Journal of Convex Analysis
Volume 17 (2010), No. 2, 643–649

Chakerian-Klamkin Type Theorems*

J. Jerónimo-Castro
Instituto de Matemáticas, UNAM, Ciudad de México, Mexico;
and: Facultad de Matemáticas, UAG, Acapulco, México
jeronimo@cimat.mx, jeronimo@matem.unam.mx

L. Montejano
Instituto de Matemáticas, UNAM, Ciudad de México, Mexico;
and: Facultad de Matemáticas, UAG, Acapulco, México
luis@matem.unam.mx

Received: September 4, 2008
Revised manuscript received: June 22, 2009

The purpose of this paper is to follow the spirit of the Chakerian-Klamkin's characterization of central symmetry to give characterizations of projective centers and hyperplanes of symmetry. We use them to give a new and unexpected characterization of ellipsoids. We also prove two geometric colored theorems in the spirit of Lovász-Bárány.

1. Introduction

Let $K \subset \mathbb{R}^n$ be a compact, convex set and let $L \subset \mathbb{R}^n$ be a compact set. If x is a point of $\mathbb{R}^n$ and K_x is the set of all vectors that translates x into K, then clearly $K_x = K - x$ is a compact, convex set. Consider the family of compact, convex sets $\mathcal{F} = \{K_x\}_{x \in L}$. A vector v belongs to the intersection of all members of $\mathcal{F}$ if and only if the vector v translates L into K, that is, $v + L \subset K$. By Helly's Theorem, there is a point in the intersection of the family $\mathcal{F}$ if and only if there is a point in the intersection of every subfamily of $\mathcal{F}$ of size $n+1$. So, if for every $\{x_0, x_1, ..., x_n\} \subset L$, there is v such that $\{x_0 + v, x_1 + v, ..., x_n + v\} \subset K$, then there exists v_0, such that $v_0 + L \subset K$. Or, in other words: if for every $\{x_0, x_1, ..., x_n\} \subset L$, there is a translated copy of K that contains $\{x_0, x_1, ..., x_n\}$, then there is a translated copy of K that contains L. This implies that a compact, convex set, K is centrally symmetric if and only if for every n-simplex $T \subset K$, there is a translated copy of $-T$ contained in K.

In fact, this result can be improved substantially if we consider the fact that L is equal to K, up to translation, if and only if $\pi(L)$ is equal to $\pi(K)$, up to translation, for every orthogonal projection π into a two dimensional plane. This implies that K is centrally symmetric if and only if for every triangle $T \subset K$, there is a translated copy of $-T$ contained in K.

It is a little surprising that Chakerian and Klamkin [5] had obtained this result for non necessarily convex sets using a very simple argument. Similarly, using Helly's Theorem again, we may show how to characterize a direction in which a orthogonal hyperplane of symmetry may occur: *If $K \subset \mathbb{R}^n$ is a compact, convex set and $v \in \mathbb{S}^{n-1}$*

*Supported by CONACYT MOD-ORD-21-07, PCI-155-05-07.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

is a fixed direction such that for every pair of points $\{a, b\} \subset K$ there is a hyperplane Γ, orthogonal to v, with the property that the orthogonal reflections of a and b with respect to Γ are also contained in K. Then K has an orthogonal hyperplane of symmetry orthogonal to v. The proof of this statement is very simple: Let L be a line parallel to v. For every $x \in K$, define K_x as the set of points $y \in L$, such that the reflected point of x with respect to the hyperplane through y, orthogonal to v, is also in K. Clearly, K_x is a segment contained in L. In this way we have obtained a pairwise intersecting family of segments contained in L. Hence, by Helly's Theorem we have that $\cap_{x\in K} K_x \neq \emptyset$. Consider $z \in \cap_{x\in K} K_x$, then the hyperplane through z, orthogonal to v, is a hyperplane of symmetry for K. It is not difficult to see that this hyperplane of symmetry is unique.

