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Estimates on the Derivative of a Polynomial with a Curved Majorant Using Convex Techniques

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A mapping $\phi : [-1,1] \to [0,\infty)$ is a curved majorant for a polynomial p in one real variable if $|p(x)| \leq \phi(x)$ for all $x \in [-1,1]$. If $\mathcal{P}_n^\phi(\mathbb{R})$ is the set of all one real variable polynomials of degree at most n having the curved majorant ϕ, then we study the problem of determining, explicitly, the best possible constant $\mathcal{M}_n^\phi(x)$ in the inequality

$$|p'(x)| \leq \mathcal{M}_n^\phi(x)\|p\|,$$

for each fixed $x \in [-1,1]$, where $p \in \mathcal{P}_n^\phi(\mathbb{R})$ and $\|p\|$ is the sup norm of p over the interval $[-1,1]$. These types of estimates are known as Bernstein type inequalities for polynomials with a curved majorant. The cases treated in this manuscript, namely $\phi(x) = \sqrt{1-x^2}$ or $\phi(x) = |x|$ for all $x \in [-1,1]$ (circular and linear majorant respectively), were first studied by Rahman in [10]. In that reference the author provided, for each $n \in \mathbb{N}$, the maximum of $\mathcal{M}_n^\phi(x)$ over $[-1,1]$ as well as an upper bound for $\mathcal{M}_n^\phi(x)$ for each $x \in [-1,1]$, where ϕ is either a circular or a linear majorant. Here we provide sharp Bernstein inequalities for some specific families of polynomials having a linear or circular majorant by means of classical convex analysis techniques (in particular we use the Krein-Milman approach).

Keywords: Bernstein and Markov inequalities, trinomials, extreme points

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1. Preliminaries

The problem of estimating the derivative of a polynomial has been studied since the end of the 19th century. Already in 1892 it was known that, for any real polynomial p of degree at most n, we have

$$|p'(x)| \leq n^2\|p\|,$$

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for all $x \in [-1, 1]$, where $\|p\|$ is the sup norm of p in the interval $[-1, 1]$. This result is due to Markov [2, 3], for which reason these uniform estimates on p' are called Markov type inequalities. Although the constant n^2 in the previous inequality is optimal, since equality is attained at $x = \pm 1$ for the polynomial $T_n(x) = \cos(n \arccos x)$ on $[-1, 1]$ (nth Chebyshev polynomial of the first kind), it may be improved for specific values of x in $(-1, 1)$. Bernstein [4] (see also [5, 6]) proved in 1912 that

$$|p'(x)| \leq \frac{n}{\sqrt{1-x^2}}\|p\|,$$

for $x \in (-1, 1)$ and p having degree at most n. In general, pointwise estimates on the derivative of a polynomial are called Bernstein type inequalities. Bernstein's estimate $\frac{n}{\sqrt{1-x^2}}\|p\|$ is far from being optimal, specially when x approaches the end points of $[-1, 1]$. If $\mathcal{M}_n(x)$ is the smallest possible constant in

$$|p'(x)| \leq \mathcal{M}_n(x)\|p\|,$$

for all polynomials p of degree at most n and $x \in [-1, 1]$ fixed, then there is a method to obtain $\mathcal{M}_n(x)$ (see [11, 12]), but no explicit formula can be given. However a number of sharp Bernstein type inequalities can be obtained for reduced families of polynomials. For instance, in [8] the authors provide, explicitly, sharp Markov and Bernstein estimates for the space of polynomials of the form $p(x) = ax^m + bx^n + c$ with $a, b, c \in \mathbb{R}$ and $m > n$ endowed with the sup norm over the interval $[-1, 1]$ ($\mathcal{P}_{m,n}(\mathbb{R})$ for short).

Another interesting question related to the study of Bernstein and Markov inequalities arises when we consider constraint families of polynomials. Indeed, if we consider a mapping $\phi : [-1, 1] \to [0, +\infty)$, called majorant in the sequel, and $\mathcal{P}_n^\phi(\mathbb{R})$ stands for the space of polynomials on the real line of degree at most n satisfying $|p(x)| \leq \phi(x)$ for all $x \in [-1, 1]$ endowed with the sup norm over that interval, what is the best constant M_n^ϕ in the inequality

$$\|p'\| \leq M_n^\phi\|p\|,$$

for all $p \in \mathcal{P}_n^\phi(\mathbb{R})$? Similarly, if $x \in [-1, 1]$ is fixed, what is the smallest $\mathcal{M}_n^\phi(x)$ that fits in the inequality

