Journal of Convex Analysis
Volume 17 (2010), No. 3&4, 1057–1075

# A Formula for the Set of Optimal Solutions of a Relaxed Minimization Problem. Applications to Subdifferential Calculus

**Marco A. López**
*Dept. of Statistics and Operations Research,*
*University of Alicante, Apt. de Correos 99, 03080 Alicante, Spain*
*marco.antonio@ua.es*

**Michel Volle**
*Université d'Avignon et des Pays de Vaucluse,*
*74 rue Louis Pasteur, 84029 Avignon, France*
*Michel.Volle@univ-avignon.fr*

*Dedicated to Hedy Attouch on the occasion of his 60th birthday.*

Received: October 4, 2008

In the infinite dimensional setting, we provide a general formula for the optimal set of a relaxed minimization problem in terms of the approximate minima of the data function. Various applications to the $\varepsilon$-subdifferential calculus are also given.

*Keywords:* Relaxed minimization problem, approximate minima, $\varepsilon$-subdifferential calculus, Legendre-Fenchel conjugate

*2000 Mathematics Subject Classification:* Primary 90C48, 90C46; Secondary 49N15, 90C25

## 1. Introduction

Let $X$ be a (real) Hausdorff locally convex space whose topological dual is denoted by $Y$. For any $(x, y) \in X \times Y$ we set $y(x) := \langle x, y \rangle$ .

Let $h \in \overline{\mathbb{R}}^X$ be an extended real valued function on $X$, with $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

The closed convex *relaxed problem* associated with

$$(\mathcal{P}) : \quad \text{minimize } \; h(x) \;\text{ s.t. } x \in X$$

is classically defined as

$$(\mathcal{P}') : \quad \text{minimize } \; h^{**}(x) \;\text{ s.t. } x \in X,$$

where $h^{**}$ denotes the Legendre-Fenchel bi-conjugate of $h$. The optimal values of both problems coincide:

$$\inf_X h = \inf_X h^{**} =: m \in \overline{\mathbb{R}}.$$

ISSN 0944-6532 / \$ 2.50 

Our purpose in this paper is to obtain the optimal set of $(\mathcal{P}')$, i.e. argmin $h^{**}$, in terms of the approximate solutions of $(\mathcal{P})$, i.e. $\varepsilon-$argmin $h$. For convenience we set $\varepsilon-$argmin $h=\emptyset$ for all $\varepsilon \geq 0$ whenever $m \notin \mathbb{R}$, i.e. if $h=+\infty$, in which case $m=+\infty$, or if $h$ is not bounded from below, which means $m=-\infty$.

The main formulas derived in this paper (Theorem 3.3 and Theorem 4.8) allow us to state new results for subdifferential calculus. More precisely, Theorem 3.5 and Corollary 4.9 provide the subdifferential of the Legendre-Fenchel conjugate of a (non necessarily convex) extended-real-valued function in terms of the data function, Theorem 4.1 and Corollary 4.11 yield a new formula for the subdifferential of the upper-envelope of an arbitrary family of (non necessarily convex) functions, while Theorem 4.5 deals with the subdifferential of the sum of two functions, one of which is nonconvex.

## 2. Notation and basic tools

In this paper $X$ is a (real) Hausdorff locally convex spaces (lcs, for short). Its topological dual space is denoted by $Y$, and for any $(x,y) \in X \times Y$ we set $y(x) := \langle x, y \rangle$ . The zero vector in both spaces is represented by $\theta$.

Let us recall some basic results of convex analysis which can be found, e.g., in [10] (see also [5] and [7]). Given two nonempty sets $A$ and $B$ in $X$ (or in $Y$), we define the algebraic (or Minkowski) sum by

$$A+B := \{a+b \mid a \in A,\ b \in B\}, \qquad A+\emptyset := \emptyset + A := \emptyset. \tag{1}$$

Moreover, if $\emptyset \neq \Lambda \subset \mathbb{R}$ we set

$$\Lambda A := \{\lambda a \mid \lambda \in \Lambda,\ a \in A\}, \qquad \Lambda\emptyset := \emptyset.$$

Furthermore, $\Lambda x := \Lambda\{x\}$, $\lambda A := \{\lambda\}A$ and $x + A := \{x\} + A$.

By $\operatorname{co} A$, $\operatorname{cone} A$, $\operatorname{lin} A$, and $\operatorname{aff} A$, we denote the *convex hull* of the set $A$, the *conic hull* of $A$ (i.e. $\operatorname{cone} A = [0,+\infty[A)$, the *linear subspace spanned by* $A$, and the *affine hull* of $A$, respectively.

In all the paper we will assume that $X$ and $Y$ are equipped with locally convex topologies compatible with the bilinear coupling $\langle \cdot,\cdot \rangle : X \times Y \to \mathbb{R}$. Such a scheme covers, of course, the Euclidean and Hilbert spaces settings, and also the case when $X$ is a normed space and its topological dual $Y$ is equipped with the weak*-topology. When $X$ is a reflexive normed space, the dual norm topology on $Y$ also works. More generally, one can deal with the weak topologies, $\sigma(X,Y)$ and $\sigma(Y,X)$, or the Mackey topologies, $\tau(X,Y)$ and $\tau(Y,X)$.

If $A \subset X$ (or $A \subset Y$), $\operatorname{int} A$ will denote the *interior* of $A$, whereas $\operatorname{cl} A$ and $\overline{A}$ are indistinctly used for denoting the *closure* of $A$. In this way, we set $\overline{\operatorname{co}}A := \operatorname{cl}(\operatorname{co} A)$ and $\overline{\operatorname{cone}}A := \operatorname{cl}(\operatorname{cone} A)$. Finally, $\operatorname{ri} A$ denotes the *relative interior* of $A$ which we define here as the interior of $A$ in the topology relative to the closed affine hull of $A$.

If $A$ is convex, we have

$$\lambda \operatorname{ri} A + (1-\lambda) \operatorname{cl} A \subset \operatorname{ri} A, \quad \text{for every } \lambda \in\ ]0,1]. \tag{2}$$

Associated with $A \neq \emptyset$ we consider the closed convex cone

$$A^- := \{x^* \in X^* \mid \langle x^*, x\rangle \leq 0 \text{ for all } x \in A\},$$

i.e. the *negative dual cone.* Further, we have

$$A^{--} = \overline{\text{cone}}(\text{co}\, A). \tag{3}$$

If $A \subset X$ and $x \in X$, we define the *normal cone* to $A$ at $x$ as

$$\text{N}_A(x) := \begin{cases} (A - x)^- & \text{if } x \in A, \\ \emptyset & \text{if } x \in X \setminus A. \end{cases}$$

Given a function $h \in \overline{\mathbb{R}}^X$, its *(effective) domain*, *epigraph*, and *level set* are defined by

$$\begin{aligned} \text{dom}\, h &:= \{x \in X \mid h(x) < +\infty\}, \\ \text{epi}\, h &:= \{(x, \alpha) \in X \times \mathbb{R} \mid h(x) \leq \alpha\}, \\ [h \leq \alpha] &:= \{x \in X \mid h(x) \leq \alpha\}. \end{aligned}$$

The function $h$ is *proper* if $\text{dom}\, h \neq \emptyset$ and $h(x) > -\infty$ for every $x \in X$.

We say that $h$ is *convex* if $\text{epi}\, h$ is convex.

