})$ is, or if this infimum is attained.

We are, therefore, lead to consider the relaxed problem

$$\inf_{u \in SBV^2(\Omega)} \mathcal{F}_{\alpha\beta}(u),$$

where

$$\mathcal{F}_{\alpha\beta}(u) := \inf_{\{u_n\}}\left\{\liminf_n I_{\alpha\beta}(u_n) :\; u_n \in SBV_0^2(\Omega), u_n \to u \text{ in } L^1(\Omega)\right\}$$

and

$$I_{\alpha\beta}(u) := \int_\Omega |\nabla u(x) - \zeta_0|^2\,dx + \alpha|D^s u|(\Omega) + \beta H^{N-1}(S_u \cap \Omega).$$

Our aim in this paper is to study the solutions of the relaxed problem, which exist by direct methods of the calculus of variations, and to relate them to solutions of the original problem.

We organise the paper as follows. We begin by presenting the notation and some general results on BV functions and sets of finite perimeter which will be used throughout the article. In Section 2 we show that any minimizing sequence for $(P_{\alpha\beta})$ is bounded in L^∞. This, together with a result of Ambrosio (cf. Theorem 1.3), yields weak compactness in $SBV^2(\Omega)$. Despite the fact that $I_{\alpha\beta}(\cdot)$ is lower semicontinuous, since we lack convergence of the traces, we cannot conclude existence of solutions of $(P_{\alpha\beta})$ by means of the direct methods of the calculus of variations.

In Section 3 we obtain an expression for the relaxed energy and we derive some relations satisfied by the solutions of the relaxed problem in Section 4. As a consequence of these properties we show that if a solution is zero in a subset Ω_1 of Ω and has gradient ζ_0 in $\Omega \setminus \Omega_1$ then it is in $W^{1,2}(\Omega)$ and is, therefore, harmonic. We point out that these results resemble the relations between the growth of harmonic functions and the growth of their nodal sets.

1. Preliminaries and notations

Throughout this paper $\Omega \subset \mathbb{R}^N$ is an open, bounded set with Lipschitz boundary, H^k denotes the k-dimensional Hausdorff measure and we use the notation $|\Omega|$ for the Lebesgue measure of the set Ω. $\mathcal{B}(\Omega)$ denotes the Borel σ-algebra of subsets of Ω and χ_A represents the characteristic function of the set A.

We use the standard notation for the Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{k,p}(\Omega)$, $C_c^\infty(\Omega)$ stands for the space of real-valued smooth functions with compact support in Ω. $B(x,\varepsilon)$ denotes the open ball centered at x with radius ε, $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ and the letter C will be used to indicate a constant whose value might change from line to line.

Given an $L^1(\Omega)$ function u the *Lebesgue set* of u, Ω_u, is defined as the set of points $x \in \Omega$ such that there exists $\tilde{u}(x) \in \mathbb{R}$ satisfying

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{B(x,\varepsilon)} |u(y) - \tilde{u}(x)|\, dy = 0.$$

The *Lebesgue discontinuity set* S_u of u is the set of points $x \in \Omega$ which are not Lebesgue points, that is $S_u := \Omega \backslash \Omega_u$. By Lebesgue's Differentiation Theorem, S_u is H^N-negligible and $\tilde{u} : \Omega \to \mathbb{R}$, which coincides with u H^N-almost everywhere in Ω_u, is called the *Lebesgue representative* of u.

The *approximate upper* and *lower limits* of u are given by

$$u^+(x) := \inf\left\{ t \in \mathbb{R} : \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} H^N(\{y \in \Omega \cap B(x,\varepsilon) : u(y) > t\}) = 0 \right\}$$

and

$$u^-(x) := \sup\left\{ t \in \mathbb{R} : \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} H^N(\{y \in \Omega \cap B(x,\varepsilon) : u(y) < t\}) = 0 \right\};$$

if $u^+(x) = u^-(x)$ then $x \in \Omega_u$ and $u^+(x) = u^-(x) = \tilde{u}(x)$. The *jump set* or *singular set* of u is defined as

$$J_u := \{x \in \Omega : u^-(x) < u^+(x)\}$$

and we denote by $[u](x)$ the *jump* of u at x, i.e. $[u](x) := u^+(x) - u^-(x)$.

We recall briefly some facts on functions of bounded variation which will be used in the sequel. We refer to [2], [13], [14] and [17] for a detailed exposition on this subject.

A function $u \in L^1(\Omega)$ is said to be of *bounded variation*, $u \in BV(\Omega)$, if for all $j = 1, ..., N$, there exists a finite Radon measure μ_j such that

$$\int_\Omega u(x) \frac{\partial \phi}{\partial x_j}(x)\, dx = -\int_\Omega \phi(x)\, d\mu_j(x)$$

for every $\phi \in C_0^1(\Omega)$. The distributional derivative Du is the vector-valued measure μ with components μ_j.

The space $BV(\Omega)$ is a Banach space when endowed with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + |Du|(\Omega),$$

where $|Du|(\Omega)$ represents the total variation of the measure Du.

If $u \in BV(\Omega)$ it is well known that S_u is countably $N-1$ rectifiable, i.e.

$$S_u = \bigcup_{n=1}^{\infty} K_n \cup E,$$

where $H^{N-1}(E) = 0$ and K_n are compact subsets of C^1 hypersurfaces. Furthermore, for H^{N-1} a.e. $x \in S_u$, $u^+(x) \neq u^-(x)$ and there exists a unit vector $\nu_u(x) \in S^{N-1}$, normal to S_u at x, such that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in B(x,\varepsilon) : (y-x)\cdot \nu_u(x) > 0\}} |u(y) - u^+(x)|\, dy = 0$$

and

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in B(x,\varepsilon) : (y-x)\cdot \nu_u(x) < 0\}} |u(y) - u^-(x)|\, dy = 0.$$

In particular, $H^{N-1}(S_u \setminus J_u) = 0$.

If $u \in BV(\Omega)$ then the distributional derivative Du may be decomposed as

$$Du = \nabla u H^N + (u^+ - u^-) \cdot \nu_u H^{N-1} \lfloor S_u + C_u, \tag{2}$$

where ∇u is the density of the absolutely continuous part of Du with respect to the Lebesgue measure and C_u is the Cantor part of Du which vanishes on all $B \in \mathcal{B}(\Omega)$ with $H^{N-1}(B) < +\infty$. The three measures appearing in (2) are mutually singular.

The space of *special functions of bounded variation*, $SBV(\Omega)$, introduced by De Giorgi and Ambrosio in [11], is the space of functions $u \in BV(\Omega)$ such that $C_u = 0$, i.e. for which

$$Du = \nabla u H^N + (u^+ - u^-) \cdot \nu_u H^{N-1} \lfloor S_u.$$

For Ω open and bounded with Lipschitz boundary the outer unit normal to $\partial\Omega$ (denoted by ν) exists H^{N-1} a.e. and we can define the trace for functions in $BV(\Omega)$. Namely, there exists a bounded linear mapping

$$T : BV(\Omega) \to L^1(\partial\Omega; H^{N-1})$$

such that

$$\int_\Omega u\mathrm{div}\phi\, dx = -\int_\Omega \phi \cdot d[Du] + \int_{\partial\Omega} (\phi\cdot\nu) Tu\, dH^{N-1},$$

for all $u \in BV(\Omega)$ and $\phi \in C^1(\mathbb{R}^N;\mathbb{R}^N)$. The function Tu is uniquely defined up to sets of $H^{N-1}\lfloor\partial\Omega$ measure zero, and is called the *trace* of u on $\partial\Omega$.

We denote by $SBV_0(\Omega)$ the set of $u \in SBV(\Omega)$ such that $Tu = 0$ on $\partial\Omega$.

We will need the following extension result:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with Lipschitz boundary. Let $f_1 \in BV(\Omega), f_2 \in BV(\mathbb{R}^N\backslash\overline{\Omega})$ and define*

$$\overline{f}(x) = \begin{cases} f_1(x), & x \in \Omega \\ f_2(x), & x \in \mathbb{R}^N\backslash\overline{\Omega}. \end{cases}$$

Then $\overline{f} \in BV(\mathbb{R}^N)$ and

$$|D\overline{f}|(\mathbb{R}^N) = |Df_1|(\Omega) + |Df_2|(\mathbb{R}^N\backslash\overline{\Omega}) + \int_{\partial\Omega} |Tf_1 - Tf_2|\, dH^{N-1}.$$

In this paper we will be concerned with functions in $SBV^2(\Omega)$ which is the space of functions $u \in SBV(\Omega)$ such that $H^{N-1}(S_u \cap \Omega) < +\infty$ and $\nabla u \in L^2(\Omega;\mathbb{R}^N)$. In this space we consider the following definition of weak convergence as introduced by Braides and Chiadò-Piat in [6].