$$|p'(x)| \leq \mathcal{M}_n^\phi(x)\|p\|,$$

for every $p \in \mathcal{P}_n^\phi(\mathbb{R})$? The cases where $\phi(x) = \sqrt{1-x^2}$ (circular majorant) and $\phi(x) = |x|$ (linear majorant) were studied by Rahman in [10] where the author provides sharp Markov constants in both cases, but not sharp Bernstein estimates. In this paper we focus our attention on the problem of finding sharp Bernstein inequalities for polynomials having a circular or a linear majorant. In Section 2 we study the problem of finding sharp Bernstein bounds for polynomials of the form $p(x) = x(ax^m+bx^n+c)$ with $a, b, c \in \mathbb{R}$ and $m, n \in \mathbb{N}$ ($m > n$) having a linear majorant. For simplicity $\mathcal{P}_{m,n}^\ell(\mathbb{R})$ will represent from now on the set of such polynomials while $M_{m,n}^\ell$ and $\mathcal{M}_{m,n}^\ell(x)$ will denote (respectively) the sharp Markov and Bernstein estimates for polynomials in $\mathcal{P}_{m,n}^\ell(\mathbb{R})$. If M_n^ℓ and $\mathcal{M}_n^\ell(x)$ are the sharp Markov and Bernstein estimates for polynomials of degree not greater than n having a linear majorant, $\mathcal{P}_n^\ell(\mathbb{R})$ for short, Rahman showed in [10] that $M_n^\ell = (n-1)^2 + 1$ and

$$\mathcal{M}_n^\ell(x) \leq \sqrt{(n-1)^2\frac{x^2}{1-x^2} + 1},$$

for all $x \in (-1,1)$. The problem of finding $\mathcal{M}_n^\ell(x)$ for every $n \in \mathbb{N}$ seems to be extremely difficult, but at least we can give $\mathcal{M}_3^\ell(x)$ (notice that $\mathcal{M}_3^\ell(x) = \mathcal{M}_{2,1}^\ell(x)$).

In Section 3 we study Bernstein inequalities for polynomials with a circular majorant. From now on $\mathcal{P}_n^c(\mathbb{R})$ will represent the set of polynomials of degree at most n having a circular majorant while M_n^c and $\mathcal{M}_n^c(x)$ will stand for the sharp Markov and Bernstein estimates for polynomials in $\mathcal{P}_n^c(\mathbb{R})$. Rahman showed in [10] that $M_n^c = 2(n-1)$ and

$$\mathcal{M}_n^c(x) \leq \sqrt{\frac{x^2}{1-x^2} + (n-1)^2},$$

for all $x \in (-1,1)$. The problem of giving an explicit formula for $\mathcal{M}_n^c(x)$ for all $n \in \mathbb{N}$ may not be solvable, but at least $\mathcal{M}_3^c(x)$ can be obtained. We also give a characterization of the polynomials of degree at most 3 with a circular majorant. This characterizations holds too for polynomials on a Hilbert space. In order to discuss polynomials on a Banach space E, the space of polynomials of degree at most n on E will be denoted by $\mathcal{P}_n(E)$.

In the following we will use the notation

$$\mathcal{R}_n^\ell(x) = \sqrt{(n-1)^2\frac{x^2}{1-x^2} + 1} \quad \text{and} \quad \mathcal{R}_n^c(x) = \sqrt{\frac{x^2}{1-x^2} + (n-1)^2},$$

where $x \in (-1,1)$ and $n \in \mathbb{N}$ for Rahman's estimates on the derivative of polynomials with linear and circular majorants, respectively. Also, B_E and S_E will stand, respectively, for the closed unit ball and the unit sphere of E.

2. Bernstein estimates for polynomials with a linear majorant

Our results in this section rely upon the following easy consequence of the Krein-Milman Theorem:

> If $C \subset \mathbb{R}^n$ is a convex nonempty set and $f : C \to \mathbb{R}$ is a convex mapping that attains its maximum in C, then there exists an extreme point e of C so that
> $$f(e) = \max\{f(x) : x \in C\}.$$

The connection between this result and the problem we are dealing with in this section is simple. Indeed, first of all notice that if $p \in \mathcal{P}_{m,n}^\ell(\mathbb{R})$, then $p(x) = xq(x)$ for all $x \in \mathbb{R}$ for some $q \in \mathsf{B}_{m,n}$, where $\mathsf{B}_{m,n}$ is the unit ball of the space $\mathcal{P}_{m,n}(\mathbb{R})$. If $x \in [-1,1]$ is fixed, taking into consideration that the mapping

$$\mathsf{B}_{m,n} \ni q \mapsto \left|\frac{d}{dx}(xq(x))\right| = |q(x) + xq'(x)| \in \mathbb{R}$$

is convex, it follows that

$$\mathcal{M}_{m,n}^\ell(x) = \sup\{|q(x)+xq'(x)| : q \in \mathsf{B}_{m,n}\} = \sup\{|q(x)+xq'(x)| : q \in \mathrm{ext}(\mathsf{B}_{m,n})\}, \quad (1)$$

where $\mathrm{ext}(\mathsf{B}_{m,n})$ is the set of extreme points of $\mathsf{B}_{m,n}$. A description of $\mathrm{ext}(\mathsf{B}_{m,n})$ can be found in [9] for all choices of $m,n \in \mathbb{N}$. However, only in a few cases the mapping $\mathcal{M}_{m,n}^\ell(x)$ can be obtained explicitly using the previous idea. We begin by studying the case where m is odd and n is even, for which the following description of $\mathrm{ext}(\mathsf{B}_{m,n})$ will be required.