The *lower semicontinuous envelope* of $h$ is the function $\overline{h} \in \overline{\mathbb{R}}^X$ defined by

$$\overline{h}(x) := \inf\{t \mid (x, t) \in \text{cl}(\text{epi}\, h)\}.$$

Clearly we have $\text{epi}\, \overline{h} = \text{cl}\,(\text{epi}\, h)$, which implies that $\overline{h}$ is the greatest lower semicontinuous (lsc, in brief) function dominated by $h$; i.e. $\overline{h} \leq h$.

If $h$ is convex, then $\overline{h}$ is also convex, and then $\overline{h}$ does not take the value $-\infty$ if and only if $h$ admits a continuous affine minorant.

Given $h \in \overline{\mathbb{R}}^X$, the *lsc convex hull* of $h$ is the convex lsc function $\overline{\text{co}}h \in \overline{\mathbb{R}}^X$ such that

$$\text{epi}(\overline{\text{co}}h) = \overline{\text{co}}(\text{epi}\, h).$$

Obviously $\overline{\text{co}}h \leq \overline{h} \leq h$.

We shall denote by $\Lambda(X)$ the set of all the proper convex functions on $X$ and by $\Gamma(X)$ the subset of $\Lambda(X)$ consisting of the lsc functions; the sets $\Lambda(Y)$ and $\Gamma(Y)$ are defined similarly.

Given $h \in \overline{\mathbb{R}}^X$, the *Legendre-Fenchel conjugate* of $h$ is the function $h^* \in \overline{\mathbb{R}}^Y$ given by

$$\begin{aligned} h^*(y) &= \sup\{\langle x, y\rangle - h(x) : \ x \in X\} \\ &= \sup\{\langle x, y\rangle - h(x) : \ x \in \text{dom}\, h\}. \end{aligned}$$

The function $h^*$ is convex and lsc. If $h$ takes the value $-\infty$, then $h^* = +\infty$, and if $\text{dom}\, h = \emptyset$ we have $h^* = -\infty$. Moreover, $h^* = \overline{h}^* = (\overline{\text{co}}h)^*$, and $h^* \in \Gamma(Y)$ if and only if $\text{dom}\, h \neq \emptyset$ and $h$ admits a continuous affine minorant.

The *bi-conjugate* of $h$ is the function $h^{**} \in \overline{\mathbb{R}}^X$ given by $h^{**} = (h^*)^*$, i.e.

$$h^{**}(x) := \sup\{\langle x, y\rangle - h^*(y) : \ y \in \operatorname{dom} h^*\}.$$

We have

$$\{h \in \overline{\mathbb{R}}^X : \ h = h^{**}\} = \Gamma(X) \cup \{+\infty\}^X \cup \{-\infty\}^X.$$

Moreover, $h^{**} \leq \overline{\mathrm{co}}h$, and the equality holds if $h$ admits a continuous affine minorant.

The *indicator* and the *support* functions of $A \subset X$ are respectively defined as

$$\mathrm{i}_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{if } x \in X \setminus A, \end{cases}$$

and

$$\mathrm{i}_A^*(y) := \sup\{\langle x, y\rangle : \ x \in A\}, \quad \text{for } y \in Y,$$

with the convention $\sup \emptyset = -\infty$. The function $\mathrm{i}_A^*$ is obviously the Legendre-Fenchel conjugate of $\mathrm{i}_A$. It is sublinear, lsc, and satisfies $\mathrm{i}_A^* = \mathrm{i}_{\overline{\mathrm{co}}A}^*$.

If $h \in \overline{\mathbb{R}}^X$ and $\varepsilon \geq 0$, the $\varepsilon-$*subdifferential* (see, for instance, [6]) of $h$ at a point $x \in X$ such that $h(x) \in \mathbb{R}$ is the $w(Y, X)$–closed convex set

$$\partial_\varepsilon h(x) := \{y \in Y : \ h(z) \geq h(x) + \langle z - x, y\rangle - \varepsilon \text{ for all } z \in X\}.$$

In particular, for $\varepsilon = 0$ we get $\partial h(x) := \partial_0 h(x)$, the *subdifferential* of $h$ at $x$.

If $h(x) \notin \mathbb{R}$ we set $\partial_\varepsilon h(x) := \emptyset$. Observe that $h$ is proper if $\partial_\varepsilon h(x) \neq \emptyset$ for a certain $x \in X$ and certain $\varepsilon \geq 0$.

Given $x \in X$ we recall the following properties:

$$\partial h(x) = \cap_{\varepsilon > 0} \partial_\varepsilon h(x)$$

and, for any $\varepsilon \geq 0$,

$$0 \in \partial_\varepsilon h(x) \Leftrightarrow x \in \varepsilon - \operatorname{argmin} h.$$

Moreover, if $h$ is proper and $x \in \operatorname{dom} h$, then

$$\partial_\varepsilon h(x) = \{y \in Y : \ h(x) + h^*(y) \leq \langle x, y\rangle + \varepsilon\}, \quad \text{for all } \varepsilon \geq 0. \tag{4}$$

If $h \in \overline{\mathbb{R}}^X$ is convex, then we have $\partial_\varepsilon h(x) \neq \emptyset$ for all $\varepsilon > 0$ if and only if $h$ is lsc and finite at $x$.

If $x \in A$,

$$\partial \mathrm{i}_A(x) = (\operatorname{cone}(A - x))^- = \mathrm{N}_A(x).$$

## 3. A formula for the optimal set of $(\mathcal{P}')$

Before establishing the main result in this paper (Theorem 3.3), we need two technical lemmas (see also [4, Lemma 1] for the first one, and [3, Lemma 1] for the second). Let us recall that the relative interior $\operatorname{ri} B$ of a set $B$ is the topological relative interior with respect to the closed affine hull of $B$.

**Lemma 3.1.** *For any convex set $B \subset Y$ and any convex function $g \in \overline{\mathbb{R}}^Y$ such that*

$$(\operatorname{ri} B) \cap \operatorname{dom} g \neq \emptyset,$$

*one has*

$$\inf_{\operatorname{ri} B} g = \inf_{\operatorname{cl} B} g = \inf_{B} g. \tag{5}$$

**Proof.** For any $y \in \operatorname{cl} B$ one has to prove that

$$\inf_{\operatorname{ri} B} g \leq g(y). \tag{6}$$

This is obvious if $\inf_{\operatorname{ri} B} g = -\infty$ or $g(y) = +\infty$. Otherwise let $v \in (\operatorname{ri} B) \cap \operatorname{dom} g$. From (2), one has

$$v_\lambda := \lambda v + (1-\lambda) y \in \operatorname{ri} B, \quad \text{for all } \lambda \in ]0,1],$$

and so, for all $\lambda \in ]0,1]$,

$$-\infty < \inf_{\operatorname{ri} B} g \leq g(v_\lambda) \leq \lambda g(v) + (1-\lambda) g(y) < +\infty.$$

Accordingly, $g(v) \in \mathbb{R}$ and $g(y) \in \mathbb{R}$. Hence, we get (6) by taking $\lambda \downarrow 0$. $\square$

**Lemma 3.2.** *For any $h \in \overline{\mathbb{R}}^X$, $\varepsilon \geq 0$, one has*

$$\operatorname{cone}(\operatorname{dom} h^*) \subset \operatorname{dom}\left(\mathrm{i}^*_{\varepsilon-\operatorname{argmin} h}\right). \tag{7}$$

**Proof.** If $m = \inf_X h \notin \mathbb{R}$, we have $\varepsilon - \operatorname{argmin} h = \emptyset$ and, so, $\mathrm{i}^*_{\varepsilon-\operatorname{argmin} h} = -\infty$, which entails $\operatorname{dom}\left(\mathrm{i}^*_{\varepsilon-\operatorname{argmin} h}\right) = Y$, and (7) is trivially satisfied. Assume, then, that $m \in \mathbb{R}$. For any $y \in \operatorname{dom} h^*$, if we take $u \in \varepsilon - \operatorname{argmin} h$ we have

$$\langle u, y \rangle \leq h(u) + h^*(y) \leq m + \varepsilon + h^*(y),$$

and therefore

$$\mathrm{i}^*_{\varepsilon-\operatorname{argmin} h}(y) \leq m + \varepsilon + h^*(y) < +\infty.$$

Hence we have

$$\operatorname{dom} h^* \subset \operatorname{dom}\left(\mathrm{i}^*_{\varepsilon-\operatorname{argmin} h}\right),$$

and we are done since the set on the right-hand side is a cone. $\square$

We now state the main result of the paper. Although it concerns any extended-real-valued function which admits a continuous affine minorant, it is mainly addressed to the functions that are proper and bounded from below.