Definition 1.2. Given $\{u_n\} \subset SBV^2(\Omega)$ and $u \in SBV^2(\Omega)$ we say that u_n converges weakly to u in $SBV^2(\Omega)$ if $u_n \to u$ in $L^1(\Omega)$, $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega;\mathbb{R}^N)$ and $\sup_n |Du_n|(\Omega) < +\infty$.

The introduction of this kind of convergence was motivated by the following compactness theorem due to Ambrosio [1].

Theorem 1.3. *Let $\{u_n\} \subset SBV^2(\Omega)$ be such that*

$$\sup_n ||u_n||_{BV(\Omega)} < +\infty$$

and

$$\sup_n \left\{ \int_\Omega |\nabla u_n(x)|^2\, dx + H^{N-1}(S_{u_n} \cap \Omega) \right\} < +\infty.$$

Then there exists a subsequence $\{u_{n_j}\} \subset \{u_n\}$ converging weakly to a function u in $SBV^2(\Omega)$. Moreover,

$$H^{N-1}(S_u \cap \Omega) \leq \liminf_{j\to\infty} H^{N-1}(S_{u_{n_j}} \cap \Omega).$$

Remark 1.4. Using this compactness result one can show the lower semicontinuity, with respect to L^1 convergence, of the functional

$$\int_\Omega |\nabla u(x) - \zeta_0|^2\, dx + \int_{S_u\cap\Omega} (\alpha[u](x) + \beta)\ dH^{N-1}(x);$$

see, for instance, [6].

An H^N-measurable set $A \subset \Omega$ is said to be of *finite perimeter* in Ω if $\chi_A \in BV(\Omega)$. The *perimeter* of A in Ω is defined by

$$\mathrm{Per}_\Omega(A) := \sup\left\{\int_A \mathrm{div}\varphi(x)\,dx : \varphi \in C_0^1(\Omega;\mathbb{R}^N),\ \|\varphi\|_\infty \leq 1\right\}.$$

Given a set $A \subset \Omega$ of locally finite perimeter the *reduced boundary* of A in Ω, $\partial^* A$, is given by

$$\partial^* A \cap \Omega = S_{\chi_A} \cap \Omega$$

and we recall that for H^{N-1} a.e. $x \in \partial^* A$, it is possible to define a measure theoretical interior normal to A, $\nu_A(x) \in S^{N-1}$, such that

$$D\chi_A(B) = \int_{B\cap\partial^* A} \nu_A(x)\, dH^{N-1}(x)$$

for every $B \in \mathcal{B}(\Omega)$.

In Section 4 we will need the following result which can be found in [13].

Proposition 1.5. *Let $E \subset \mathbb{R}^N$ be a set of locally finite perimeter. Then there exists a positive constant A, depending only on N, such that, for each $x_0 \in \partial^* E$*

$$\liminf_{r\to 0^+} \frac{H^{N-1}\left(\partial^* E \cap B(x_0, r)\right)}{r^{N-1}} \geq A > 0.$$

We say that a sequence (E_i) is a *Borel partition* of a given set $B \in \mathcal{B}(\mathbb{R}^N)$ if and only if

$$E_i \in \mathcal{B}(\mathbb{R}^N),\ \ \forall i \in \mathbb{N}; \qquad E_i \cap E_j = \emptyset\ \text{ if } i \neq j; \qquad \bigcup_{i=1}^{\infty} E_i = B.$$

The following result, whose proof can be found in [9] (see Lemma 1.11), will be used in Section 4 to characterize the solutions of the relaxed problem.

Lemma 1.6. *If $u \in SBV(\Omega)$, $\nabla u = 0$ a.e. in Ω and $H^{N-1}(S_u \cap \Omega) < +\infty$ then there exist a Borel partition (E_i) of Ω, and a sequence (u_i) in $\mathbb{R}$ with $u_i \neq u_j$ for $i \neq j$, such that*

$$u = \sum_{i=1}^{+\infty} u_i \chi_{E_i} \text{ a.e. in } \Omega \quad \text{and} \quad \sum_{i=1}^{+\infty} \mathrm{Per}_\Omega(E_i) < +\infty.$$

2. Weak compactness

Recall that the problem under consideration is the following

$$(P_{\alpha\beta}) \qquad \inf_{u\in SBV_0^2(\Omega)} I_{\alpha\beta}(u)$$

where

$$I_{\alpha\beta}(u) = \int_\Omega |\nabla u(x) - \zeta_0|^2\, dx + \alpha|D^s u|(\Omega) + \beta H^{N-1}(S_u \cap \Omega)$$

for $\alpha \geq 0$, $\beta > 0$, $\zeta_0 \in \mathbb{R}^N \setminus \{0\}$.

Since Ω is bounded, let B denote an open ball such that $\Omega \subset B$. By taking x_i, $i = 1, 2$, to be symmetrical points on ∂B along the direction determined by ζ_0, it is easy to see that one can construct affine functions p_i, $i = 1, 2$, such that

$$\nabla p_i = \zeta_0, \ \ p_i(x_i) = 0, \ i = 1, 2$$

and

$$p_1(x) > 0, \ \ \forall x \in \overline{B} \setminus \{x_1\}, \qquad p_2(x) < 0, \ \ \forall x \in \overline{B} \setminus \{x_2\}.$$

Given $u \in SBV_0^2(\Omega)$, let

$$u_1(x) := \min\{u(x), p_1(x)\}, \ x \in \Omega.$$

Clearly $u_1 \in SBV^2(\Omega)$ and, by definition of the trace and of p_1, $Tu_1(x) = 0$ for H^{N-1} a.e $x \in \partial\Omega$. Also, one has

$$S_{u_1} \subset \{x \in S_u : u^-(x) < p_1(x)\} \subset S_u.$$

Due to the definition of the approximate upper and lower limits and to the continuity of p_1 it follows that

$$[u_1](x) = \begin{cases} [u](x), & \text{if } u^+(x) \leq p_1(x) \\ p_1(x) - u^-(x), & \text{if } u^-(x) < p_1(x) < u^+(x) \\ 0, & \text{if } u^-(x) \geq p_1(x). \end{cases}$$

Thus, comparing the energies, we conclude that

$$I_{\alpha\beta}(u_1) \leq I_{\alpha\beta}(u).$$

Similarly, defining

$$u_2(x) := \max\{u_1(x), p_2(x)\}, \quad x \in \Omega,$$

we have $u_2 \in SBV_0^2(\Omega)$,

$$S_{u_2} \subset \{x \in S_{u_1} : u_1^+(x) > p_2(x)\} \subset S_u$$

and

$$[u_2](x) = \begin{cases} [u_1](x), & \text{if } u_1^-(x) \geq p_2(x) \\ u_1^+(x) - p_2(x), & \text{if } u_1^-(x) < p_2(x) < u_1^+(x) \\ 0, & \text{if } u_1^+(x) \leq p_2(x) \end{cases}$$

and so

$$I_{\alpha\beta}(u_2) \leq I_{\alpha\beta}(u_1) \leq I_{\alpha\beta}(u)$$

where $p_2 \leq u_2 \leq p_1$.

Due to this truncation argument we conclude that a minimizing sequence $\{u_n\}$ for $(P_{\alpha\beta})$ is bounded in L^∞.

This implies, in particular, that

$$\sup_n \|u_n\|_{L^1} < +\infty$$

and that

$$\sup_n |D^s u_n|(\Omega) < +\infty$$

(if $\alpha > 0$ the second condition follows immediately from the fact that $\{u_n\}$ is a minimizing sequence), so $\{u_n\}$ satisfies the hypotheses of Theorem 1.3. This result thus yields that a minimizing sequence for $(P_{\alpha\beta})$ has a subsequence $\{u_{n_k}\}$ that converges weakly to some $u \in SBV^2(\Omega)$. Since the functional $I_{\alpha\beta}$ is L^1 lower semicontinuous (cf. Remark 1.4), the infimum would be attained if we could ensure that $Tu = 0$. However this is not the case.