Moreover, after some definitions, we will be ready to use Helly's Theorem to give a Chakerian-Klamkin type characterization of projective centers and hyperplanes of projective symmetry. In order to do this, we complete $\mathbb{R}^n$ to the n-dimensional projective space $\mathbb{P}^n$ by adding the hyperplane at infinity. Let $O \in \{\mathbb{P}^n \setminus \operatorname{bd} K\}$. According to [7], we say that O is a *pole* of K if there is a hyperplane H of $\mathbb{P}^n$ with the property that for every line L through O such that $\operatorname{bd} K \cap L = \{A, B\}$, we have that the cross ratio of A, B, O and the intersection of L and H, $C = L \cap H$, is minus one. That is,

$$[A, B; O, C] = \frac{\overline{AO}}{\overline{OB}} \cdot \frac{\overline{CB}}{\overline{AC}} = -1,$$

where $\overline{XY}$ stands for the directed segment from X to Y. If this is so, we say that A and B are harmonic conjugate points with respect to O and C. Clearly, O and C are also harmonic conjugate with respect to A and B. Moreover, in this case we say that H is a *polar hyperplane* of K and also that H is the polar of the pole O. If $O \in \operatorname{int} K$ is a pole of K, we then say that O is a *projective center of symmetry* of K because, in this case, the polar of K is a hyperplane H that does not intersect K and if π is a projective isomorphism that sends H to the hyperplane at infinity, then $\pi(K)$ is a centrally symmetric, convex body with centre $\pi(O)$. If $O \in \{\mathbb{P}^n \setminus K\}$ is a pole of K, then its polar H will be called a *projective hyperplane of symmetry* of K, because in this case $K \cap H \neq \emptyset$ and there is a projective isomorphism π that sends O to the hyperplane at infinity and we get that $\pi(H)$ is an affine hyperplane of symmetry for the convex body $\pi(K)$.

Proposition 1.1. *Let $K \subset \mathbb{R}^n$ be a convex body and let H be a fixed hyperplane. Then H is a polar of K if and only if for every $(n+1)$-tuple of points $\{\alpha_0, ..., \alpha_n\} \subset K$ there is a point β such that the corresponding harmonic conjugate points $\{\alpha'_0, ...\alpha'_n\}$ with respect to β and H, are also contained in K.*

Proof. For every point $x \in K$, let K_x be the set of all points β such that the harmonic conjugate of x with respect to β and H lies in K. First note that K_x is convex, because if $\beta, \gamma \in K_x$, then the harmonic conjugate x' of x with respect to β and H lies in K and the harmonic conjugate x'' of x with respect to γ and H lies in K. So, by elementary geometry, if $\delta \in [\beta, \gamma]$, then the harmonic conjugate of x with respect to δ and H lies in $[x', x'']$ which is contained in K. Next observe that K_x is compact, because it is clearly closed and for every line L through K_x, the set $L \cap K_x$ is bounded. Let us consider the family of compact, convex sets $\mathcal{F} = \{K_x \mid x \in K\}$. By hypothesis, given $\{\alpha_0, ..., \alpha_n\} \subset K$, there exists $\beta \in K_{\alpha_0} \cap ... \cap K_{\alpha_n}$, so by Helly's Theorem there is a point $\beta_0 \in \cap\{K_x \mid x \in K\}$. This implies that the harmonic conjugate K' of K with respect to β_0 and H is contained in K. Therefore, β_0 is a pole of K and H its polar. $\square$

The purpose of this paper is to follow the spirit of the Chakerian-Klamkin's characterization of central symmetry to give new characterizations of ellipsoids and spheres (in Section 2). Furthermore, in Section 3 we will prove two colored theorems in the spirit of Lovász-Bárány.

2. Characterizations of Ellipsoids

We start by proving the following characterization of ellipsoids. We shall give two different proofs, the first one of them uses Proposition 1.1 and the second one uses a very simple extremal principle.