Let us recall that the current topologies on $X$ and $Y$ are locally convex topologies compatible with the primal-dual bilinear coupling $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$.

**Theorem 3.3.** *For any function $h \in \overline{\mathbb{R}}^X$ such that $\operatorname{dom} h^* \neq \emptyset$ one has*

$$\operatorname{argmin}\ h^{**} = \bigcap_{\substack{\varepsilon > 0 \\ y \in \operatorname{dom} h^*}} \overline{\operatorname{co}}\left(\varepsilon - \operatorname{argmin} h + \{y\}^-\right). \tag{8}$$

*If* $\operatorname{cone}(\operatorname{dom} h^*)$ *is closed or* $\operatorname{ri}(\operatorname{cone}(\operatorname{dom} h^*)) \neq \emptyset$, *then*

$$\operatorname{argmin}\ h^{**} = \bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left((\varepsilon - \operatorname{argmin} h) + (\operatorname{dom} h^*)^-\right). \tag{9}$$

*In particular, if* $\operatorname{cone}(\operatorname{dom} h^*) = Y$, *we have*

$$\operatorname{argmin}\ h^{**} = \bigcap_{\varepsilon>0} \overline{\operatorname{co}}\,(\varepsilon - \operatorname{argmin} h). \tag{10}$$

**Proof.** First we analyze the case $h = +\infty$, i.e. $h^* = -\infty$ and $h^{**} = +\infty$. It is obvious (by the conventions taken above), that both (8) and (9) hold trivially (all the sets are empty).

If, alternatively, $h^* \neq -\infty$, the assumption $\operatorname{dom} h^* \neq \emptyset$ gives rise to the existence of $y_0 \in Y$ such that $h^*(y_0) \in \mathbb{R}$, and $\langle \cdot, y_0\rangle - h^*(y_0)$ is a continuous affine minorant of $h$ (which, consequently, will be proper). Now two possibilities arise. The first one correspond to the case that $m = \inf_X h = \inf_X h^{**} = -\infty$, in which case (8) and (9) hold again due to the convention on $\varepsilon - \operatorname{argmin} h$.

So, we have only to study the unique remaining case in which $h$ is bounded from below ($m \in \mathbb{R}$). We shall decompose the proof in different steps.

*Step 1.* We proof first the inclusion "$\supseteq$" in (8).

We are assuming that $m = \inf_X h = \inf_X h^{**} \in \mathbb{R}$. Then, for any $\varepsilon > 0$, $y \in \operatorname{dom} h^*$, $u \in \varepsilon - \operatorname{argmin} h$, and $z \in \{y\}^-$ one has

$$\langle u + z, y\rangle - h^*(y) \le \langle z, y\rangle + h(u) \le h(u) \le m + \varepsilon;$$

in other words

$$\varepsilon - \operatorname{argmin} h + \{y\}^- \subseteq [\langle \cdot, y\rangle - h^*(y) \le m + \varepsilon].$$

Since the set on the right-hand side is closed and convex one has

$$\overline{\operatorname{co}}\left(\varepsilon - \operatorname{argmin} h + \{y\}^-\right) \subseteq [\langle \cdot, y\rangle - h^*(y) \le m + \varepsilon].$$

By taking the intersection over $y \in \operatorname{dom} h^*$ we get

$$\bigcap_{y\in\operatorname{dom} h^*} \overline{\operatorname{co}}\left(\varepsilon - \operatorname{argmin} h + \{y\}^-\right) \subseteq \left[\sup_{y\in\operatorname{dom} h^*} \{\langle \cdot, y\rangle - h^*(y)\} \le m + \varepsilon\right].$$

The set on right-hand side is nothing else but

$$[h^{**} \le m + \varepsilon] = \varepsilon - \operatorname{argmin} h^{**}.$$

We finish this part of the proof by taking the intersection over $\varepsilon > 0$.

*Step 2.* Next we proof the converse inclusion "$\subseteq$" in (8).

Let $\varepsilon_0 > 0$ and $y_0 \in \operatorname{dom} h^*$ such that

$$a \notin \overline{\operatorname{co}}\left(\varepsilon_0 - \operatorname{argmin} h + \{y_0\}^-\right).$$

We have to check that

$$h^{**}(a) > m.$$

By the Hahn-Banach separation theorem, there is $v \in Y \diagdown \{\theta\}$ such that

$$\begin{aligned}\langle a, v\rangle &> \mathrm{i}^*_{(\varepsilon_0-\operatorname{argmin} h+\{y_0\}^-)}(v) \\ &= \mathrm{i}^*_{\varepsilon_0-\operatorname{argmin} h}(v) + \mathrm{i}^*_{\{y_0\}^-}(v),\end{aligned}$$

and this implies that $v \in \mathbb{R}_+ y_0$ and

$$\langle a, y_0\rangle > \mathrm{i}^*_{\varepsilon_0-\operatorname{argmin} h}(y_0). \tag{11}$$

Now, for any $\rho > 0$ one gets

$$h^{**}(a) \geq \langle a, \rho y_0\rangle - h^*(\rho y_0), \tag{12}$$

with

$$-h^*(\rho y_0) = \alpha(\rho) \wedge \beta(\rho) := \min\{\alpha(\rho), \beta(\rho)\},$$

where

$$\alpha(\rho) := \inf_{u\in\varepsilon_0-\operatorname{argmin} h} \{h(u) - \langle u, \rho y_0\rangle\},$$

and

$$\beta(\rho) := \inf_{u\in X\diagdown\varepsilon_0-\operatorname{argmin} h} \{h(u) - \langle u, \rho y_0\rangle\}.$$

On one hand, from (11):

$$\begin{aligned}\alpha(\rho) &\geq \left(\inf_{u\in\varepsilon_0-\operatorname{argmin} h} h(u)\right) - \rho \mathrm{i}^*_{\varepsilon_0-\operatorname{argmin} h}(y_0) \\ &> \left(\inf_{u\in\varepsilon_0-\operatorname{argmin} h} h(u)\right) - \rho\langle a, y_0\rangle \\ &\geq m - \rho\langle a, y_0\rangle.\end{aligned} \tag{13}$$

On the other hand, if we take $\rho \in ]0, 1[$,

$$\begin{aligned}\beta(\rho) &= \inf_{u\in X\diagdown\varepsilon_0-\operatorname{argmin} h} \{\rho(h(u) - \langle u, y_0\rangle) + (1-\rho)h(u)\} \\ &\geq -\rho h^*(y_0) + (1-\rho)(m+\varepsilon_0).\end{aligned} \tag{14}$$