Therefore, our aim in what follows is to see what can be said about the infimum of the problem and under what conditions it is attained, without relying on the direct method of the calculus of variations.

In order to gain some insight on what to expect we start with a simple example. Assume that $N = 1, \Omega = (0,1)$, $\alpha = 0$ and $\beta = 1$. Clearly,

$$\inf(P_{01}) = \min\{|\zeta_0|^2, 1\}.$$

In fact,

$$I_{01}(u) \geq 1, \quad \forall u \in SBV_0^2(\Omega) \setminus W_0^{1,2}(\Omega),$$

and, by Jensen's inequality, if $u \in W_0^{1,2}(\Omega)$,

$$I_{01}(u) \geq |\zeta_0|^2 |\Omega|.$$

Considering, for $\varepsilon \in (0, \frac{1}{2})$, u_ε such that $\nabla u_\varepsilon = \zeta_0$ on the interval $(0, 1-2\varepsilon)$ and $D^s u_\varepsilon = 0$ we have

$$\begin{aligned} I_{01}(u_\varepsilon) &= \int_{1-2\varepsilon}^1 \left| \frac{2\varepsilon - 1}{2\varepsilon}\zeta_0 - \zeta_0 \right|^2 dx \\ &= \frac{1}{2\varepsilon}|\zeta_0|^2 < 1 \quad \text{for } \varepsilon > \frac{|\zeta_0|^2}{2}. \end{aligned}$$

For $|\zeta_0|^2 \leq 1$ and $\varepsilon \in (|\zeta_0|^2, 1)$, since $\frac{d}{d\varepsilon} I_{01}(u_\varepsilon) < 0$, we conclude that the infimum of the energy is attained at $\varepsilon = \frac{1}{2}$, i.e. at $u = 0$. For $|\zeta_0|^2 > 1$ the infimum is attained at any $u \in SBV_0^2(\Omega)$ such that $\nabla u = \zeta_0$ a.e. in Ω, having only one jump point.

This simple example shows that in some cases the classical solution is prefered, whereas in others we obtain solutions with discontinuity points. For an example with $N > 1$ we refer to Section 4.

In both cases of the above example the set Ω can be written as the union of two subsets Ω_1 and Ω_2 such that $u = 0$ in Ω_1 and $\nabla u = \zeta_0$ in Ω_2. Of course the case $N = 1$ is very particular and we cannot derive any general conclusions from these examples. However, we conjecture that in many situations the minimizers should have the appropriate gradient in a measurable subset of Ω and be zero in the remainder. If this is so, we can show that the minimizer belongs necessarily to $W^{1,2}(\Omega)$ (cf. Theorem 4.4).

3. Relaxation of the problem

In view of the previous discussion, we now look for the relaxation in $SBV^2(\Omega)$ of the problem $(P_{\alpha\beta})$. For $\alpha \geq 0$, $\beta > 0$, and for $u \in SBV^2(\Omega)$ we define

$$\mathcal{F}_{\alpha\beta}(u) = \inf_{\{u_n\}} \left\{ \liminf_n I_{\alpha\beta}(u_n) : \ u_n \in SBV_0^2(\Omega), \ u_n \to u \text{ in } L^1(\Omega) \right\}.$$

Our goal is to show that

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set, with Lipschitz boundary. For any $u \in SBV^2(\Omega)$ we have*

$$\begin{aligned} \mathcal{F}_{\alpha\beta}(u) &= \int_\Omega |\nabla u(x) - \zeta_0|^2 \, dx + \alpha |D^s u|(\Omega) + \beta H^{N-1}(S_u \cap \Omega) \\ &\quad + \int_{\partial\Omega \cap \{Tu \neq 0\}} (\alpha |Tu(x)| + \beta) \ dH^{N-1}(x) =: J_{\alpha\beta}(u). \end{aligned}$$

Proof. We begin by noticing that given any $u \in SBV^2(\Omega)$ and any sequence $\{u_n\} \subset SBV_0^2(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$, we have

$$J_{\alpha\beta}(u) \leq \mathcal{F}_{\alpha\beta}(u).$$

If $\liminf_n I_{\alpha\beta}(u_n) = +\infty$ the result is trivial. On the other hand, if $\liminf_n I_{\alpha\beta}(u_n) < +\infty$, we consider Ω_1 such that $\Omega \subset\subset \Omega_1$ and we extend u, u_n by 0 to Ω_1. As $u_n \to u$ in $L^1(\Omega)$, by the L^1 lower semicontinuity of $I_{\alpha\beta}$ (extended as a functional $\widehat{I_{\alpha\beta}}$ in $SBV_0^2(\Omega_1)$), we obtain

$$J_{\alpha\beta}(u) + |\zeta_0|^2 |\Omega_1 \setminus \Omega| = \widehat{I_{\alpha\beta}}(u) \leq \liminf_n \widehat{I_{\alpha\beta}}(u_n) = \liminf_n I_{\alpha\beta}(u_n) + |\zeta_0|^2 |\Omega_1 \setminus \Omega|.$$

In order to show the reverse inequality and complete the proof, for each $u \in SBV^2(\Omega)$, we must produce a sequence $\{u_n\} \subset SBV_0^2(\Omega)$, converging to u in $L^1(\Omega)$, such that

$$\liminf_n I_{\alpha\beta}(u_n) \leq J_{\alpha\beta}(u).$$

The idea is to locally contract u near the boundary $\partial\Omega$. In what follows we adopt the notation of Theorem 1, Section 5.3 in [13].

Given $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$, write $x = (x', x_N)$ where $x' = (x_1, x_2, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$. For $x \in \mathbb{R}^N$, $r, h > 0$, we define the open cylinder

$$C(x, r, h) = \{y \in \mathbb{R}^N : |y' - x'| < r, \ |y_N - x_N| < h\}.$$

Since $\partial\Omega$ is locally Lipschitz, for each $x \in \partial\Omega$ there exist $r, h > 0$ and a Lipschitz function

$$\gamma : \mathbb{R}^{N-1} \to \mathbb{R}$$

such that

$$\max_{|x'-y'| \leq r} |\gamma(y') - x_N| \leq \frac{h}{4}$$

and, upon rotating and relabeling the coordinate axes if necessary,

$$\Omega \cap C(x,r,h) = \{y \in \mathbb{R}^N : |x'-y'| < r,\ \gamma(y') < y_N < x_N + h\}.$$

Let $u \in SBV^2(\Omega)$. Select a point $x \in \partial\Omega$ and let r, h, γ be as above. In what follows we write $C \equiv C(x,r,h)$ for simplicity of notation. For n large enough (so that $\frac{1}{n} \leq \frac{h}{2}$) define

$$C_n = \left\{ y \in C : \gamma(y') + \frac{1}{2n} \leq y_N \leq \gamma(y') + \frac{1}{n} \right\}$$

and for $y \in C_n$ let

$$u_n(y) = u\left(y', 2y_N - \gamma(y') - \frac{1}{n} \right).$$

Clearly we have $u_n(y) = u(y)$ in $\left\{ y \in C : y_N = \gamma(y') + \frac{1}{n} \right\}$ and $u_n(y) = u(y', \gamma(y'))$ in $\left\{ y \in C : y_N = \gamma(y') + \frac{1}{2n} \right\}$. We now define

$$u_n(y) = u(y) \ \text{ in } (\Omega \setminus C) \cup \left\{ y \in C : y_N > \gamma(y') + \frac{1}{n} \right\}$$

and

$$u_n(y) = 0 \ \text{ in } \left\{ y \in \Omega \cap C : \gamma(y') < y_N < \gamma(y') + \frac{1}{2n} \right\}.$$

It is easy to see that this sequence satisfies $u_n \in SBV^2(C \cap \Omega)$, $Tu_n = 0$ on $C \cap \partial\Omega$, $u_n \to u$ in $L^1(C \cap \Omega)$ and

$$\begin{aligned}
&\liminf_n \left[\int_{C\cap\Omega} |\nabla u_n(y) - \zeta_0|^2 \, dy + \alpha |D^s u_n|(C \cap \Omega) + \beta H^{N-1}(S_{u_n} \cap C \cap \Omega) \right] \\
\leq \ & \int_{C\cap\Omega} |\nabla u(y) - \zeta_0|^2 \, dy + \alpha |D^s u|(C \cap \Omega) + \beta H^{N-1}(S_u \cap C \cap \Omega) \\
&\quad + \int_{C\cap\partial\Omega\cap\{Tu\neq 0\}} (\alpha |Tu(y)| + \beta) \ dH^{N-1}(y). \qquad (3)
\end{aligned}$$

As $\partial\Omega$ is compact, we may cover it by a finite number of cylinders of the above form, in each of which we construct a sequence satisfying (3). The result now follows by a standard partition of unity argument. □

4. Characterization of solutions of the relaxed problem

In this section we prove several properties satisfied by minimizers of $\mathcal{F}_{\alpha\beta}$ in $SBV^2(\Omega)$. Using perturbations of the minimizers we derive inequalities in which level sets and planes with normal ζ_0 play a particularly relevant role. In the case where $\alpha = 0$ these inequalities become equalities.