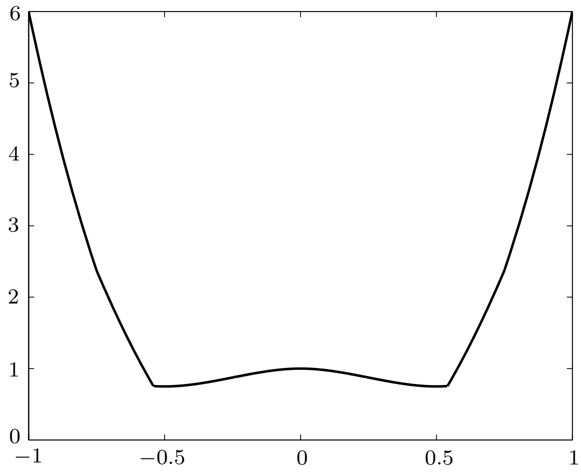

Figure 2.1: $\mathcal{M}^{\ell}_{3,2}(x)$. Here $M^{\ell}_{3,2} = 6$.

Lemma 2.1 (Muñoz-Fernández and Seoane-Sepúlveda, [9]). *If $m, n \in \mathbb{N}$ are such that m is odd, n is even and $m > n$, then*

$$\operatorname{ext}(\mathsf{B}_{m,n}) = \{\pm(2x^n - 1), \pm(x^m + x^n - 1), \pm(x^m - x^n + 1), \pm 1\}.$$

Now we are ready to give the expression of the function $\mathcal{M}^{\ell}_{m,n}(x)$ for m odd, n even, $m > n$.

Theorem 2.2. *Let $m, n \in \mathbb{N}$ be such that m is odd, n is even and $m > n$. Then*

$$\mathcal{M}^{\ell}_{m,n}(x) = \begin{cases} (m+1)|x|^m - (n+1)x^n + 1 & \text{if } |x| \le t_1, \\ 2(n+1)x^n - 1 & \text{if } t_1 \le |x| \le \sqrt[m-n]{\frac{n+1}{m+1}}, \\ (m+1)|x|^m + (n+1)x^n - 1 & \text{if } \sqrt[m-n]{\frac{n+1}{m+1}} \le |x| \le 1, \end{cases} \tag{2}$$

where $t_1 \in \mathbb{R}$ is the unique solution of

$$(m+1)x^m - 3(n+1)x^n + 2 = 0 \tag{3}$$

in the interval $\left(\frac{1}{\sqrt[n]{2(n+1)}}, \frac{1}{\sqrt[n]{n+1}} \right)$.

Proof. If $x \in [-1, 1]$, by definition we have that

$$\mathcal{M}^{\ell}_{m,n}(x) = \sup_{p \in \mathcal{P}^{\ell}_{m,n}(\mathbb{R})} |p'(x)|.$$

Since p is a trinomial and has a linear majorant then it is necessarily of the form $p(x) = xq(x)$ for some trinomial $q \in \mathsf{B}_{m,n}$. By means of the Krein-Milman theorem, it suffices to work just with the extreme polynomials of $\mathsf{B}_{m,n}$, which are given in the previous Lemma. Notice that the contribution of ± 1 to $\mathcal{M}^{\ell}_{m,n}(x)$ is irrelevant. Thus, it suffices to consider the polynomials

$$q_1(x) = \pm(2x^n - 1), \quad q_2(x) = \pm(x^m + x^n - 1) \quad \text{and} \quad q_3(x) = \pm(x^m - x^n + 1).$$

Without loss of generality, and for simplicity, we can assume that $x \geq 0$. Therefore

$$\begin{aligned}
\mathcal{M}^{\ell}_{m,n}(x) &= \max\{|p'(x)| : p(x) = xq(x); q \in \mathsf{B}_{m,n}\} \\
&= \max\{|q(x) + xq'(x)| : q \in \mathrm{ext}(\mathsf{B}_{m,n})\} \\
&= \max\{|q_1(x) + xq_1'(x)|, |q_2(x) + xq_2'(x)|, |q_3(x) + xq_3'(x)|\} \\
&= \max\{|2(n+1)x^n - 1|, |(m+1)x^m + (n+1)x^n - 1|, \\
&\qquad\quad |-(m+1)x^m + (n+1)x^n - 1|\} \\
&= \max\{|2(n+1)x^n - 1|, (m+1)x^m + |(n+1)x^n - 1|\}.
\end{aligned}$$