Theorem 2.1. *Let $K \subset \mathbb{R}^n$ be a convex body. Then, K is an ellipsoid if and only if for every triple $T := \{a, b, c\} \subset K$, there is an ellipse $\mathcal{E}$ such that $T \subset \mathcal{E} \subset K$.*

First proof. We will prove first the case $n = 2$. In order to prove that K is an ellipse it is sufficient to prove, by a theorem proved in [7], that for any line L that does not intersect K, the line L is a polar of K. So, consider a fixed line L with $L \cap K = \emptyset$. We will prove that L is a polar of K by using our Proposition 1.1. Consider a triple $T := \{a, b, c\} \subset K$ and by hypothesis let $\mathcal{E}$ be an ellipse such that $T \subset \mathcal{E} \subset K$. Since L is a polar for $\mathcal{E}$, there exists the corresponding pole β. Hence, the corresponding harmonic conjugate points $\{a', b', c'\}$ with respect to β and L are also contained in $\mathcal{E} \subset K$. This proves that L is a polar of K, therefore K is an ellipse.

Now, for the case $n > 2$, we proceed as follows: let H be a 2-dimensional plane and let $K_H = K \cap H$. If we restrict ourselves to consider triples of points contained in K_H then it is easy to prove that K_H is an ellipse. It follows that every 2-dimensional section of K is an ellipse, therefore K is an ellipsoid. $\square$

Second proof. We will prove only the case $n = 2$, the proof for the case $n > 2$ is exactly the same as above. For this, let $T \subset K$ be a triangle with the largest area and let $\{a, b, c\}$ be its vertices. Clearly, $\{a, b, c\} \subset \operatorname{bd} K$. Now, let T' be the triangle with vertices $a+b-c$, $b+c-a$, and $c+a-b$. This triangle is inversely homothetic to T and its sides are tangent to K at the points a, b, c. The condition of the theorem implies the existence of an ellipse E with $T \subset E \subset K \subset T'$. Since E contains the midpoints of the sides of T' then it is the unique ellipse contained in T' and passing through the points a, b, c. It is well-known that this ellipse is the one with maximum area contained in T'. Let $x \in \operatorname{bd} E$ be an arbitrary point different of a, b, and c. It is very easy to see that there are points $y, z \in \operatorname{bd} E$ such that the triangle with vertices x, y, z has maximum area. Since $|xyz| = |abc|$, where $|mnp|$ stands for the area of a triangle with vertices m, n, p, we then have that $\{x, y, z\} \subset \operatorname{bd} K$. Since x is an arbitrary point in $\operatorname{bd} E$ we have that $\operatorname{bd} E = \operatorname{bd} K$. $\square$

Similarly, we have the following.

Theorem 2.2. *Let $K \subset \mathbb{R}^2$ be a convex body. Then, K is an ellipse if and only if for every triangle T which contains K there is an ellipse $\mathcal{E}$ such that $K \subset \mathcal{E} \subset T$.*

Proof. The proof is very similar to the second proof of the previous theorem but in this case we must consider the triangle T with minimum area which contains K. Let a, b, c, be the midpoints of the sides of T. It is a well-known result that $\{a, b, c\} \subset \operatorname{bd} K$. By the condition of the theorem there is an ellipse E such that $K \subset E \subset T$. Again, E is the unique ellipse which contains a, b, c, and is contained in T. Let T' be a triangle such that $|T'| = |T|$, $T' \supset E$, and makes contact with E in the points x, y, z, different of a, b, and c. Clearly $\{x, y, z\} \subset \operatorname{bd} K$, otherwise we may obtain a triangle with its sides parallel to the sides of T' and with smaller area. Therefore, K coincides with E. □

We also may add an additional restriction to the triples of considered points.