As a result of (4) below we can show that if a minimizer of $\mathcal{F}_{\alpha\beta}$ belongs to $W^{1,2}(\Omega)$ then it is harmonic. We also derive formulas envolving the bulk energy above and below a plane Γ with normal ζ_0. These relations are specific of this particular problem but they are in the spirit of the formulas that connect the growth of harmonic functions and the growth of their nodal sets (see [15] for a survey on this matter), since they relate the size of the nodal set intersecting a section of Ω with normal ζ_0 and the growth of u.

In the following result we make a (nonexhaustive) list of relations obtained through perturbations of minimizers.

Proposition 4.1. *Let P be an affine function with gradient ζ_0 and set*

$$\Omega^+ := \Omega \cap \{P > 0\}, \qquad \Gamma := \Omega \cap \{P = 0\}, \qquad \Omega^- := \Omega \cap \{P < 0\}.$$

Then any minimizer u of $\mathcal{F}_{\alpha\beta}$ satisfies

$$\int_\Omega < \nabla u(x) - \zeta_0, \nabla\phi(x) > \, dx = 0, \quad \forall \phi \in C_c^\infty(\Omega). \tag{4}$$

If, in addition, $\alpha = 0$ then the following hold:

$$\int_{\{t_1 \le u < t_2\}} < \nabla u(x) - \zeta_0, \nabla u(x) > \, dx = 0, \quad \text{for } L^1 \text{ a.e. } t_1 < t_2; \tag{5}$$

for all $n \in \mathbb{N}_0$ and for L^1 a.e. $t > 0$,

$$-\frac{1}{t^n} \int_{\{u \ge t\}} < \nabla u(x) - \zeta_0, \nabla(u^n P)(x) > \, dx = \int_{\{u \ge t\}} |\nabla u(x) - \zeta_0|^2 \, dx; \tag{6}$$

for all $n \in \mathbb{N}$ and for L^1 a.e. $t > 0$,

$$\frac{1}{t^n} \int_{\{0 < u < t\}} < \nabla u(x) - \zeta_0, \nabla(u^n P)(x) > \, dx = \int_{\{u \ge t\}} |\nabla u(x) - \zeta_0|^2 \, dx; \tag{7}$$

for L^1 a.e. $t > 0$,

$$\int_{\{u \ge t\}} |\nabla u - \zeta_0|^2 \, dx = \lim_{h \to 0^+} \lim_{n \to +\infty} \frac{n}{t} \int_{\{t-h < u < t\}} < \nabla u - \zeta_0, \nabla u > \left(\frac{u}{t}\right)^{n-1} P \, dx; \tag{8}$$

$$\int_{\Omega^+ \cap \{0 < u < P\}} |\nabla u(x) - \zeta_0|^2 \, dx = -\int_{\Omega^+ \cap \{u > 0\}} < \nabla u(x) - \zeta_0, \zeta_0 > \, dx; \tag{9}$$

$$\int_{\Omega^+ \cap \{u \ge P\}} |\nabla u(x) - \zeta_0|^2 \, dx = -\frac{1}{|\zeta_0|} \int_{\Gamma \cap \{u > 0\}} < \nabla u(x) - \zeta_0, \zeta_0 > u(x) \, dH^{N-1}(x). \tag{10}$$

Remark 4.2. Results similar to (6), (7) and (8) also hold for L^1 a.e. $t < 0$ and equivalents for (9) and (10) hold for Ω^- and $\{u < 0\}$.

Proof. *Step 1.* We start by describing in generality the method we will use to derive the above relations. Let u be a minimizer of $\mathcal{F}_{\alpha\beta}$. For $\varepsilon > 0$ and g regular enough, we consider perturbations

$$u_\varepsilon := u \pm \varepsilon g$$

such that

1) $u_\varepsilon \in SBV^2(\Omega)$,
2) $S_{u_\varepsilon} \subseteq S_u$,
3) $\{x \in \partial\Omega : Tu_\varepsilon(x) \neq 0\} \subseteq \{x \in \partial\Omega : Tu(x) \neq 0\}$,
4) $|[u_\varepsilon]|(x) \leq |[u]|(x)$, for H^{N-1} a.e. $x \in S_u$.

Then, the comparison of the energies yields

$$\begin{aligned}\int_\Omega |\nabla u(x) - \zeta_0|^2\,dx &\leq \int_\Omega |\nabla u_\varepsilon(x) - \zeta_0|^2\,dx \\ = \int_\Omega |\nabla u(x) - \zeta_0|^2 &\pm 2\varepsilon < \nabla u(x) - \zeta_0, \nabla g(x) > + \varepsilon^2 |\nabla g(x)|^2\,dx\end{aligned}$$

so that

$$\int_\Omega \pm 2\varepsilon < \nabla u(x) - \zeta_0, \nabla g(x) > + \varepsilon^2 |\nabla g(x)|^2\,dx \geq 0.$$

Dividing by ε and letting $\varepsilon \to 0^+$ we obtain

$$\int_\Omega < \nabla u(x) - \zeta_0, \nabla g(x) > \,dx = 0. \tag{11}$$

Equation (4) follows by taking $g = \phi \in C_c^\infty(\Omega)$ in (11). In this case we also have

$$\int_\Omega < \nabla u(x), \nabla \phi(x) > \,dx = 0, \quad \forall \phi \in C_c^\infty(\Omega),$$

which, in particular, means that if $u \in W^{1,2}(\Omega)$ then u is harmonic.

Notice that if $\alpha = 0$ item 4) in the above list is unnecessary so we can also consider perturbations where $g = u^n \psi$, with $\psi \in C^\infty(\Omega)$, $n \in \mathbb{N}$. For $\alpha \neq 0$, compliance of perturbations with item 4), implies that in general we can only derive the inequality

$$\int_\Omega < \nabla u(x) - \zeta_0, \nabla g(x) > \,dx \leq 0.$$

This is the case for $g = u^n \psi$, since $u_\varepsilon = u + \varepsilon u^n \psi$ leads to an increase of the jump term.

For the remainder of this proof we will restrict ourselves to the case $\alpha = 0$. In this case, if P is an affine function with $\nabla P = \zeta_0$ and $n \in \mathbb{N}, m \in \mathbb{N}_0$, we have

$$\int_\Omega < \nabla u(x) - \zeta_0, \nabla(u^n P^m)(x) > \,dx = 0. \tag{12}$$

Step 2. For $t \geq 0$ we now consider the perturbation

$$u_{\varepsilon,t} := \begin{cases} u \pm \varepsilon u, & \text{if } u < t \\ u \pm \varepsilon t, & \text{if } u \geq t. \end{cases}$$

Comparing energies and following the argument described in *Step 1* we obtain

$$\int_{\{u<t\}} < \nabla u(x) - \zeta_0, \nabla u(x) > \,dx = 0, \tag{13}$$

which, given the arbitrariness of $t \geq 0$, leads to

$$\int_{\{t_1 \leq u < t_2\}} < \nabla u(x) - \zeta_0, \nabla u(x) > \, dx = 0, \tag{14}$$

for L^1 a.e. $0 \leq t_1 < t_2$. A similar argument can be done for $t \leq 0$ so (5) is proved.

Notice also that from (12) (for $n = 1, m = 0$) and (13) we have that

$$\int_{\{u \geq t\}} |\nabla u(x) - \zeta_0|^2 \, dx = - \int_{\{u \geq t\}} < \nabla u(x) - \zeta_0, \zeta_0 > \, dx, \tag{15}$$

for L^1 a.e. $t > 0$.