Therefore, for any $\rho \in ]0, 1[$, (12) and (14) yield

$$\begin{aligned}h^{**}(a) &\geq \{\rho\langle a, y_0\rangle + \alpha(\rho)\} \wedge \{\rho\langle a, y_0\rangle + \beta(\rho)\} \\ &\geq \{\rho\langle a, y_0\rangle + \alpha(\rho)\} \wedge \{\rho\langle a, y_0\rangle - \rho h^*(y_0) + (1-\rho)(m+\varepsilon_0)\}.\end{aligned}$$

By (13) one has

$$\rho\langle a, y_0\rangle + \alpha(\rho) > m.$$

Therefore, in order to achieve the aimed conclusion, one has to choose $\rho \in ]0, 1[$ such that

$$\rho\langle a, y_0\rangle - \rho h^*(y_0) + (1-\rho)(m+\varepsilon_0) > m,$$

or equivalently,

$$\rho\left\{\langle a, y_0\rangle - h^*(y_0) - m - \varepsilon_0\right\} > -\varepsilon_0,$$

which is always possible by taking $\rho$ small enough.

*Step 3.* Now we prove (9) under the assumption that $\operatorname{cone}(\operatorname{dom} h^*)$ is $\sigma(Y, X)-$closed or $\operatorname{ri}(\operatorname{cone}(\operatorname{dom} h^*)) \neq \emptyset$.

From (8) it is obvious that the inclusion "$\supseteq$" in (9) holds. Now let us consider $\varepsilon_0 > 0$ such that

$$a \notin \overline{\operatorname{co}}\left((\varepsilon_0 - \operatorname{argmin} h) + (\operatorname{dom} h^*)^-\right).$$

By the Hahn-Banach separation theorem, there must exist $w \in Y \diagdown \{\theta\}$ such that

$$\begin{aligned}\langle a, w\rangle &> \mathrm{i}^*_{(\varepsilon_0 - \operatorname{argmin} h)+(\operatorname{dom} h^*)^-}(w)\\ &= \mathrm{i}^*_{\varepsilon_0 - \operatorname{argmin} h}(w) + \mathrm{i}^*_{(\operatorname{dom} h^*)^-}(w),\end{aligned}$$

and this entails that

$$w \in (\operatorname{dom} h^*)^{--} = \overline{\operatorname{cone}}(\operatorname{dom} h^*), \tag{15}$$

and that

$$\langle a, w\rangle > \mathrm{i}^*_{\varepsilon_0 - \operatorname{argmin} h}(w). \tag{16}$$

Now we make the following discussion:

*Case 1.* If $\operatorname{cone}(\operatorname{dom} h^*)$ is closed, we have that $w_0 = w \in \operatorname{cone}(\operatorname{dom} h^*)$ satisfies

$$\langle a, w_0\rangle > \mathrm{i}^*_{\varepsilon_0 - \operatorname{argmin} h}(w_0). \tag{17}$$

*Case 2.* If $\operatorname{ri}(\operatorname{cone}(\operatorname{dom} h^*)) \neq \emptyset$, the reasoning is the following:

From Lemma 3.2 one has

$$\operatorname{cone}(\operatorname{dom} h^*) \subseteq \operatorname{dom}\left(\mathrm{i}^*_{\varepsilon_0 - \operatorname{argmin} h}\right).$$

Now we can apply Lemma 3.1 to the function $g = \mathrm{i}^*_{\varepsilon_0 - \operatorname{argmin} h}(\cdot) - \langle a, \cdot\rangle$ and the convex set $B = \operatorname{cone}(\operatorname{dom} h^*)$ to conclude, from (15) and (16), the existence of $w_0 \in \operatorname{cone}(\operatorname{dom} h^*)$ such that (17) is also satisfied.

Hence, in both cases, there will exist $\lambda > 0$ and $y_0 \in \operatorname{dom} h^*$ such that $w_0 = \lambda y_0$, and so

$$\langle a, y_0\rangle > \mathrm{i}^*_{\varepsilon_0 - \operatorname{argmin} h}(y_0).$$

As a consequence of that

$$\mathrm{i}^*_{(\varepsilon_0 - \operatorname{argmin} h + \{y_0\}^-)}(y_0) = \mathrm{i}^*_{\varepsilon_0 - \operatorname{argmin} h}(y_0) < \langle a, y_0\rangle,$$

and again, by Hahn-Banach theorem we reach the aimed conclusion

$$a \notin \overline{\operatorname{co}}\left((\varepsilon_0 - \operatorname{argmin} h) + \{y_0\}^-\right)$$

so that, by (8),

$$a \notin \operatorname{argmin}\ h^{**}.$$

Hence, we have proved the inclusion

$$\operatorname{argmin} \ h^{**} \subseteq \overline{\operatorname{co}}\left((\varepsilon - \operatorname{argmin} h) + (\operatorname{dom} h^*)^-\right), \quad \text{for all } \varepsilon > 0.$$

*Step 4.* If $\operatorname{cone}(\operatorname{dom} h^*) = Y$, we have $(\operatorname{dom} h^*)^- = \{\theta\}$, and (10) is a trivial consequence of (9). □

**Remark 3.4.** In the finite dimensional setting it has been observed in [1, Comment 4-8, 3, p. 1672] that under the additional assumptions

a) $h$ is lsc,
b) $h$ is asymptotically epi-pointed,

the following formula holds:

$$\operatorname{argmin} h^{**} = \operatorname{co}(\operatorname{argmin} h) + \operatorname{co}(\operatorname{argmin} h_\infty),$$

where $h_\infty$ is the asymptotic function of $h$, namely

$$h_\infty(x) = \liminf_{\substack{u \to x \\ t \downarrow 0}} t h(u/t),$$

and asymptotically epi-pointed means

$$\operatorname{co}(\operatorname{epi} h_\infty) \cap (-\operatorname{co}(\operatorname{epi} h_\infty)) = \{(0,0)\}.$$

Now we proceed with a relevant application of Theorem 3.3 to obtain the subdifferential of the Legendre-Fenchel conjugate of $h \in \overline{\mathbb{R}}^X$, not necessarily convex, in terms of the inverse multivalued mappings

$$M_\varepsilon h = (\partial_\varepsilon h)^{-1}, \quad \varepsilon \geq 0,$$

of the $\varepsilon-$subdifferentials of $h$.

If $-\infty \in h(X)$, one has $\partial_\varepsilon h(x) = \emptyset$ for all $x \in X$ and, so, $\operatorname{dom}(M_\varepsilon h) = \emptyset$. If $h = +\infty$, we also have $\operatorname{dom}(M_\varepsilon h) = \emptyset$. If $h$ is proper, for any $y \in Y$,

$$\begin{aligned} M_\varepsilon h(y) &= \{x \in X : \ y \in \partial_\varepsilon h(x)\} \\ &= \{x \in X : \ h(x) - \langle x, y\rangle \leq -h^*(y) + \varepsilon\} \\ &= \varepsilon - \operatorname{argmin}\left(h(\cdot) - \langle \cdot, y\rangle\right). \end{aligned}$$

It is obvious that, according to our convention, $M_\varepsilon h(y) = \emptyset$ whenever $h^*(y) \notin \mathbb{R}$ and $\varepsilon \geq 0$. Moreover, $M_\varepsilon h(y) \neq \emptyset$ whenever $h^*(y) \in \mathbb{R}$ and $\varepsilon > 0$. Finally, we also have (see, for instance, [10, Theorem 2.4.2 (ii)])

$$M_\varepsilon h(y) \subset \partial_\varepsilon h^*(y), \quad \text{for all } y \in Y \text{ and every } \varepsilon \geq 0,$$

and the inclusion above becomes an equality whenever $h = h^{**}$.