Theorem 2.3. *Let $K \subset \mathbb{R}^n$, $n \geq 3$, be a convex body. Suppose that there exists a number $\epsilon > 0$, such that for every triangle T with vertices $a, b, c \in K$, of area smaller than ϵ, there is an ellipse $\mathcal{E}$ such that $T \subset \mathcal{E} \subset K$. Then K is an ellipsoid.*

Proof. First note that our hypotheses clearly imply that K is strictly convex and smooth. Let H be a two dimensional plane such that $K \cap H$ has area smaller than ϵ. Given any triangle $T \subset K \cap H$, since the area of T is smaller than ϵ, there is an ellipse $\mathcal{E}$ such that $T \subset \mathcal{E} \subset K \cap H$. Therefore, by Theorem 2.1, we have that $K \cap H$ is an ellipse, but the same holds for every section which is close enough to a supporting hyperplane of K. Hence, by Theorem 2 of [3], K is an ellipsoid. □

Consider a triangle T, outside a convex body $K \subset \mathbb{R}^3$, sufficiently small in comparison with K. We would like to know how much could the triangle penetrate into the convex body K without the sides of T touching K.

Let T by a directed triangle, that is, we consider T together with a normal vector for it. So, if T is a directed triangle, by T_+ we denote rotation of T by $90°$, where the rotation takes in count the normal vector of T and the right-hand rule. So, the directed triangle $-T = T_{++}$ has the same normal vector as T.

Let K be a convex body and let T be a directed triangle with normal vector v. Consider a translated copy T' of T on the supporting plane H of K that shares v as its outer normal vector, in such a way that the relative interior of T' contains $H \cap K$. Let $\iota(T, K)$ be the largest number such that $H' = H + \iota(T, K)(-v)$ contains a translated copy T'' of T, in such a way that we can move continuously the triangle T from the position of T' to the position of T'', and keeping always the translated copy of T parallel to T. We will call the number $\iota(T, K)$ the *penetration depth* of T in K.

Formally and in all dimensions, if $v \in \mathbb{S}^{n-1}$ is the unit normal vector of the directed $(n-1)$-simplex T and H is the supporting hyperplane of $K \subset \mathbb{R}^n$ with normal vector v, then $\iota(T, K)$ is the supreme of all positive numbers $t > 0$, such that for every $0 \leq t' < t$, there is a translated copy T' of T contained in the hyperplane $(H - t'v)$ and with the property that the relative interior of T' contains $T' \cap K$. Furthermore, as above, if T is a directed $(n-1)$-simplex, by $-T$ we denote the usual inverse $(n-1)$-simplex but with the same normal vector as T.

We shall prove that if for every sufficiently small directed $(n-1)$-simplex T, $\iota(T, K) = \iota(-T, K)$, then K is an ellipsoid. Furthermore, if $K \subset \mathbb{R}^3$ and if for every sufficiently small directed triangle T, $\iota(T, K) = \iota(T_+, K)$, then K is a solid sphere.

For that purpose, we will give another proof of a theorem of Lutwak [6], that claims that if for every n-simplex σ that contains L, we have that $K \subset \sigma + v$, for some translation vector v, then $K \subset L + v_0$ for some vector v_0.

Theorem 2.4. *Let L and K be convex bodies in $\mathbb{R}^n$. If for every n-simplex σ that contains L there is a translated copy of σ that contains K, then there is a translated copy of L that contains K.*