Now, let us divide the interval $[0, 1]$ as follows:

$$[0,1] = \left[0, \frac{1}{\sqrt[n]{2(n+1)}}\right) \cup \left[\frac{1}{\sqrt[n]{2(n+1)}}, \frac{1}{\sqrt[n]{n+1}}\right] \cup \left(\frac{1}{\sqrt[n]{n+1}}, 1\right] := A \cup B \cup C,$$

and let us work on each of the elements of the partition separately. Clearly, when $x \in A$, it follows immediately that

$$\begin{aligned}
(m+1)x^m + |(n+1)x^n - 1| &= (m+1)x^m - (n+1)x^n + 1 \\
&> 1 - 2(n+1)x^n \\
&= |2(n+1)x^n - 1|,
\end{aligned}$$

thus, $\mathcal{M}^{\ell}_{m,n}(x) = (m+1)x^m - (n+1)x^n + 1$ over A. Next, when working on B, let us notice that, if we define

$$f(x) = (m+1)x^m - 3(n+1)x^n + 2,$$

then a straight forward calculation gives that

$$f\left(\frac{1}{\sqrt[n]{2(n+1)}}\right) > 0.$$

On the other hand, and by performing simple calculations, one can also arrive at the fact that

$$f\left(\frac{1}{\sqrt[n]{n+1}}\right) < 0$$

if and only if $(m+1)^n < (n+1)^m$, which is always true since the sequence $\left(\frac{\log(k+1)}{k}\right)_k$ is strictly decreasing. Bolzano's Theorem then gives that $f(x) = 0$ (equation (3)) has, at least, one solution in B. On the other hand, if (3) had two solutions then, necessarily, it would have at least three and this is impossible, since the functions $3(n+1)x^n - 2$ and $(m+1)x^m$ are both convex on $[0, \infty)$. Then, let $t_1 \in B$ be the unique solution of equation (3) which, in general, cannot be obtained explicitly. Thus, now we can write

$$B := B_1 \cup B_2 := \left[\frac{1}{\sqrt[n]{2(n+1)}}, t_1\right) \cup \left[t_1, \frac{1}{\sqrt[n]{n+1}}\right].$$

Also, since $f(B_1) \subset \mathbb{R}^+$ and $f(B_2) \subset \mathbb{R}^-$ it follows that $\mathcal{M}^\ell_{m,n}(x) = (m+1)x^m - (n+1)x^n + 1$ over B_1 and $\mathcal{M}^\ell_{m,n}(x) = 2(n+1)x^n - 1$ over B_2. Finally, if $x \in C$, some simple calculations show that $|2(n+1)x^n - 1|$ and $(m+1)x^m + |(n+1)x^n - 1|$ only intercept at $t_0 = \sqrt[m-n]{\dfrac{n+1}{m+1}} \geq \dfrac{1}{\sqrt[n]{n+1}}$, from which C can be expressed as

$$C := C_1 \cup C_2 := \left(\frac{1}{\sqrt[n]{n+1}}, t_0\right] \cup (t_0, 1].$$

To finish the proof it suffices with noticing that on C_1 we have $\mathcal{M}^\ell_{m,n}(x) = 2(n+1)x^n - 1$ and $\mathcal{M}^\ell_{m,n}(x) = (m+1)x^m + (n+1)x^n - 1$ over C_2. □

The graph of $\mathcal{M}^\ell_{3,2}(x)$ can be seen in Figure 2.1. From the previous result we can obtain straightforwardly the following.

Corollary 2.3. *If $m, n \in \mathbb{N}$ with m odd and n even, then*

$$M^\ell_{m,n} = m + n + 1.$$

The same idea can be applied to other choices of m and n, but in those cases we haven't been able to obtain an explicit solution. However we can still derive $\mathcal{M}^\ell_{m,n}(x)$ for specific choices of m and n in relatively simple terms. For instance, the following characterization of the extreme points of $\mathsf{B}_{2,1}$ will help us to obtain $\mathcal{M}^\ell_{2,1}(x)$.

Lemma 2.4 (Aron and Klimek, [1]). *The extreme points of the unit ball of $\mathcal{P}_{2,1}(\mathbb{R})$ are given by*

$$\left\{tx^2 \pm 2\left(\sqrt{2t} - t\right)x + 1 + t - 2\sqrt{2t} : t \in [1/2, 2]\right\}.$$

Theorem 2.5. *If $x \in [-1, 1]$ then*

$$\mathcal{M}^\ell_{2,1}(x) = \begin{cases} \left|\frac{3x^2-1}{2}\right| + 2|x| & \text{if } |x| \in \left[\frac{\sqrt{13}-2}{9}, \frac{\sqrt{13}+2}{9}\right], \\ |6x^2 - 1| & \text{if } |x| \in \left[0, \frac{\sqrt{13}-2}{9}\right] \cup \left[\frac{\sqrt{13}+2}{9}, 1\right]. \end{cases}$$

Proof. Let us fix $x \in [0, 1]$. We have

$$\begin{aligned} \mathcal{M}^\ell_{2,1}(x) &= \sup\{|q(x) + xq'(x)| : q \in \mathrm{ext}(\mathsf{B}_{2,1})\} \\ &= \sup\left\{\left|3tx^2 \pm 4\left(\sqrt{2t} - t\right)x + 1 + t - 2\sqrt{2t}\right| : t \in [1/2, 2]\right\}. \end{aligned}$$

If we define $f_+(t) = 3tx^2 + 4(\sqrt{2t} - t)x + 1 + t - 2\sqrt{2t}$ and $f_-(t) = 3tx^2 - 4(\sqrt{2t} - t)x + 1 + t - 2\sqrt{2t}$, we obtain that $f'_+(t) = 0$ if and only if $t = t_+ = 2\left(\frac{2x+1}{3x^2+4x+1}\right)^2$, and $f'_-(t) = 0$ if and only if $t = t_- = 2\left(\frac{-2x+1}{3x^2-4x+1}\right)^2$. It is easy to check that $t_+ \in [1/2, 2]$ if and only if $x \in \left[0, \frac{\sqrt{3}}{3}\right]$, and $t_- \in [1/2, 2]$ if and only if $x \in \left[\frac{3-\sqrt{3}}{3}, \frac{4-\sqrt{7}}{3}\right] \cup \left[\frac{\sqrt{3}}{3}, \frac{2}{3}\right]$.