Step 3. For $t > 0$ we now set

$$u_{\varepsilon,t} := \begin{cases} u \pm \varepsilon(u - t)P, & \text{if } u \geq t \\ u, & \text{if } u < t. \end{cases}$$

The comparison of the energies argument in this case yields

$$\int_{\{u \geq t\}} < \nabla u(x) - \zeta_0, \nabla[(u - t)P](x) > \, dx = 0, \tag{16}$$

for L^1 a.e. $t > 0$ and so, by (15),

$$\begin{aligned} \frac{1}{t} \int_{\{u \geq t\}} < \nabla u(x) - \zeta_0, \nabla(uP)(x) > \, dx &= \int_{\{u \geq t\}} < \nabla u(x) - \zeta_0, \zeta_0 > \, dx \\ &= - \int_{\{u \geq t\}} |\nabla u(x) - \zeta_0|^2 \, dx. \end{aligned} \tag{17}$$

Arguing similarly, but now for

$$u_{\varepsilon,t} := \begin{cases} u \pm \varepsilon(u - t)u^n P, & \text{if } u \geq t \\ u, & \text{if } u < t, \end{cases}$$

where $n \in \mathbb{N}_0$, it follows, by expanding the gradient of $u_{\varepsilon,t}$ and using an induction argument together with (17), that

$$-\frac{1}{t^n} \int_{\{u \geq t\}} < \nabla u(x) - \zeta_0, \nabla(u^n P)(x) > \, dx = \int_{\{u \geq t\}} |\nabla u(x) - \zeta_0|^2 \, dx \tag{18}$$

as claimed in equation (6).

Now, by (18) and (12) with $m = 1$, it follows that

$$\frac{1}{t^n} \int_{\{u < t\}} < \nabla u(x) - \zeta_0, \nabla(u^n P)(x) > \, dx = \int_{\{u \geq t\}} |\nabla u(x) - \zeta_0|^2 \, dx,$$

for all $n \in \mathbb{N}$.

Considering

$$u_\varepsilon := \begin{cases} u \pm \varepsilon u^n P^m, & \text{if } u > 0 \\ u, & \text{if } u \leq 0 \end{cases}$$

we obtain the equivalent of equation (12) in $\Omega \cap \{u > 0\}$ (which, together with (12), leads to the same conclusion in $\Omega \cap \{u \leq 0\}$). Hence, from (18) we have

$$\frac{1}{t^n} \int_{\{0<u<t\}} < \nabla u(x) - \zeta_0, \nabla(u^n P)(x) > \, dx = \int_{\{u \geq t\}} |\nabla u(x) - \zeta_0|^2 \, dx,$$

for all $n \in \mathbb{N}$, which proves (7).

Expanding $\nabla(u^n P)$ in the previous equation and passing to the limit as $n \to +\infty$, we conclude that, for L^1 a.e. $t > 0$,

$$\int_{\{u \geq t\}} |\nabla u - \zeta_0|^2 \, dx = \lim_{n \to +\infty} \frac{n}{t} \int_{\{0<u<t\}} < \nabla u - \zeta_0, \nabla u > \left(\frac{u}{t}\right)^{n-1} P \, dx \tag{19}$$

since

$$\lim_{n \to +\infty} \frac{1}{t^n} \int_{\{0<u<t\}} < \nabla u(x) - \zeta_0, \zeta_0 > u^n(x) \, dx = 0$$

by the dominated convergence theorem. Let $0 < t_1 < t_2$. Then, from (19),

$$\begin{aligned} \int_{\{u \geq t_2\}} |\nabla u - \zeta_0|^2 \, dx &= \lim_{n \to +\infty} \frac{n}{t_2} \int_{\{0<u<t_2\}} < \nabla u - \zeta_0, \nabla u > \left(\frac{u}{t_2}\right)^{n-1} P \, dx \\ &= \lim_{n \to +\infty} \frac{n}{t_2} \left[\int_{\{0<u \leq t_1\}} < \nabla u - \zeta_0, \nabla u > \left(\frac{u}{t_2}\right)^{n-1} P \, dx \right. \\ &\qquad \left. + \int_{\{t_1<u<t_2\}} < \nabla u - \zeta_0, \nabla u > \left(\frac{u}{t_2}\right)^{n-1} P \, dx \right]. \end{aligned}$$

Since $|\frac{u}{t_2}| \leq |\frac{t_1}{t_2}| < 1$ and $\lim_{n \to +\infty} n y^{n-1} = 0$ for $0 < y < 1$, the first term converges to zero. Thus, from the arbitrariness of t_1, t_2, we have

$$\begin{aligned} \int_{\{u \geq t\}} |\nabla u - \zeta_0|^2 \, dx &= \lim_{h \to 0^+} \lim_{n \to +\infty} \frac{n}{t} \int_{\{t-h<u<t\}} < \nabla u - \zeta_0, \nabla u > \left(\frac{u}{t}\right)^{n-1} P \, dx \\ &= \lim_{n \to +\infty} \frac{n}{t} \int_{\{t-\frac{1}{n}<u<t\}} < \nabla u - \zeta_0, \nabla u > \left(\frac{u}{t}\right)^{n-1} P \, dx, \end{aligned}$$

for L^1 a.e. $t > 0$.

Step 4. We now derive relations where the sections of Ω with normal ζ_0 play a special role.

Let P be an affine function with $\nabla P = \zeta_0$ and set

$$\Omega^+ := \Omega \cap \{P > 0\}, \qquad \Gamma := \Omega \cap \{P = 0\}, \qquad \Omega^- := \Omega \cap \{P < 0\}.$$

For

$$\overline{P} := \begin{cases} P, & \text{in } \Omega^+ \cup \Gamma \\ 0, & \text{in } \Omega^- \end{cases}$$

consider

$$\tilde{u}_P := \max(0, \min(u, \overline{P}))$$

and

$$u_{\varepsilon,P} := u \pm \varepsilon \tilde{u}_P.$$

Comparing energies as before, and taking into account that $u_{\varepsilon,P} = u$ in $\{u \leq 0\}$ and in $(\Omega^- \cup \Gamma) \cap \{u > 0\}$, we are lead to

$$\int_{\Omega^+ \cap \{0<u<P\}} < \nabla u(x) - \zeta_0, \nabla u(x) > \, dx + \int_{\Omega^+ \cap \{u \geq P\}} < \nabla u(x) - \zeta_0, \zeta_0 > \, dx = 0,$$

which can be written equivalently in the form

$$\int_{\Omega^+ \cap \{0<u<P\}} |\nabla u(x) - \zeta_0|^2 \, dx = - \int_{\Omega^+ \cap \{u>0\}} < \nabla u(x) - \zeta_0, \zeta_0 > \, dx \tag{20}$$

and we have thus obtained (9).

Finally, in order to prove (10) we let

$$\overline{P_t} := \begin{cases} P - t, & \text{in } \Omega \cap \{P > t\} \\ 0, & \text{in } \Omega \cap \{P \leq t\} \end{cases}$$

and set

$$\tilde{u}_{P,t} := \max(0, u\overline{P_t})$$

and

$$u_{\varepsilon,P,t} := u \pm \varepsilon \tilde{u}_{P,t}.$$

Once again, comparing energies, as $u_{\varepsilon,P,t} = u$ in $\{P \leq t\}$ and in $\{u \leq 0\}$, we obtain

$$\int_{\{u>0\} \cap \{P>t\}} < \nabla u(x) - \zeta_0, \nabla[u(P-t)](x) > \, dx = 0$$

which, differentiating with respect to t (cf. [13], Prop. 3 page 118), leads to

$$-\frac{1}{|\zeta_0|} \int_{\{P=t\} \cap \{u>0\}} < \nabla u - \zeta_0, \zeta_0 > u \, dH^{N-1}(x) = \int_{\{u>0\} \cap \{P>t\}} < \nabla u - \zeta_0, \nabla u > \, dx.$$

Setting $t = 0$ in the above expression yields

$$-\frac{1}{|\zeta_0|} \int_{\Gamma \cap \{u>0\}} < \nabla u - \zeta_0, \zeta_0 > u \, dH^{N-1}(x) = \int_{\Omega^+ \cap \{u>0\}} < \nabla u - \zeta_0, \nabla u > \, dx. \tag{21}$$