We are now in a position to state the following result:

**Theorem 3.5.** *For any function* $h \in \overline{\mathbb{R}}^X$ *such that* $\operatorname{dom} h^* \neq \emptyset$ *one has*

$$\partial h^*(y) = \bigcap_{\substack{\varepsilon>0 \\ v \in \operatorname{dom} h^*}} \overline{\operatorname{co}}\left(M_\varepsilon h(y) + \{v-y\}^-\right), \quad \textit{for all } y \in Y. \tag{18}$$

*If* $\operatorname{cone}(\operatorname{dom} h^* - y)$ *is closed or* $\operatorname{ri}(\operatorname{cone}(\operatorname{dom} h^* - y)) \neq \emptyset$, *then*

$$\partial h^*(y) = \bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left(M_\varepsilon h(y) + \mathrm{N}_{\operatorname{dom} h^*}(y)\right). \tag{19}$$

**Proof.** We first observe that

$$\begin{aligned} \partial h^*(y) &= \operatorname{argmin}\{h^{**}(\cdot) - \langle ., y\rangle\} \\ &= \operatorname{argmin}\{h(\cdot) - \langle ., y\rangle\}^{**}. \end{aligned}$$

Then we apply Theorem 3.3 to the function $f(\cdot) := h(\cdot) - \langle ., y\rangle$ taking into account that $f^*(\cdot) = h^*(\cdot + y)$. $\square$

All the results above can be equivalently stated for functions defined on $Y$, instead of $X$. In particular we have a dual version of Theorem 3.5:

**Theorem 3.6.** *For any function* $g \in \overline{\mathbb{R}}^Y$ *such that* $\operatorname{dom} g^* \neq \emptyset$ *one has*

$$\partial g^*(x) = \bigcap_{\substack{\varepsilon>0 \\ z \in \operatorname{dom} g^*}} \overline{\operatorname{co}}\left(M_\varepsilon g(x) + \{z-x\}^-\right), \quad \textit{for all } x \in X. \tag{20}$$

*If* $\operatorname{cone}(\operatorname{dom} g^* - x)$ *is closed or* $\operatorname{ri}(\operatorname{cone}(\operatorname{dom} g^* - x)) \neq \emptyset$, *then*

$$\partial g^*(x) = \bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left(M_\varepsilon g(x) + \mathrm{N}_{\operatorname{dom} g^*}(x)\right). \tag{21}$$

## 4. Some applications to subdifferential calculus

Next we give an important application of Theorem 3.6.

**Theorem 4.1.** *For a given nonempty family of functions* $\{f_t,\ t \in T\} \subset \overline{\mathbb{R}}^X$, *let us consider the* supremum function

$$f := \sup_{t \in T} f_t,$$

*and assume that* $\operatorname{dom} f \neq \emptyset$. *If the following condition is satisfied*

$$f^{**} \equiv \left(\sup_{t \in T} f_t\right)^{**} = \sup_{t \in T} f_t^{**}, \tag{22}$$

*the subdifferential of the supremum function at any point* $x \in X$ *is given by the formula*

$$\partial f(x) = \bigcap_{\substack{\varepsilon>0 \\ z \in \operatorname{dom} f}} \overline{\operatorname{co}}\left(\bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \{z-x\}^-\right), \tag{23}$$

*where*

$$T_\varepsilon(x) := \begin{cases} \{t \in T : \ f_t(x) \geq f(x) - \varepsilon\}, & \textit{if } f(x) \in \mathbb{R}, \\ \emptyset, & \textit{if } f(x) \notin \mathbb{R}. \end{cases} \tag{24}$$

*If, moreover,* cone(dom $f - x$)) *is convex and either* cone(dom $f - x$)) *is closed or* ri(cone(dom $f - x$)) $\neq \emptyset$, *then*

$$\partial f(x) = \bigcap_{\varepsilon>0} \overline{\text{co}} \left( \bigcup_{t\in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \mathrm{N}_{\mathrm{dom}\, f}(x) \right). \tag{25}$$

**Proof.** First, we shall prove the inclusion "$\supseteq$" in (23). To this aim, let $\varepsilon > 0$, $z \in$ dom $f$,

$$v \in \bigcup_{t\in T_\varepsilon(x)} \partial_\varepsilon f_t(x) \quad \text{and} \quad w \in \{z - x\}^-.$$

Accordingly, there will exist $t_0 \in T_\varepsilon(x)$ such that

$$v \in \partial_\varepsilon f_{t_0}(x),$$

and therefore, for every $z \in X$,

$$f(z) \geq f_{t_0}(z) \geq f_{t_0}(x) + \langle z - x, v\rangle - \varepsilon \geq f(x) + \langle z - x, v\rangle - 2\varepsilon,$$

i.e.

$$v \in \partial_{2\varepsilon} f(x). \tag{26}$$

As a consequence of (26),

$$\begin{aligned} \langle z, v + w\rangle - f(z) &= \langle z - x, v + w\rangle + \langle x, v + w\rangle - f(z) \\ &\leq \langle z - x, v\rangle - f(z) + \langle x, v + w\rangle \\ &\leq \langle x, v + w\rangle - f(x) + 2\varepsilon. \end{aligned}$$

In other words, for any $\varepsilon > 0$ and $z \in$ dom $f$, one has

$$\bigcup_{t\in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \{z - x\}^- \subseteq [\langle z - x, \cdot\rangle - f(z) \leq -f(x) + 2\varepsilon],$$

and since the set on the right-hand side is closed and convex, we have

$$\overline{\text{co}} \left( \bigcup_{t\in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \{z - x\}^- \right) \subseteq [\langle z - x, \cdot\rangle - f(z) \leq -f(x) + 2\varepsilon].$$

By taking the intersection over $z \in$ dom $f$ we get

$$\bigcap_{z\in\mathrm{dom}\, f} \overline{\text{co}} \left( \bigcup_{t\in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \{z - x\}^- \right) \subseteq [f^*(\cdot) - \langle x, \cdot\rangle \leq -f(x) + 2\varepsilon],$$

and so

$$\bigcap_{\substack{z\in\operatorname{dom} f\\ \varepsilon>0}} \overline{\operatorname{co}}\left(\bigcup_{t\in T_\varepsilon(x)} \partial_\varepsilon f_t(x)+\{z-x\}^-\right) \subseteq \left[f^*(\cdot)-\langle x,\cdot\rangle \leq -f(x)\right].$$

Since the set on the right hand side is just $\partial f(x)$ we are done.