On the other hand, we have $f_+(1/2) = \frac{3x^2-1}{2} + 2x$, $f_-(1/2) = \frac{3x^2-1}{2} - 2x$ and $f_+(2) = f_-(2) = 6x^2 - 1$. Since $|6x^2 - 1| \geq \left|\frac{3x^2-1}{2}\right| + 2|x|$ if and only if $x \in \left[0, \frac{\sqrt{13}-2}{9}\right] \cup \left[\frac{\sqrt{13}+2}{9}, 1\right]$,

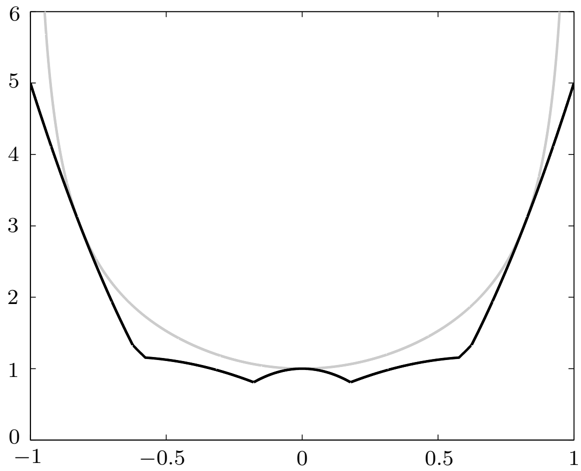

Figure 2.2: $\mathcal{M}_3^\ell(x)$ (in black) vs. $\mathcal{R}_3^\ell(x)$ (in gray). Here $M_3^\ell = 5$.

we obtain that

$$\mathcal{M}_{2,1}^\ell(x) = \begin{cases} \max\{|6x^2-1|, |f_+(t_+)|\} & \text{if } |x| \in \left[0, \frac{\sqrt{13}-2}{9}\right], \\ \max\left\{\left|\frac{3x^2-1}{2}\right| + 2|x|, |f_+(t_+)|\right\} & \text{if } |x| \in \left[\frac{\sqrt{13}-2}{9}, \frac{3-\sqrt{3}}{3}\right], \\ \max\left\{\left|\frac{3x^2-1}{2}\right| + 2|x|, |f_+(t_+)|, |f_-(t_-)|\right\} & \text{if } |x| \in \left[\frac{3-\sqrt{3}}{3}, \frac{4-\sqrt{7}}{3}\right], \\ \max\left\{\left|\frac{3x^2-1}{2}\right| + 2|x|, |f_+(t_+)|\right\} & \text{if } |x| \in \left[\frac{4-\sqrt{7}}{3}, \frac{\sqrt{3}}{3}\right], \\ \max\left\{\left|\frac{3x^2-1}{2}\right| + 2|x|, |f_-(t_-)|\right\} & \text{if } |x| \in \left[\frac{\sqrt{3}}{3}, \frac{\sqrt{13}+2}{9}\right], \\ \max\{|6x^2-1|, |f_-(t_-)|\} & \text{if } |x| \in \left[\frac{\sqrt{13}+2}{9}, \frac{2}{3}\right], \\ |6x^2-1| & \text{if } |x| \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

Performing some technical, but simple, calculations we achieve the desired result. □

The graphs of $\mathcal{M}_3^\ell(x)$ and $\mathcal{R}_3^\ell(x)$ are compared in Figure 2.2.

3. Bernstein estimates for polynomials with a circular majorant

Next we investigate the norm $\|\cdot\|_{\infty,c}$ on $\mathbb{R}^2$ defined by

$$\|(a,b)\|_{\infty,c} := \sup\left\{\left|\sqrt{1-x^2}(ax+b)\right| : x \in [-1,1]\right\}.$$

If we identify (a,b) with the polynomial $p_{a,b}(x) := (1-x^2)(ax+b)$, then the fact that (a,b) is in the unit ball of the space $(\mathbb{R}^2, \|\cdot\|_{\infty,c})$ is equivalent to

$$\left|\sqrt{1-x^2}(ax+b)\right| \leq 1, \quad \text{for every } x \in [-1,1],$$

i.e. $|(1-x^2)(ax+b)| \leq \sqrt{1-x^2}$ for every $x \in [-1,1]$. In other words, the polynomial $p_{a,b}$ has a *circular majorant.* On the other hand, all polynomials of degree not greater than 3 with a circular majorant are of the form $p_{a,b}$ for some $a, b \in \mathbb{R}$. In order to see this just notice that if p has a circular majorant then p has roots at ± 1. Thus we can identify the unit ball of the space $(\mathbb{R}^2, \|\cdot\|_{\infty,c})$ with the set of polynomials of degree not greater than 3 with a circular majorant. Polynomials in one real variable with a circular majorant have been studied several times in the past, see for instance Rahman's work [10]. Rahman's results were generalized for polynomials on a Hilbert space in [7]. At the end of this section we will also characterize the set of polynomials in $\mathcal{P}_3^c(H)$ where H is any real Hilbert space.