Proof. Let $\mathcal{L}$ be the family of all supporting closed half-spaces of L. For every $\Delta \in \mathcal{L}$, let K_Δ be the set of vectors that translate Δ to a closed half-space that contains K, that is: $K_\Delta = \{v \in \mathbb{R}^n \mid K \subset v + \Delta\}$. Notice that K_Δ is a closed half-space. If $v_0 \in \cap\{K_\Delta \mid \Delta \in \mathcal{L}\}$, then clearly $K \subset L + v_0$, as we wish. By Helly's Theorem such a vector v_0 exists if $\cap\{K_\Delta \mid \Delta \in \mathcal{L}\}$ is bounded and for every subfamily $\{\Delta_0, ..., \Delta_n\} \subset \mathcal{L}$, we have that $K_{\Delta_0} \cap ... \cap K_{\Delta_n} \neq \emptyset$. If $\Delta_0 \cap ... \cap \Delta_n$ is an n-simplex σ, then $L \subset \sigma$ and by hypothesis there is v such that $K \subset v + \sigma$, but this implies that $v \in K_{\Delta_0} \cap ... \cap K_{\Delta_n} \neq \emptyset$. If $\Delta_0 \cap ... \cap \Delta_n$ is an unbounded set then it is easy to find v such that $K \subset v + (\Delta_0 \cap ... \cap \Delta_n)$. So, in both cases $K_{\Delta_0} \cap ... \cap K_{\Delta_n} \neq \emptyset$. Finally, we have to prove that $\cap\{K_\Delta \mid \Delta \in \mathcal{L}\}$ is bounded, but this is so if for some $\{\Delta_0, ..., \Delta_n\} \subset \mathcal{L}$, we have that $K_{\Delta_0} \cap ... \cap K_{\Delta_n}$ is bounded. Assume that $\Delta_0 \cap ... \cap \Delta_n$ is an n-simplex σ, then the set of vectors v such that $K \subset v + \sigma$ is clearly bounded, therefore $K_{\Delta_0} \cap ... \cap K_{\Delta_n}$ is bounded. □

As a corollary we have the following result proved in [4].

Corollary 2.5. *Let $K \subset \mathbb{R}^n$ be a convex body. Assume that for every n-simplex σ that contains K, there is a translated copy of $-\sigma$ that contains K. Then K is centrally symmetric.*

In what follows we will say that every section of a convex body, which is close enough to a supporting hyperplane of the body, is a *superficial section*. We use the penetration depth to give the following characterization of the ellipsoid.

Theorem 2.6. *Let $K \subset \mathbb{R}^n$ be a convex body and suppose that for every directed $(n-1)$-simplex T,*

$$\iota(T, K) = \iota(-T, K).$$

Then K is an ellipsoid.

Proof. The idea is simple. Our assumption precisely implies, using Corollary 2.5, that every superficial section of K is centrally symmetric, hence, by Theorem 2 of Burton [3] we have that K is an ellipsoid. □

Theorem 2.7. *Let $K \subset \mathbb{R}^3$ be a convex body and suppose that for every directed triangle T,*

$$\iota(T, K) = \iota(T_+, K).$$

Then K is a solid sphere.

Proof. If $\iota(T, K) = \iota(T_+, K)$, then $\iota(T, K) = \iota(T_+, K) = \iota(T_{++}, K) = \iota(-T, K)$, and hence by the above theorem K is an ellipsoid. Now, notice that for an ellipsoid K, different of a sphere, we have that $\iota(T, K) \neq \iota(T_+, K)$. □

3. Colored Theorems

An interesting role in combinatorial geometry is played now by the colored Helly's theorem. Results of the type "if every subfamily of size at most k of a family F has property P then F has property P" are called Helly-type theorems. Associated with every Helly-type theorem we have its colorful version. Suppose in addition that every object of the family F is painted with at least one color among the colors $\{1, ..., k\}$ and also that every rainbow subset of size at most k of F has property P. Then it is too much to expect that the whole family F has property P. What usually happens, but not always, is that there is a color $i \in \{1, ..., k\}$ with the property that the subfamily of all members of F of color i has property P. If this is so, we say that this Helly-type theorem is colorable (see [1]). The first colorable theorem was discovered by L. Lovász and it is the colorful version of Helly's Theorem. Independently, searching for a mathematical game, Bárány found the colorful version of Caratheódory's Theorem [2]. To be more precise, what is usually called the colored Helly's Theorem states that if a family of compact convex sets in $\mathbb{R}^d$ is painted with $d+1$ colors and if every rainbow subfamily of size $d+1$ has a non-empty intersection, then there is a color with the property that all convex sets painted with this color have a non-empty intersection. Using this colored version instead of the regular Helly's Theorem, and arguing as in the introduction we end up with the following unexpected result: Let A_1, A_2 and A_3 be three compact convex sets in the plane. Suppose that for every triple $T := \{a_1, a_2, a_3\}$, with $a_1 \in A_1$, $a_2 \in A_2$ and $a_3 \in A_3$, there is $T' := \{-a_1 + v, -a_2 + v, -a_3 + v\}$ a translation of $-T$, such that $-a_1 + v \in A_1$, $-a_2 + v \in A_2$ and $-a_3 + v \in A_3$. Then one of the sets A_i, for some $i \in \{1, 2, 3\}$, is centrally symmetric. It is doubly unexpected that in fact what is true is the following theorem.