The proof of the converse inclusion "$\subseteq$" in (23) is based on the application of Theorem 3.6 to the function

$$g := \inf_{t\in T} f_t^*.$$

From (22) one has $g^* = f^{**}$. Moreover,

$$\operatorname{dom} g^* = \operatorname{dom}\left(\sup_{t\in T} f_t^{**}\right) \supseteq \operatorname{dom}\left(\sup_{t\in T} f_t\right) = \operatorname{dom} f \neq \emptyset.$$

If $\partial f(x)\neq\emptyset$, one has, by (22) and Theorem 2.4.1(ii) in [10],

$$\partial f(x) = \partial f^{**}(x) = \partial g^*(x).$$

In this proof we shall use the indices set

$$T_\varepsilon'(x) := \{t\in T:\ f_t^{**}(x)\geq f(x)-\varepsilon\}.$$

Obviously $T_\varepsilon'(x)\subseteq T_\varepsilon(x)$, and we observe that, for any $\varepsilon>0$,

$$M_\varepsilon g(x) \subseteq \bigcup_{t\in T_{2\varepsilon}'(x)} M_{2\varepsilon} f_t^*(x). \tag{27}$$

In fact

$$y\in M_\varepsilon g(x) \Leftrightarrow x\in\partial_\varepsilon g(y) \Leftrightarrow g(u)\geq g(y)+\langle x,u-y\rangle-\varepsilon, \quad \text{for all } u\in Y.$$

If $t_0\in T$ satisfies

$$g(y) = \inf_{t\in T} f_t^*(y) \geq f_{t_0}^*(y)-\varepsilon, \tag{28}$$

we can write, for all $u\in Y$,

$$f_{t_0}^*(u)\geq g(u)\geq g(y)+\langle x,u-y\rangle-\varepsilon \geq f_{t_0}^*(y)+\langle x,u-y\rangle-2\varepsilon,$$

i.e.

$$x\in\partial_{2\varepsilon} f_{t_0}^*(y),$$

and

$$y\in M_{2\varepsilon} f_{t_0}^*(x).$$

Moreover, also by Theorem 2.4.1(ii) in [10],

$$x\in\partial_\varepsilon g(y) \Leftrightarrow g(y)+g^*(x) = g(y)+f(x)\leq\langle x,y\rangle+\varepsilon,$$

and, from the last inequality and from (28)

$$\begin{aligned} f_{t_0}^{**}(x) &\geq \langle x, y\rangle - f_{t_0}^{*}(y) \\ &\geq \langle x, y\rangle - g(y) - \varepsilon \\ &\geq f(x) - 2\varepsilon. \end{aligned}$$

Thus

$$t_0 \in T'_{2\varepsilon}(x),$$

and we have proved the inclusion (27).

Now, for any $t \in T'_{2\varepsilon}(x)$ one has

$$\begin{aligned} & y \in \partial_{2\varepsilon} f_t^{**}(x) \Leftrightarrow f_t^{**}(z) \geq f_t^{**}(x) + \langle z - x, y\rangle - 2\varepsilon \quad \text{for all } z \in X \\ \Rightarrow\ & f_t^{**}(z) \geq f(x) + \langle z - x, y\rangle - 4\varepsilon \quad \text{for all } z \in X \\ \Rightarrow\ & f_t^{**}(z) \geq f_t(x) + \langle z - x, y\rangle - 4\varepsilon \quad \text{for all } z \in X, \end{aligned}$$

and we have shown that

$$M_{2\varepsilon} f_t^{*}(x) = \partial_{2\varepsilon} f_t^{**}(x) \subseteq \partial_{4\varepsilon} f_t(x).$$

This inclusion leads us to

$$\bigcup_{t \in T'_{2\varepsilon}(x)} M_{2\varepsilon} f_t^{*}(x) \subseteq \bigcup_{t \in T'_{2\varepsilon}(x)} \partial_{4\varepsilon} f_t(x) \subseteq \bigcup_{t \in T_{4\varepsilon}(x)} \partial_{4\varepsilon} f_t(x).$$

From Theorem 3.6 we get (taking into account that $\operatorname{dom} g^* \supseteq \operatorname{dom} f$)

$$\partial f(x) = \partial g^*(x) \subseteq \bigcap_{\substack{\varepsilon > 0 \\ z \in \operatorname{dom} f}} \overline{\operatorname{co}} \left( \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \{z - x\}^- \right).$$

Finally, (25) follows from (23) by a similar reasoning to the one used in the proof of (9). $\square$

**Remark 4.2.** Observe that in (23) one may have $\partial f(x) = \emptyset$ (this is for instance the case if $f(x) \notin \mathbb{R}$).

**Remark 4.3.** Under the assumption (22) and if $\operatorname{cone}(\operatorname{dom} f - x) = X$ we obtain from (23) or (25)

$$\partial f(x) = \bigcap_{\varepsilon > 0} \overline{\operatorname{co}} \left( \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) \right).$$

This formula was established by Volle in [9] when the functions $f_t,\ t \in T$, are convex and $f$ is finite and continuous at $x$. (See also [4, Corollary 10] and [8] for a related formula.)

Our purpose now is to illustrate condition (22) by a relevant example. Recall that for any *convex* function $k \in \overline{\mathbb{R}}^X$ such that $\operatorname{dom} k^* \neq \emptyset$, one has

$$\overline{k} = k^{**} \in \Gamma(X) \cup \{+\infty\}^X. \tag{29}$$

In variational analysis there are important functions that satisfy (29) despite that they are *nonconvex* (see, for instance, [2, Propositions (IX)1.2 (X)2.15.]).

We begin with an elementary lemma.

**Lemma 4.4.** *Let $h, k \in \overline{\mathbb{R}}^X$ be proper functions. Assume that $h$ is continuous on $X$ and that $\overline{k}$ is also proper. Then*

$$\overline{h+k} = h + \overline{k}.$$

**Proof.** Since $h$ and $\overline{k}$ are both proper and lsc, $h+\overline{k}$ is lsc too (it is well-defined). As $h+\overline{k} \leq h+k$, we thus have

$$h+\overline{k} \leq \overline{h+k}.$$

In order to prove the opposite inequality, let $x \in X$ and $r \in \mathbb{R}$ such that $r > h(x)+\overline{k}(x)$. One has to prove that $r \geq (\overline{h+k})(x)$. To this aim, let $U$ be an arbitrary neighborhood of $x$; one has to check that $r \geq \inf_U (h+k)$.

Now it is possible to choose $s, t \in \mathbb{R}$ such that $r = s+t,\ s > h(x)$, and $t > \overline{k}(x)$. Since $h$ is continuous at $x$, there must exist a neighborhood $V \subset U$ of $x$ such that $s > \sup_V h$, while there must exist $v \in V$ such that $t > k(v)$. Finally

$$\inf_U (h+k) \leq \inf_V (h+k) \leq h(v) + k(v) \leq s+t = r.$$

□

**Theorem 4.5.** *Let $h \in \Gamma(X)$ and $k \in \overline{\mathbb{R}}^X$ satisfying $\overline{k} = k^{**} \in \Gamma(X)$. Let us assume that $(\operatorname{dom} h) \cap (\operatorname{dom} k) \neq \emptyset$ and that $h$ is continuous on $X$. Then, for any $x \in X$,*

$$\partial(h+k)(x) = \bigcap_{\substack{\varepsilon>0,\\ u \in (\operatorname{dom} h)\cap(\operatorname{dom} k)}} \operatorname{cl}\left(\partial_\varepsilon h(x) + \partial_\varepsilon k(x) + \{u-x\}^-\right). \tag{30}$$

*If, moreover,* $\operatorname{cone}((\operatorname{dom} h)\cap(\operatorname{dom} k)-x))$ *is convex and either* $\operatorname{cone}((\operatorname{dom} h)\cap(\operatorname{dom} k)-x))$ *is closed or* $\operatorname{ri}(\operatorname{cone}((\operatorname{dom} h)\cap(\operatorname{dom} k)-x)) \neq \emptyset$, *then*

$$\partial(h+k)(x) = \bigcap_{\varepsilon>0} \operatorname{cl}\left(\partial_\varepsilon h(x) + \partial_\varepsilon k(x) + \mathrm{N}_{(\operatorname{dom} h)\cap(\operatorname{dom} k)}(x)\right). \tag{31}$$