First of all we will find a formula for $\|\cdot\|_{\infty,c}$.

Theorem 3.1. *For every $(a,b)\in\mathbb{R}^2$ we have*

$$\|(a,b)\|_{\infty,c} = \begin{cases} \frac{(3|b|+\sqrt{8a^2+b^2})\sqrt{4a^2-b^2+|b|\sqrt{8a^2+b^2}}}{8\sqrt{2}|a|} & \text{if } a\neq 0, \\ |b| & \text{if } a=0. \end{cases} \tag{4}$$

Proof. If $a=0$, the proof is trivial. Otherwise, since f vanishes at ± 1, the sup norm of $f(x)=\sqrt{1-x^2}(ax+b)$, namely $\|f\|_\infty$, is attained at a critical point of f in $(-1,1)$. It can be easily seen that the only critical points of f are

$$r_1 = \frac{-b-\sqrt{8a^2+b^2}}{4a} \quad \text{and} \quad r_2 = \frac{-b+\sqrt{8a^2+b^2}}{4a}.$$

It is also simple to check that

$$r_1 \in (-1,1) \Leftrightarrow b < |a| \quad \text{and} \quad r_2 \in (-1,1) \Leftrightarrow b > -|a|.$$

Therefore

$$\|(a,b)\|_{\infty,c} = \|f\|_\infty = \begin{cases} |f(r_1)| & \text{if } b \le -|a|, \\ \max\{|f(r_1)|,|f(r_2)|\} & \text{if } |b|<|a|, \\ |f(r_2)| & \text{if } b \ge |a|. \end{cases}$$

Now, since $|f(r_1)| \le |f(r_2)|$ if and only if $b \ge 0$, it follows that

$$\begin{aligned} \|(a,b)\|_{\infty,c} &= \begin{cases} |f(r_1)| & \text{if } b\le 0, \\ |f(r_2)| & \text{if } b \ge 0, \end{cases} \\ &= \begin{cases} \frac{|-3b+\sqrt{8a^2+b^2}|\sqrt{\frac{4a^2-b^2-b\sqrt{8a^2+b^2}}{a^2}}}{8\sqrt{2}} & \text{if } b\le 0, \\ \frac{|3b+\sqrt{8a^2+b^2}|\sqrt{\frac{4a^2-b^2+b\sqrt{8a^2+b^2}}{a^2}}}{8\sqrt{2}} & \text{if } b \ge 0, \end{cases} \end{aligned}$$

which concludes the proof. □

By means of (4) we obtain the following characterization of the unit ball of the space $(\mathbb{R}^2, \|\cdot\|_{\infty,c})$.

Corollary 3.2. *If $a,b\in\mathbb{R}$, then $\|(a,b)\|_{\infty,c}\le 1$ if and only if*

$$\left(\sqrt{8a^2+b^2}+3|b|\right)^3 \le 32\left(\sqrt{8a^2+b^2}+|b|\right).$$

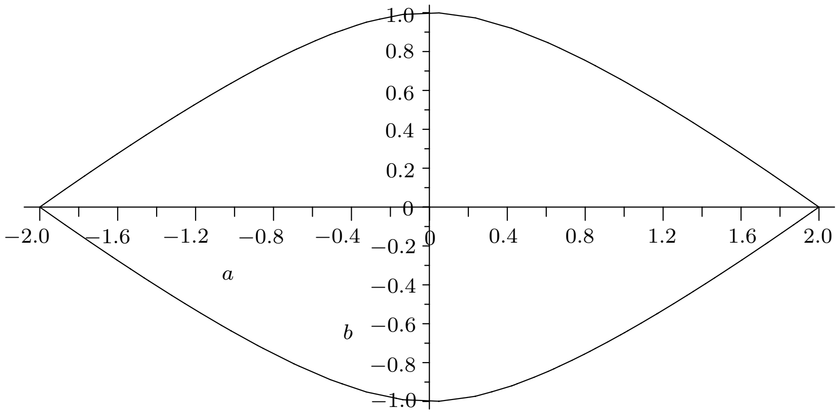

Figure 3.1: Unit sphere of the space $(\mathbb{R}^2, \|\cdot\|_{\infty,c})$.