Theorem 3.1. *Let A_1, A_2 and A_3 be three compact sets in $\mathbb{R}^n$ and suppose that for every triple $T := \{a_1, a_2, a_3\}$ with $a_1 \in A_1$, $a_2 \in A_2$ and $a_3 \in A_3$, there is a translated copy $\{-a_1 + a, -a_2 + a, -a_3 + a\}$ of $-T$, with $-a_1 + a \in A_1$, $-a_2 + a \in A_2$ and $-a_3 + a \in A_3$. Then A_1, A_2 and A_3 are concentric centrally symmetric sets.*

Proof. Let $[x_1, x_2]$, with $x_1 \in A_1$ and $x_2 \in A_2$, be the directed, closed interval with maximum length d, among all directed intervals with extreme points in A_1 and A_2. Let $v = \frac{x_2 - x_1}{||x_2 - x_1||} \in \mathbb{S}^{n-1}$ be the direction of $[x_1, x_2]$. Notice that if $[y_1, y_2]$, with $y_1 \in A_1$ and $y_2 \in A_2$, has the same length and is parallel to $[x_1, x_2]$, then its direction $w = \frac{y_2 - y_1}{||y_2 - y_1||} \in \mathbb{S}^{n-1}$ is equal to $-v$, otherwise, one of the diagonals: $[x_1, y_2]$ or $[y_2, x_1]$, of the parallelogram determined by $[x_1, x_2]$ and $[y_1, y_2]$ had length greater than d.

Let $x \in A_3$, and let us consider the triple $T := \{x_1, x_2, x\}$. Then, by hypothesis, there is a vector $a(x) \in \mathbb{R}^n$, depending on x such that $-x_1 + a(x) \in A_1$, $-x_2 + a(x) \in A_2$ and $-x + a(x) \in A_3$. But hence, the parallel closed intervals $[-x_1 + a(x), -x_2 + a(x)]$ have length d and the same direction. This implies that $a(x) = a$, a constant vector a, for every $x \in A_3$. Therefore A$_3$ is centrally symmetric with centre $\frac{a}{2}$. Similarly, A_1 and A_2 are centrally symmetric sets.

Now we will prove that A_1, A_2, and A_3 are concentric. In order to prove this, first note that $\operatorname{conv} A_i$ is a centrally symmetric convex set for $i = 1, 2$. Moreover, the centre of A_i and $\operatorname{conv} A_i$ is the same. So, let H_1 and H_2, be the hyperplanes orthogonal to the interval $[x_1, x_2]$, through x_1 and x_2, respectively. Clearly, H_i is a supporting hyperplane of $\operatorname{conv} A_i$, and $H_i \cap \operatorname{conv} A_i = \{x_i\}$, $i = 1, 2$. Similarly, let Γ_1 and Γ_2, be the hyperplanes orthogonal to the interval $[x_1, x_2]$, through $-x_1+a$ and $-x_2+a$, respectively. Again, Γ_i is a supporting hyperplane of $\operatorname{conv} A_i$, and $\Gamma_i \cap \operatorname{conv} A_i = \{-x_i+a\}$, $i = 1, 2$. Therefore, the chord $[x_1, -x_1 + a]$ is a diametral chord of $\operatorname{conv} A_1$, furthermore, it is the unique diametral chord of $\operatorname{conv} A_1$, with respect to the parallel supporting hyperplanes H_1 and Γ_1. This implies that the centre of $\operatorname{conv} A_1$ is the middle point of $[x_1, -x_1 + a]$. The above proves that $\frac{a}{2}$ is the centre of A_1 and similarly, $\frac{a}{2}$ is the centre of A_2. □