**Proof.** Observe that $h+\overline{k} = h + k^{**} \in \Gamma(X)$ and

$$\begin{aligned} h+\overline{k} &= \sup_{y \in \operatorname{dom} h^*} \left(\langle \cdot, y\rangle - h^*(y) + k^{**}\right) \qquad (32)\\ &= \sup_{y \in \operatorname{dom} h^*} \left(\langle \cdot, y\rangle - h^*(y) + k\right)^{**}. \end{aligned}$$

On the other hand, since $h$ is continuous, one has by Lemma 4.4

$$h + \overline{k} = \overline{h+k} \geq (h+k)^{**} \geq (h+\overline{k})^{**} = h + \overline{k}. \tag{33}$$

Setting

$$f := h + k,$$

we get

$$f = \sup_{y \in \operatorname{dom} h^*} \left\{ \langle \cdot, y \rangle - h^*(y) + k \right\},$$

and from (32) and (33),

$$f^{**} = \sup_{y \in \operatorname{dom} h^*} \left\{ \langle \cdot, y \rangle - h^*(y) + k \right\}^{**}.$$

Therefore, we can apply Theorem 4.1 to the function $f$ above. To this aim, we identify $T := \operatorname{dom} h^*$ and, for any $y \in T$,

$$f_y := \langle \cdot, y \rangle - h^*(y) + k.$$

Then we have

$$\operatorname{dom} f = (\operatorname{dom} h) \cap (\operatorname{dom} k) \neq \emptyset,$$

together with

$$\begin{aligned} T_\varepsilon(x) &= \{ y \in T \mid f_y(x) \geq f(x) - \varepsilon \} \\ &= \{ y \in \operatorname{dom} h^* \mid \langle x, y \rangle - h^*(y) + k(x) \geq f(x) - \varepsilon \} \\ &= \{ y \in \operatorname{dom} h^* \mid h(x) + h^*(y) \leq \langle x, y \rangle + \varepsilon \} \\ &= \partial_\varepsilon h(x), \end{aligned}$$

and

$$\partial_\varepsilon f_y(x) = y + \partial_\varepsilon k(x).$$

Replacing in (23) and in (25) we get (30) and (31), respectively. □

**Remark 4.6.** Let us observe that in Theorem 4.5 the function $k \in \overline{\mathbb{R}}^X$ is continuous but not necessarily finite-valued.

We shall proceed by giving some consequences of Theorem 4.1.

**Corollary 4.7.** *Let $C \subset Y$ and $h \in \overline{\mathbb{R}}^X$ be a convex set and a function, respectively, such that*

$$\operatorname{ri}\left(\operatorname{cone}(C \cap \operatorname{dom} h^*)\right) \neq \emptyset. \tag{34}$$

*Then we have*

$$\operatorname{argmin}\ h^{**} \subseteq \bigcap_{\varepsilon > 0} \overline{\operatorname{co}} \left( \varepsilon - \operatorname{argmin} h + (C \cap \operatorname{dom} h^*)^- \right).$$

**Proof.** We shall approach the non-trivial case in which $h$ is bounded from below and $h \neq +\infty$, i.e. $\inf_X h = m \in \mathbb{R}$.

Let $\varepsilon_0 > 0$ and

$$a \notin \overline{\mathrm{co}}\left(\varepsilon_0 - \operatorname{argmin} h + (C \cap \operatorname{dom} h^*)^-\right). \tag{35}$$

By Theorem 3.3, it will be enough to find $y_0 \in \operatorname{dom} h^*$ such that

$$a \notin \overline{\mathrm{co}}\left(\varepsilon_0 - \operatorname{argmin} h + \{y_0\}^-\right).$$

By the Hahn-Banach separation theorem applied to (35), we know that there exists $w \in Y \diagdown \{\theta\}$ such that

$$\begin{aligned}\langle a, w\rangle &> \mathrm{i}^*_{(\varepsilon_0 - \operatorname{argmin} h + (C\cap \operatorname{dom} h^*)^-)}(w)\\ &= \mathrm{i}^*_{\varepsilon_0 - \operatorname{argmin} h}(w) + \mathrm{i}^*_{(C\cap \operatorname{dom} h^*)^-}(w),\end{aligned}$$

and this entails that

$$w \in (C \cap \operatorname{dom} h^*)^{--} = \overline{\mathrm{cone}}(C \cap \operatorname{dom} h^*),$$

and additionally we have

$$\langle a, w\rangle > \mathrm{i}^*_{\varepsilon_0 - \operatorname{argmin} h}(w). \tag{36}$$

From Lemma 3.2 the following inclusions hold

$$\mathrm{cone}(C \cap \operatorname{dom} h^*) \subseteq \mathrm{cone}(\operatorname{dom} h^*) \subseteq \operatorname{dom} \mathrm{i}^*_{\varepsilon_0 - \operatorname{argmin} h}.$$

Now we apply Lemma 3.1 to the function $g = \mathrm{i}^*_{\varepsilon_0 - \operatorname{argmin} h}(\cdot) - \langle a, \cdot\rangle$ and the convex set $B = \mathrm{cone}(C \cap \operatorname{dom} h^*)$ to conclude the existence of $w_0 \in \mathrm{cone}(C \cap \operatorname{dom} h^*)$ such that

$$\langle a, w_0\rangle > \mathrm{i}^*_{\varepsilon_0 - \operatorname{argmin} h}(w_0).$$

Hence, there will exist $\lambda > 0$ and $y_0 \in C \cap \operatorname{dom} h^* \subseteq \operatorname{dom} h^*$ such that $w_0 = \lambda y_0$, and so

$$\langle a, y_0\rangle > \mathrm{i}^*_{\varepsilon_0 - \operatorname{argmin} h}(y_0).$$

As a consequence of that

$$\mathrm{i}^*_{(\varepsilon_0 - \operatorname{argmin} h + \{y_0\}^-)}(y_0) = \mathrm{i}^*_{\varepsilon_0 - \operatorname{argmin} h}(y_0) < \langle a, y_0\rangle,$$

and again, by Hahn-Banach theorem we reach the aimed conclusion

$$a \notin \overline{\mathrm{co}}\left(\varepsilon_0 - \operatorname{argmin} h + \{y_0\}^-\right).$$

□

**Theorem 4.8.** *For any function $h \in \overline{\mathbb{R}}^X$ and any family $\{C_i,\ i \in I\}$ of convex subsets of $Y$ satisfying*

$$\operatorname{dom} h^* \subseteq \bigcup_{i\in I} C_i, \tag{37}$$

*and*

$$\mathrm{ri}\,(\mathrm{cone}(C_i \cap \operatorname{dom} h^*)) \neq \emptyset, \quad \text{for all } i \in I, \tag{38}$$

*one has*

$$\operatorname{argmin}\ h^{**} = \bigcap_{\substack{\varepsilon>0\\ i\in I}} \overline{\mathrm{co}}\left(\varepsilon - \operatorname{argmin} h\ +\ (C_i \cap \operatorname{dom} h^*)^-\right).$$