From (20) we conclude that

$$\begin{aligned}
&\int_{\Omega^+\cap\{u>0\}} < \nabla u(x) - \zeta_0, \nabla u(x) > \, dx - \int_{\Omega^+\cap\{u>0\}} < \nabla u(x) - \zeta_0, \zeta_0 > \, dx \\
= &\int_{\Omega^+\cap\{u>0\}} |\nabla u(x) - \zeta_0|^2 \, dx \\
= &\int_{\Omega^+\cap\{0<u<P\}} |\nabla u(x) - \zeta_0|^2 \, dx + \int_{\Omega^+\cap\{u\geq P\}} |\nabla u(x) - \zeta_0|^2 \, dx \\
= &- \int_{\Omega^+\cap\{u>0\}} < \nabla u(x) - \zeta_0, \zeta_0 > \, dx + \int_{\Omega^+\cap\{u\geq P\}} |\nabla u(x) - \zeta_0|^2 \, dx
\end{aligned}$$

and hence, by (21),

$$\begin{aligned}
\int_{\Omega^+\cap\{u\geq P\}} |\nabla u(x) - \zeta_0|^2 \, dx &= \int_{\Omega^+\cap\{u>0\}} < \nabla u(x) - \zeta_0, \nabla u(x) > \, dx \\
&= -\frac{1}{|\zeta_0|} \int_{\Gamma\cap\{u>0\}} < \nabla u(x) - \zeta_0, \zeta_0 > u(x) \, dH^{N-1}(x).
\end{aligned}$$

□

It is well known that the growth of a harmonic function u is related to the growth of its nodal set (the set of points where $u = 0$), we refer to [15] for more details on this matter. In particular, the growth of a harmonic function is measured by its frequency M. If u is harmonic in the unit ball $B(0,1) \subset \mathbb{R}^N$ its frequency is defined by

$$M = \frac{\displaystyle\int_{B(0,1)} |\nabla u(x)|^2 \, dx}{\displaystyle\int_{\partial B(0,1)} u^2(x) \, dH^{N-1}(x)}. \tag{22}$$

The following theorem, whose proof may be found in [15], estimates the measure of the nodal set of a harmonic function in terms of its frequency.

Theorem 4.3. *Suppose u is a harmonic function in $B(0,1)$. Then*

$$H^{N-1}\left\{x \in B\left(0, \frac{1}{2}\right) : u(x) = 0\right\} \leq c(N)M,$$

where M is the frequency of u in $B(0,1)$ defined by (22) and $c(N)$ is a positive constant depending only on the dimension N.

The properties proved in Proposition 4.1 establish a relation between growth of bulk energy and of level sets. This is in accordance with the fact that our minimizers, if regular enough, are harmonic.

We point out that in our case the frequency is directly related to ζ_0. Indeed, if $\Omega = B(0,1)$, by (12) with $n = 1$ and $m = 0$, we have

$$M = \frac{\displaystyle\int_{B(0,1)} < \nabla u(x), \zeta_0 > \, dx}{\displaystyle\int_{\partial B(0,1)} u^2(x)\, dH^{N-1}(x)} \leq \frac{\displaystyle\int_{B(0,1)} |\zeta_0|^2\, dx}{\displaystyle\int_{\partial B(0,1)} u^2(x)\, dH^{N-1}(x)}.$$

We can also view equation (10),

$$-\frac{1}{|\zeta_0|}\int_{\Gamma\cap\{u>0\}} < \nabla u(x) - \zeta_0, \zeta_0 > u(x)\, dH^{N-1}(x) = \int_{\Omega^+\cap\{u\geq P\}} |\nabla u(x) - \zeta_0|^2\, dx,$$

as a result along the same lines as Theorem 4.3. Notice in particular that if $u = 0$ along a strip orthogonal to ζ_0 then u must grow less than P, i.e. $|u|(x) \leq P(x)$, in $\Omega^+ \cap \{u > 0\}$. Likewise in $\Omega^- \cap \{u < 0\}$.

The relations stated in Proposition 4.1 strongly suggest that a minimizer will try to be either zero or to have the appropriate gradient ζ_0. To that effect we are able to prove the following partial result:

Theorem 4.4. *Let $u \in SBV^2(\Omega)$ be a minimizer for $\mathcal{F}_{\alpha\beta}$. Suppose that there exist $\Omega_1, \Omega_2 \subseteq \Omega$ such that $|\Omega_1| + |\Omega_2| = |\Omega|$ and $u = 0$ in Ω_1, $\nabla u = \zeta_0$ in Ω_2. Then $u \in W^{1,2}(\Omega)$ and the sets Ω_1 and Ω_2 are sets of finite perimeter in Ω.*

Proof. *Step 1.* We start by showing that under the stated hypotheses the sets Ω_1 and Ω_2 are sets of finite perimeter in Ω.

By means of a rotation argument we can assume, without loss of generality, that $\zeta_0 = e_1$. Moreover, for simplicity of notation, we take $N = 2$. The general case $N > 2$ follows by applying *Step 2* below to the vectors of the canonical basis e_i, $i = 2, \dots, N$.

Extend u by 0 to all of $\mathbb{R}^2$ (where we still denote the extension by u) and set

$$E = \left\{(x,y) \in \mathbb{R}^2 : u(x,y) = 0\right\}.$$

Then $\nabla u = \zeta_0 = e_1$ a.e. in $\mathbb{R}^2 \backslash E$, $u \in SBV(\mathbb{R}^2)$ and $H^1(J_u) < +\infty$, since $u \in SBV^2(\Omega)$ and Ω is a set of finite perimeter.

For $y \in \mathbb{R}$, define

$$u_y(x) = u(x,y).$$

It is well known (cf. [2], Theorem 3.103 and Remark 3.109) that, for a.e. $y \in \mathbb{R}$, $u_y \in SBV(\mathbb{R})$ and $J_{u_y} = (J_u)_y$ and hence $H^0(J_{u_y}) < +\infty$. By hypothesis we also have that $\frac{du_y}{dx} = 1$ for a.e. $x \in \mathbb{R} \setminus E_y$ where

$$E_y := \{x \in \mathbb{R} : (x,y) \in E\}.$$

We want to show that E_y is a set of finite perimeter, that is, that $\chi_{E_y} \in SBV(\mathbb{R})$. Let a, b be two consecutive points in J_{u_y}. Replacing u by its good representative if necessary, we can assume that u_y is continuous in $]a,b[$. Hence,

$$\{x \in]a,b[: u_y(x) \neq 0\}$$

is open and therefore can be written as a countable union of open intervals $I_k =]\alpha_k, \beta_k[$, $k \in \mathbb{N}$ where $u_y(\beta_k) - u_y(\alpha_k) = \beta_k - \alpha_k$. By continuity of u_y in $]a, b[$ it is clear that we can have at most two intervals I_k and therefore

$$P_1(E_y) \leq 2P_1(J_{u_y}) = 2P_1((J_u)_y).$$

Step 2. As in *Step 1*, for $x \in \mathbb{R}$ define

$$u_x(y) = u(x, y).$$

We know that, for a.e. $x \in \mathbb{R}$, $u_x \in SBV(\mathbb{R})$ and $J_{u_x} = (J_u)_x$ and hence $H^0(J_{u_x}) < +\infty$. By hypothesis we also have that $\frac{du_x}{dy} = 0$ for a.e. $y \in \mathbb{R} \setminus E_x$ where

$$E_x := \{y \in \mathbb{R} : (x, y) \in E\}.$$

Hence, reasoning as in the previous step, we conclude that

$$P_1(E_x) \leq P_1(J_{u_x}) = P_1((J_u)_x).$$

Thus, for a.e. $y \in \mathbb{R}$, $(\chi_E)_y = \chi_{E_y} \in SBV(\mathbb{R})$ and

$$\begin{aligned} \int_{-\infty}^{+\infty} |D(\chi_E)_y|(\mathbb{R})\, dy &= \int_{-\infty}^{+\infty} P_1(E_y)\, dy \\ &\leq 2\int_{-\infty}^{+\infty} P_1((J_u)_y)\, dy \leq 2H^1(J_u) \end{aligned}$$

and for a.e. $x \in \mathbb{R}$, $(\chi_E)_x = \chi_{E_x} \in SBV(\mathbb{R})$ and

$$\begin{aligned} \int_{-\infty}^{+\infty} |D(\chi_E)_x|(\mathbb{R})\, dx &= \int_{-\infty}^{+\infty} P_1(E_x)\, dx \\ &\leq \int_{-\infty}^{+\infty} P_1((J_u)_x)\, dx \leq H^1(J_u). \end{aligned}$$

We conclude (cf. [2], Remark 3.104) that $\chi_E \in BV(\mathbb{R}^2)$, that is Ω_1, and therefore Ω_2, are sets of finite perimeter in Ω.