Remark 3.3. Equation

$$\left(\sqrt{8a^2+b^2}+3|b|\right)^3 = 32\left(\sqrt{8a^2+b^2}+|b|\right) \tag{5}$$

defines implicitly an even function $\Gamma : [-2,2] \to [0,1]$, $b = \Gamma(a)$, such that $\Gamma(\pm 2) = 0$, $\Gamma(0) = 1$ and the unit sphere of the space $(\mathbb{R}^2, \|\cdot\|_{\infty,c})$ is given by

$$\{(a, \pm\Gamma(a)) : a \in [-2,2]\}.$$

A sketch of the unit sphere of $(\mathbb{R}^2, \|\cdot\|_{\infty,c})$ can be found in Figure 3.1. It is straightforward that the polynomials $p_{a,b}$ with a and b satisfying (5) are the extreme points of the convex set $\mathcal{P}_3^c(\mathbb{R})$.

If H is a real Hilbert space, using (4) we can obtain an explicit formula for the norm of any polynomial $P \in \mathcal{P}_3(H)$ with a circular majorant, i.e., satisfying $|P(x)| \leq \sqrt{1-\|x\|^2}$ for all $x \in \mathsf{B}_H$. Indeed, a formula for the norm of such a P can be deduced from the following result.

Proposition 3.4. *If E is a real Banach space, L is a continuous linear form on E, $b \in \mathbb{R}$ and we define $f(x) := \sqrt{1-\|x\|^2}(L(x)+b)$, then*

$$\sup\{|f(x)| : \ x \in \mathsf{B}_E\} = \|(\|L\|, |b|)\|_{\infty,c},$$

where $\|L\|$ denotes the sup norm of L over B_E.

Proof. Let $\|f\|_\infty := \sup\{|f(x)| : \ x \in \mathsf{B}_E\}$. A simple application of the triangle inequality shows that

$$\|f\|_\infty \leq \sup\left\{\sqrt{1-r^2}(\|L\|r+|b|) : r \in [-1,1]\right\} = \|(\|L\|, |b|)\|_{\infty,c}.$$

Now, let $\varepsilon > 0$ be arbitrarily small with $\varepsilon < \|L\|$ (the case $L \equiv 0$ is trivial) and choose $x_0 \in \mathsf{S}_E$ such that $\|L\| - \varepsilon < L(x_0)$. Then for $0 \leq r \leq 1$ we have

$$\begin{aligned}\|f\|_\infty \geq |f(\operatorname{sign}(b)x_0 r)| &= \sqrt{1-r^2}|L(\operatorname{sign}(b)x_0 r)+b| \\ &= \sqrt{1-r^2}(rL(x_0)+|b|) > \sqrt{1-r^2}[(\|L\|-\varepsilon)r+|b|].\end{aligned}$$

Finally, letting $\varepsilon \to 0$ and taking the sup over all r's in $[0,1]$, we obtain

$$\begin{aligned}\|f\|_\infty &\geq \sup\left\{\sqrt{1-r^2}(\|L\|r+|b|) : r \in [0,1]\right\}\\ &= \sup\left\{\sqrt{1-r^2}(\|L\|r+|b|) : r \in [-1,1]\right\}.\end{aligned}$$

□

It was proved in [7, Lemma 9] that, if H is a real Hilbert space, then for any polynomial $P \in \mathcal{P}_n(H)$ with a circular majorant there exists $Q \in \mathcal{P}_{n-2}(H)$ such that

$$P(x) = (1-\|x\|^2)Q(x),$$

for every $x \in H$. Hence, if $P \in \mathcal{P}_3(H)$, then there exist a linear form L on H and $b \in \mathbb{R}$ such that

$$P(x) = (1-\|x\|^2)(L(x)+b).$$

Using this together with Corollary 3.2, we derive the following consequence.

Corollary 3.5. *If H is a real Hilbert space and $P \in \mathcal{P}_3(H)$ is of the form $P(x) = (1-\|x\|^2)(L(x)+b)$ with L being a bounded linear form on H and $b \in \mathbb{R}$, then P has circular majorant if and only if*

$$\left(\sqrt{8\|L\|^2+b^2}+3|b|\right)^3 = 32\left(\sqrt{8\|L\|^2+b^2}+|b|\right).$$

Now we are going to obtain a sharp Bernstein type inequality for the polynomials in $\mathcal{P}_3^c(\mathbb{R})$.

Theorem 3.6. *For each $x \in [-1,1]$ we have*

$$\mathcal{M}_3^c(x) = \begin{cases} 2|1-3x^2| & \text{if } |x| \in \left[0, \frac{\sqrt{4-\sqrt{7}}}{3}\right] \cup \left[\frac{\sqrt{4+\sqrt{7}}}{3}, 1\right], \\ \dfrac{4x^2}{\sqrt{-9x^4+10x^2-1}} & \text{if } |x| \in \left[\frac{\sqrt{4-\sqrt{7}}}{3}, \frac{\sqrt{4+\sqrt{7}}}{3}\right]. \end{cases}$$