**Proof.** From Corollary 4.7 and (38) the inclusion "$\subseteq$" follows.

Moreover, for any $y \in \operatorname{dom} h^*$ there will exist, in virtue of (37), $i_0 \in I$ such that $y \in C_{i_0}$. Hence,

$$\{y\}^- \supseteq (C_{i_0} \cap \operatorname{dom} h^*)^-,$$

and the reverse inclusion "$\supseteq$" follows from Theorem 3.3. $\square$

**Corollary 4.9.** *For any function $h \in \overline{\mathbb{R}}^X$ with $\operatorname{dom} h^* \neq \emptyset$, if*

$$\mathcal{F}_y := \{L \subset Y \mid L \text{ is a finite-dimensional linear subspace such that } y \in L\},$$

*one has*

$$\partial h^*(y) = \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}_y}} \overline{\operatorname{co}}\,(M_\varepsilon h(y) + \mathrm{N}_{L\cap \operatorname{dom} h^*}(y)) \quad \text{for all } y \in Y, \tag{39}$$

*where*

$$M_\varepsilon h(y) = \varepsilon - \operatorname{argmin}\,(h(\cdot) - \langle \cdot, y\rangle)\,.$$

**Proof.** Since

$$\partial h^*(y) = \operatorname{argmin}\{h(\cdot) - \langle ., y\rangle\}^{**},$$

then we shall apply Theorem 4.8 to the function $f(\cdot) := h(\cdot) - \langle ., y\rangle$ taking into account that $f^*(\cdot) = h^*(\cdot + y)$ and, consequently, $\operatorname{dom} f^* = \operatorname{dom} h^* - y$.

For this purpose, let us verify that $\mathcal{F}_y$ satisfies conditions (37) and (38) of the family $\{C_i,\ i \in I\}$ in Theorem 4.8. In fact, (37) now reads

$$\operatorname{dom} f^* = \operatorname{dom} h^* - y \subseteq \bigcup_{L\in\mathcal{F}_y} L,$$

and this inclusion holds trivially because, for any $v \in \operatorname{dom} h^*$, one has

$$v - y \in \operatorname{lin}\{v, y\} \in \mathcal{F}_y.$$

Additionally, (38) also holds automatically because, for every $L \in \mathcal{F}_y$, the nonempty convex set

$$L \cap (\operatorname{dom} h^* - y)$$

is finite dimensional.

Now we apply Theorem 4.8 and get

$$\begin{aligned}
\partial h^*(y) &= \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}_y}} \overline{\operatorname{co}}\left(M_\varepsilon h(y) + \{L \cap (\operatorname{dom} h^* - y)\}^-\right) \\
&= \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}_y}} \overline{\operatorname{co}}\left(M_\varepsilon h(y) + \{(L \cap \operatorname{dom} h^*) - y\}^-\right) \\
&= \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}_y}} \overline{\operatorname{co}}\,(M_\varepsilon h(y) + \mathrm{N}_{L\cap \operatorname{dom} h^*}(y))\,.
\end{aligned}$$

$\square$

**Corollary 4.10.** *For any function* $g \in \overline{\mathbb{R}}^Y$ *with* $\operatorname{dom} g^* \neq \emptyset$, *if*

$$\mathcal{F}_x := \{L \subset X \mid L \text{ is a finite-dimensional linear subspace such that } x \in L\},$$

*one has*

$$\partial g^*(x) = \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}_x}} \overline{\operatorname{co}} \left( M_\varepsilon g(x) + \mathrm{N}_{L\cap \operatorname{dom} g^*}(x) \right), \quad \text{for all } x \in X, \tag{40}$$

*where*

$$M_\varepsilon g(x) = \varepsilon - \operatorname{argmin} \left( g(\cdot) - \langle x, \cdot \rangle \right).$$

Now we derive another version of Theorem 4 in [4], in which the functions $f_t,\ t \in T$, are not necessarily convex.

**Corollary 4.11.** *Given a non-empty family* $\{f_t : t \in T\} \subset \overline{\mathbb{R}}^X$, *and the associated supremum function,* $f := \sup_{t\in T} f_t$, *if we assume that* $\operatorname{dom} f \neq \emptyset$ *and that* (22) *is satisfied, i.e.*

$$f^{**} = \sup_{t\in T} f_t^{**},$$

*then we have, for any* $x \in X$,

$$\partial f(x) = \bigcap_{L\in\mathcal{F}_x, \varepsilon>0} \operatorname{cl} \left( \operatorname{co} \left( \bigcup_{t\in T_\varepsilon(x)} \partial_\varepsilon f_t(x) \right) + \mathrm{N}_{L\cap \operatorname{dom} f}(x) \right). \tag{41}$$

**Proof.** The proof of the inclusion "$\supseteq$" is straightforward (see [4, p. 871]), and the proof of the converse inclusion "$\subseteq$" is based on the application of Corollary 4.10 to the function

$$g := \inf_{t\in T} f_t^*.$$

In fact, one can assume that $\partial f(x) \neq \emptyset$. Therefore

$$f(x) = f^{**}(x) = g^*(x) \quad \text{and} \quad \partial f(x) = \partial g^*(x).$$

Let us now observe that

$$M_{\varepsilon/2} g(x) \subset \bigcup_{t\in T_\varepsilon(x)} \partial_\varepsilon f_t(x). \tag{42}$$

In fact, if $y \in M_{\varepsilon/2} g(x)$, then

$$g(y) + g^*(x) < \langle x, y \rangle + \varepsilon,$$

and there will exist $t \in T$ such that

$$f_t^*(y) + g^*(x) = f_t^*(y) + f(x) \leq \langle x, y \rangle + \varepsilon. \tag{43}$$

By the Fenchel-Legendre inequality, it follows that

$$-f_t(x) + f(x) \leq \varepsilon,$$

and so $t \in T_\varepsilon(x)$.

From (43) one also has

$$f_t^*(y) + f_t(x) \leq \langle x, y \rangle + \varepsilon,$$

and this means $y \in \partial_\varepsilon f_t(x)$. Finally, the inclusion "$\subset$" in (41) follows from (40), (42), and from the inclusion $\operatorname{dom} g^* \supseteq \operatorname{dom} f$. □

## References

[1] J. Benoist, J.-B. Hiriart-Urruty: What is the subdifferential of the closed convex hull of a function?, SIAM J. Math. Anal. 27 (1996) 1661–1679.

[2] I. Ekeland, R. Temam: Convex Analysis and Variational Problems, North-Holland, Amsterdam (1976).

[3] E. Ernst, M. Thèra, M. Volle: Optimal boundedness criteria for extended-real-valued functions, Optimization 56 (2007) 323–338.

[4] A. Hantoute, M. A. López, C. Zălinescu: Subdifferential calculus rules in convex analysis: a unifying approach via pointwise supremum functions, SIAM J. Optim. 19 (2008) 863–882.

[5] J.-B. Hiriart-Urruty, C. Lemaréchal: Convex Analysis and Minimization Algorithms. Part I: Fundamentals; Part II: Advanced Theory and Bundle Methods, Springer, Berlin (1993).

[6] J.-B. Hiriart-Urruty, M. Moussaoui, A. Seeger, M. Volle: Subdifferential calculus without qualification conditions, using approximate subdifferentials: a survey, Nonlinear Anal. 24 (1995) 1727–1754.

[7] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton (1970).

[8] M. Valadier: Sous-differentiels d'une borne supérieure et d'une somme continue de fonctions convexes, C. R. Acad. Sci., Paris, Sér. A 268 (1969) 39–42.

[9] M. Volle: Sous-différentiel d'une enveloppe supérieure de fonctions convexes, C. R. Acad. Sci., Paris, Sér. I 317 (1993) 845–849.

[10] C. Zălinescu: Convex Analysis in General Vector Spaces, World Scientific, Singapore (2002).