Step 3. Combining the conclusion of *Steps 1* and *2* with the characterization of SBV functions whose gradient is zero a.e. (see Lemma 1.6), it follows immediately that there is a Borel partition $\{U_i\}$ of Ω_2 and a sequence $\{t_i\} \subset \mathbb{R}$ with $t_i \neq t_j$ for $i \neq j$, such that

$$\sum_{i=1}^{+\infty} \mathrm{Per}_\Omega(U_i) < +\infty$$

and

$$u(x) =< \zeta_0, x > + \sum_{i=1}^{+\infty} t_i \chi_{U_i}(x) \text{ a.e. in } \Omega_2.$$

Indeed, defining $v(x) = u(x) - < \zeta_0, x >$ in Ω_2 and $v(x) = 0$ in Ω_1 this result follows immediately from Lemma 1.6 applied to v.

Step 4. Taking into account the previous results, we are now in position to further characterize the solutions of the relaxed problem. In fact, we can prove that

$$H^{N-1}(\partial^*\Omega_i \cap S_u) = 0, \quad i = 1, 2.$$

This step of the proof follows closely that in [5], Theorem 3.3, but we include it here for the sake of completeness.

Since Ω_1 is a set of finite perimeter, we know that $\overline{\partial^*\Omega_1} = \partial\Omega_1$, and so we can choose a point x_0 in $\partial^*\Omega_1 \cap \Omega$, and suppose that this point belongs to S_u. Assume, without loss of generality, that $x_0 = 0$, $u^-(0) = 0$ and $u^+(0) > 0$. We can also assume that $u^+(x) > 0, \forall x \in S_u \cap B(0, \varepsilon)$.

For $\varepsilon, k > 0$ such that $\frac{\varepsilon}{k} < u^+(0)$ define

$$g_{\varepsilon,k}(x) = \frac{\varepsilon - |x|}{k}.$$

Moreover, take k so that $k < \frac{1}{|\zeta_0|}$ in order to ensure that, for any $\lambda > 1$, we have

$$H^{N-1}\left\{x \in B(0,\varepsilon) : g_{\varepsilon,k}(x) = \frac{\varepsilon}{\lambda k} + < \zeta_0, x >\right\} > 0. \tag{23}$$

For ε, k, λ as above and $x \in B(0, \varepsilon)$ set

$$f_{\varepsilon,\lambda,k}(x) := \min\left\{g_{\varepsilon,k}(x), \frac{\varepsilon}{\lambda k} + < \zeta_0, x >\right\}.$$

Fix $\lambda_0 = \lambda_0(\varepsilon, k, \zeta_0)$ so that

$$\begin{aligned} m_{\varepsilon,k,\zeta_0} &:= \min_{B(0,\frac{\varepsilon}{2})} \left\{\frac{\varepsilon}{\lambda_0 k} + < \zeta_0, x >, \frac{\varepsilon}{2k}\right\} \\ &= \min\left\{f_{\varepsilon,\lambda_0,k}(x) : \ x \in B\left(0, \frac{\varepsilon}{2}\right)\right\} \geq \frac{\varepsilon}{2k}, \end{aligned}$$

and define

$$u_{\varepsilon,k}(x) = \begin{cases} u(x), & \text{if } x \in \Omega \setminus B(0,\varepsilon) \\ \max\{u(x), f_{\varepsilon,\lambda_0,k}(x)\}, & \text{if } x \in B(0,\varepsilon). \end{cases}$$

Clearly,

$$u_{\varepsilon,k} \in SBV^2(\Omega),$$

and, by (23),

$$S_{u_{\varepsilon,k}} = \left\{x \in S_u : u^+(x) > f_{\varepsilon,\lambda_0,k}(x)\right\} \subseteq S_u.$$

Therefore, $u_{\varepsilon,k}$ is admissible for $(\overline{P_{\alpha\beta}})$. Also, due to the definition of the approximate upper and lower limits and the continuity of $f_{\varepsilon,\lambda_0,k}$, we have

$$[u_{\varepsilon,k}](x) = \begin{cases} [u](x) & \text{if } u^-(x) \geq f_{\varepsilon,\lambda_0,k}(x) \\ u^+(x) - f_{\varepsilon,\lambda_0,k}(x) & \text{if } u^-(x) < f_{\varepsilon,\lambda_0,k}(x) < u^+(x) \\ 0 & \text{if } u^+(x) \leq f_{\varepsilon,\lambda_0,k}(x). \end{cases}$$

Comparing the energies of u and $u_{\varepsilon,k}$ (which are equal outside $\overline{B(0,\varepsilon)}$), we claim that

$$J_{\alpha\beta}(u_{\varepsilon,k}) < J_{\alpha\beta}(u),$$

for some appropriate choice of ε, k, thus contradicting the minimality of u. Indeed, taking for simplicity of writing, from now on, $\alpha = \beta = 1$,

$$\begin{aligned}
& J_{\alpha\beta}(u_{\varepsilon,k}) < J_{\alpha\beta}(u) \\
\Leftrightarrow \quad & \int_{B(0,\varepsilon)} |\nabla u_{\varepsilon,k} - \zeta_0|^2 - |\nabla u - \zeta_0|^2 \, dx \\
\leq \quad & \int_{S_u \cap B(0,\varepsilon)} (1 + [u](x)) \, dH^{N-1}(x) - \int_{S_{u_{\varepsilon,k}} \cap B(0,\varepsilon)} (1 + [u_{\varepsilon,k}](x)) \, dH^{N-1}(x).
\end{aligned}$$

Since $S_{u_{\varepsilon,k}} \subseteq S_u$ and

$$\int_{B(0,\varepsilon)} |\nabla u_{\varepsilon,k} - \zeta_0|^2 - |\nabla u - \zeta_0|^2 \, dx \leq \frac{4\varepsilon^N w_N}{k^2},$$

it suffices to show that

$$\begin{aligned}
\frac{4\varepsilon^N w_N}{k^2} \quad \leq \quad & \int_{S_{u_{\varepsilon,k}} \cap B(0,\varepsilon)} ([u](x) - [u_{\varepsilon,k}](x)) \, dH^{N-1}(x) \\
& + \int_{(S_u \setminus S_{u_{\varepsilon,k}}) \cap B(0,\varepsilon)} [u](x) \, dH^{N-1}(x) \\
= \quad & \int_{S_u \cap B(0,\varepsilon) \cap \{u^- < f_{\varepsilon,\lambda_0,k} < u^+\}} (f_{\varepsilon,\lambda_0,k}(x) - u^-(x)) \, dH^{N-1}(x) \\
& + \int_{S_u \cap \{u^+ \leq f_{\varepsilon,\lambda_0,k}\} \cap B(0,\varepsilon)} [u](x) \, dH^{N-1}(x).
\end{aligned}$$

Since the last integral is nonnegative and as, by definition, $\frac{\varepsilon}{k} \geq f_{\varepsilon,\lambda_0,k} \geq m_{\varepsilon,k,\zeta_0} \geq \frac{\varepsilon}{2k}$ in $B(0, \frac{\varepsilon}{2})$ and $u^-(x) = 0$ for $x \in \partial^*\Omega_1$, it is enough to prove that

$$\frac{4\varepsilon^N w_N}{k^2} \leq \int_{\partial^*\Omega_1 \cap B(0,\frac{\varepsilon}{2}) \cap \{u^+ > \frac{\varepsilon}{k}\}} \frac{\varepsilon}{2k} \, dH^{N-1}(x).$$

Therefore, if we can show that

$$\frac{8 w_N}{k} \leq \frac{H^{N-1}(\partial^*\Omega_1 \cap B(0, \frac{\varepsilon}{2}) \cap \{u^+ > \frac{\varepsilon}{k}\})}{\varepsilon^{N-1}}, \tag{24}$$

for some appropriate choice of ε, k, the desired contradiction follows. By Proposition 1.5, there exists $C > 0$, such that

$$\liminf_{r \to 0^+} \frac{H^{N-1}(\partial^*\Omega_1 \cap B(0, \frac{r}{2}))}{r^{N-1}} > C > 0.$$

Hence we can fix ε_1 satisfying the initial restrictions, and such that

$$0 < C < \frac{H^{N-1}(\partial^*\Omega_1 \cap B(0, \frac{\varepsilon_1}{2}))}{\varepsilon_1^{N-1}}.$$