Proof. Let us fix $x \in [0,1]$ and consider $p_{a,b} \in \mathcal{P}_3^c(\mathbb{R})$ with $p_{a,b}(x) = (1-x^2)(ax+b)$. Then according to Remark 3.3 and using the symmetry of $\mathcal{P}_3^c(\mathbb{R})$,

$$\begin{aligned}&\mathcal{M}_3^c(x)\\ &= \sup\{|p'_{a,b}(x)| : p_{a,b} \in \mathcal{P}_3^c(\mathbb{R})\}\\ &= \sup\left\{|(1-3x^2)a-2xb| : \left(\sqrt{8a^2+b^2}+3|b|\right)^3 = 32\left(\sqrt{8a^2+b^2}+|b|\right)\right\}\\ &= \sup\left\{|1-3x^2|a+2|x|b : \left(\sqrt{8a^2+b^2}+3|b|\right)^3 = 32\left(\sqrt{8a^2+b^2}+|b|\right), a, b \geq 0\right\}.\end{aligned}$$

Call $F(a,b) = |1-3x^2|a+2|x|b$. In order to maximize F over the a, b's such that

$$\left(\sqrt{8a^2+b^2}+3|b|\right)^3 = 32\left(\sqrt{8a^2+b^2}+|b|\right) \quad \text{and } a, b \geq 0,$$

we perform the following change of variables

$$a = \frac{r}{2\sqrt{2}} \cos t,$$
$$b = r \sin t,$$

where $t \in [0, \pi/2]$ and $r \geq 0$. These new variables applied to

$$\left(\sqrt{8a^2 + b^2} + 3|b|\right)^3 = 32\left(\sqrt{8a^2 + b^2} + |b|\right) \quad \text{and } a, b \geq 0,$$

yield

$$r = \sqrt{\frac{32(1 + \sin t)}{(1 + 3\sin t)^3}}.$$

Hence, maximizing $F(a, b)$ over the considered domain reduces to maximize

$$g(t) = \sqrt{\frac{32(1 + \sin t)}{(1 + 3\sin t)^3}} \left(\frac{|1 - 3x^2| \cos t}{2\sqrt{2}} + 2|x| \sin t\right),$$

for $t \in [0, \pi/2]$. It can be proved that $g'(t) = 0$ is equivalent to

$$2|x| \cos t - \sqrt{2}|1 - 3x^2|(1 + \sin t) = 0,$$

from which we have that g has a unique critical point at a $t_0 \in [0, \pi/2]$ such that

$$\sin t_0 = \frac{-9x^4 + 8x^2 - 1}{9x^4 - 4x^2 + 1}.$$

It can be shown that $\sin t_0 \geq 0$ on the interval $\left[\frac{\sqrt{4-\sqrt{7}}}{3}, \frac{\sqrt{4+\sqrt{7}}}{3}\right]$ and

$$g(t_0) = \frac{4x^2}{\sqrt{-9x^4 + 10x^2 - 1}}.$$

Therefore

$$\begin{aligned}
\mathcal{M}_3^c(x) &= \max\{g(t) : t \in [0, \pi/2]\} \\
&= \max\{|g(0)|, |g(\pi/2)|, |g(t_0)|\} \\
&= \max\left\{2|1 - 3x^2|, 2|x|, \frac{4x^2}{\sqrt{-9x^4 + 10x^2 - 1}}\right\}.
\end{aligned}$$

From here the interested reader can easily verify, after some technical but simple calculations, that the result follows. □

Remark 3.7. Notice that the estimate $\mathcal{R}_3^c(x) = \sqrt{\frac{x^2}{1-x^2} + 4}$ only agrees with the optimal $\mathcal{M}_3^c(x)$ at the points $x = 0, \pm\sqrt{\frac{5}{6}}$.

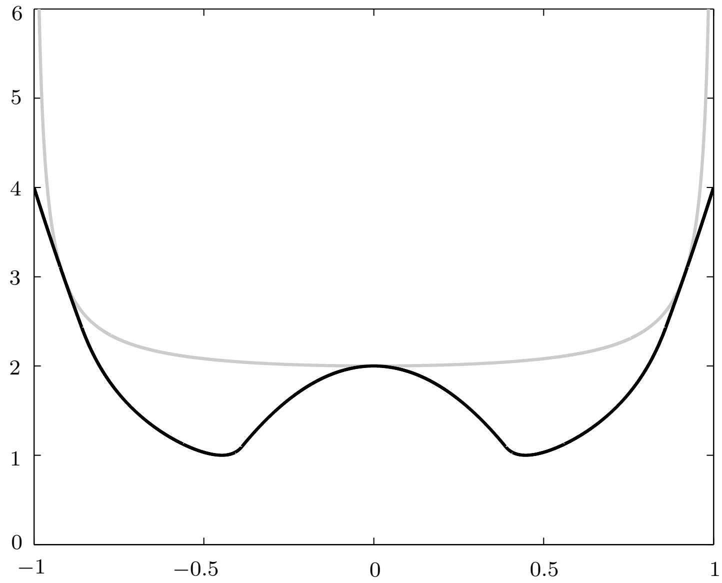

Figure 3.2: $\mathcal{M}_3^c$ (in black) vs. $\mathcal{R}_3^c$ (in gray).

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