Theorem 3.1 has the following interesting corollary.

Corollary 3.2. *Let A_1, A_2 and A_3 be three compact, convex sets in $\mathbb{R}^n$ and let F_i be the family of homothetic copies of A_i with centre of homothecy in A_i and ratio of homothecy $\frac{1}{2}$, $i = 1, 2, 3$. Suppose that for every triple of sets $\{\alpha_1, \alpha_2, \alpha_3\}$, with $\alpha_1 \in F_1$, $\alpha_2 \in F_2$ and $\alpha_3 \in F_3$, the intersection $\alpha_1 \cap \alpha_2 \cap \alpha_3 \neq \emptyset$. Then the intersection of all members of the family $F_1 \cup F_2 \cup F_3$ is non-empty.*

Proof. A set α_i belongs to the family F_i if and only if $\alpha_i = \frac{a_i}{2} + \frac{A_i}{2}$, for some $a_i \in A_i$. Furthermore, a triple of sets $\{\alpha_1, \alpha_2, \alpha_3\}$, with $\alpha_1 \in F_1$, $\alpha_2 \in F_2$ and $\alpha_3 \in F_3$, is such that $\alpha_1 \cap \alpha_2 \cap \alpha_3 \neq \emptyset$ if and only if given a triple $T := \{\frac{a_1}{2}, \frac{a_2}{2}, \frac{a_3}{2}\}$ with $\frac{a_1}{2} \in \frac{A_1}{2}$, $\frac{a_2}{2} \in \frac{A_2}{2}$ and $\frac{a_3}{2} \in \frac{A_3}{2}$, there is a translated copy $\{-\frac{a_1}{2} + a, -\frac{a_2}{2} + a, -\frac{a_3}{2} + a\}$ of $-T$, with $-\frac{a_1}{2} + a \in \frac{A_1}{2}$, $-\frac{a_2}{2} + a \in \frac{A_2}{2}$ and $-\frac{a_3}{2} + a \in \frac{A_3}{2}$. Consequently, by Theorem 3.1, $\frac{A_1}{2}$, $\frac{A_2}{2}$ and $\frac{A_3}{2}$ are concentric centrally symmetric convex sets. But if this is so, the intersection of all members of the family $F_1 \cup F_2 \cup F_3$ is non-empty. □

Acknowledgements. We thank to the unknown referee for the interesting comments and the idea for the second proof of Theorem 2.1.

References

[1] J. Arocha, I. Bárány, J. Bracho, R. Fábila, L. Montejano: Very colorful theorems, Discrete Comput. Geom. 42 (2009) 142–154.

[2] I. Bárány: A generalization of Carathéodory's theorem, Discrete Math. 40 (1982) 141–152.

[3] G. Burton: Some characterisations of the ellipsoid, Israel J. Math. 28 (1977) 339–349.

[4] V. Boltyanski, J. Jerónimo: Centrally symmetric convex sets, J. Convex Analysis 14 (2007) 345–351.

[5] G. D. Chakerian, M. S. Klamkin: A three point characterization of central symmetry, Amer. Math. Monthly 111 (2004) 903–905.

[6] E. Lutwak: Containment and circumscribing simplices, Discrete Comput. Geom. 19 (1998) 229–235.

[7] L. Montejano, E. Morales: Characterization of ellipsoids and polarity in convex sets, Mathematika 50 (2003) 63–72.