Choose k satisfying our initial conditions. Then, by the general properties of nested families of measurable sets,

$$\frac{H^{N-1}(\partial^*\Omega_1 \cap B(0, \frac{\varepsilon_1}{2}))}{\varepsilon_1^{N-1}} = \lim_{\varepsilon \to 0^+} \frac{H^{N-1}(\partial^*\Omega_1 \cap B(0, \frac{\varepsilon_1}{2}) \cap \{u^+ > \frac{\varepsilon}{k}\})}{\varepsilon_1^{N-1}}$$

and therefore, there exists $\varepsilon_2 \leq \varepsilon_1$ such that

$$\frac{H^{N-1}(\partial^*\Omega_1 \cap B(0, \frac{\varepsilon_1}{2}) \cap \{u^+ > \frac{\varepsilon_2}{k}\})}{\varepsilon_1^{N-1}} > C > 0.$$

Thus,

$$\begin{aligned} 0 < C\varepsilon_1^{N-1} &= H^{N-1}\left(\partial^*\Omega_1 \cap B\left(0, \frac{\varepsilon_1}{2}\right) \cap \left\{u^+ > \frac{\varepsilon_2}{k}\right\}\right) \\ &= H^{N-1}\left(\partial^*\Omega_1 \cap B\left(0, \frac{\varepsilon_1}{2}\right) \cap \left\{u^+ > \frac{\varepsilon_1}{k}\right\}\right) \\ &\qquad + H^{N-1}\left(\partial^*\Omega_1 \cap B\left(0, \frac{\varepsilon_1}{2}\right) \cap \left\{\frac{\varepsilon_2}{k} < u^+ \leq \frac{\varepsilon_1}{k}\right\}\right). \end{aligned}$$

Since

$$\begin{aligned} & H^{N-1}\Big(\partial^*\Omega_1 \cap B\left(0, \frac{\varepsilon_1}{2}\right) \cap \left\{\frac{\varepsilon_2}{k} < u^+ \leq \frac{\varepsilon_1}{k}\right\}\Big) \\ \leq\ & H^{N-1}\Big(\partial^*\Omega_1 \cap B\left(0, \frac{\varepsilon_1}{2}\right) \cap \left\{0 < u^+ \leq \frac{\varepsilon_1}{k}\right\}\Big), \end{aligned}$$

again, due to the fact that this is a nested family of measurable sets, we conclude that

$$\lim_{k \to +\infty} H^{N-1}\left(\partial^*\Omega_1 \cap B\left(0, \frac{\varepsilon_1}{2}\right) \cap \left\{\frac{\varepsilon_2}{k} < u^+ \leq \frac{\varepsilon_1}{k}\right\}\right) = 0$$

and so we can choose k_1 sufficiently large so that

$$H^{N-1}\left(\partial^*\Omega_1 \cap B\left(0, \frac{\varepsilon_1}{2}\right) \cap \left\{u^+ > \frac{\varepsilon_1}{k_1}\right\}\right) \geq C\varepsilon_1^{N-1} > 0. \tag{25}$$

Finally, letting $k_2 > k_1$ be such that

$$k_2 > \frac{8w_N}{C},$$

where C is the constant appearing in (25), it follows from (25) that

$$\begin{aligned} & \frac{H^{N-1}(\partial^*\Omega_1 \cap B(0, \frac{\varepsilon_1}{2}) \cap \{u^+ > \frac{\varepsilon_1}{k_2}\})}{\varepsilon_1^{N-1}} \\ \geq\ & \frac{H^{N-1}(\partial^*\Omega_1 \cap B(0, \frac{\varepsilon_1}{2}) \cap \{u^+ > \frac{\varepsilon_1}{k_1}\})}{\varepsilon_1^{N-1}} \geq C > \frac{8w_N}{k_2}, \end{aligned}$$

and (24) is proved.

An immediate consequence of these results is that the sets Ω_1 and Ω_2 are connected and there is no jump between them. Thus, $u \in W^{1,2,}(\Omega)$ and the proof is complete. $\square$

Remark 4.5. We conclude, therefore, that if a solution of the relaxed problem, satisfying the hypotheses of Theorem 4.4, is in fact a solution of the initial problem then it is necessarily the trivial one.

Remark 4.6. Notice that, if u is a minimizer satisfying the hypotheses of Theorem 4.4, it doesn't necessarily follow that $H^{N-1}(S_u \cap \Omega_2) = 0$. In fact, if Ω is a square and if the solution satisfies $|\Omega_1| = 0$, then, if $\alpha > 0$, it is clearly better to have a jump in some middle section of Ω, as in the one dimensional example given in Section 2. If $\alpha = 0$ the energies obtained by the continuous and discontinuous functions are the same.

We conclude this paper with two examples for which there is a threshold, depending on ζ_0 and a relation between the volume and the perimeter of the domain, such that, below this threshold the solution seems to be trivial and above it seems to be a function satisfying the hypotheses of Theorem 4.4.

Example 4.7. For simplicity we take $N = 2$ and $\alpha = 0$ and we let $\Omega = B(0, R)$.

For $0 \le r \le R$, let u_r be the function in $SBV^2(B(0, R))$ defined by

$$u_r := \begin{cases} \zeta_0 \cdot x, & \text{if } x \in B(0, r) \\ 0, & \text{if } x \in \Omega \setminus B(0, r). \end{cases}$$

Then,

$$J_{0\beta}(u_r) = \int_{B(0,R)\setminus B(0,r)} |\zeta_0|^2 \, dx + 2\pi\beta r = |\zeta_0|^2 \pi (R^2 - r^2) + 2\pi\beta r$$

and a simple calculation shows that

$$r = \frac{\beta}{|\zeta_0|^2}$$

is a local maximum for the energy, whereas $r = 0$ and $r = R$ correspond to local minima. Hence, if $|\zeta_0|^2 R > 2\beta$, the solution is not identically equal to zero. Thus, for R above the threshold $\frac{2\beta}{|\zeta_0|^2}$, the solution(s) of the relaxed problem are not classical, in the sense that they are not harmonic satisfying $Tu = 0$. This means that the solution necessarily has jumps inside the domain Ω and/or its trace is not zero everywhere on the boundary. Naively, we might expect these jumps/trace to enclose a region where the solution has the right gradient.

In this next example we take, without loss of generality, $\zeta_0 = (1, 0)$ and for $\theta \in (0, \frac{\pi}{2})$ we consider

$$W_\theta := \{x = (x_1, x_2) \in B(0, R) : x_1 > R\cos\theta\}$$

and

$$u_\theta(x_1, x_2) := \begin{cases} x_1 - R\cos\theta, & \text{if } x \in W_\theta \\ 0, & \text{if } x \in B(0, R) \setminus W_\theta. \end{cases}$$

Notice that $u_\theta(x)$ has no jumps inside Ω. Then,

$$J_{0\beta}(u_\theta) = \left(\pi R^2 - \theta R^2 + \frac{R^2}{2}\sin(2\theta)\right) + 2\beta\theta R$$

and it is easy to see that there are no local minima for $\theta \in (0, \frac{\pi}{2})$ although, if $R > \beta$, there is a local maximum. Also, if $R > 2\beta$, the functions u_θ, for $\theta = 0$ and $\theta = \frac{\pi}{2}$, have higher energy than any affine function u satisfying $\nabla u = \zeta_0$ in Ω.

The fact that, in this case, the functions u_θ do not correspond to minima of the energy is consistent with Theorem 4.4. Indeed, a function defined similarly, but with the equivalent of the subset W_θ shifted inside the domain Ω, would have the same energy as u_θ but would exhibit a discontinuity between the sets where its gradient takes the values 0 and ζ_0. Thus we would obtain a solution contradicting the hypotheses of the above mentioned theorem.

Repeating the previous calculations with

$$u_\theta(x_1, x_2) := \begin{cases} 0, & \text{if } x \in W_\theta \\ x_1 - R\cos\theta, & \text{if } x \in B(0,R) \setminus W_\theta \end{cases}$$

we have

$$J_{0\beta}(u_\theta) = R^2\left(\theta - \frac{\sin(2\theta)}{2}\right) + 2\beta R(\pi - \theta)$$

so, if $R > \beta$, there is a local minimum for $\theta \in (0, \frac{\pi}{2})$.

Therefore, in this example, the threshold above which the solution is not trivial, has been lowered to $R > \beta$.

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