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Asymptotically Bounded Multifunctions and the MCP Beyond Copositivity*

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Given a multifunction $F : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ and $q \in \mathbb{R}^n$, the multivalued complementarity problem (MCP) on the positive orthant consists in finding

$$\bar{x} \geq 0,\ \bar{y} \in F(\bar{x}) :\ \bar{y} + q \geq 0,\ \langle \bar{y} + q, \bar{x} \rangle = 0.$$

Such a formulation appears in many applications in Science and Engineering and therefore was the object of many investigations in the last three decades. Most of the works existing in the literature deal with the case when F is pseudomonotone (in the Karamardian sense) or quasimonotone, and only a few assume copositivity. In this work we introduce the notion of asymptotic multifunction with respect to a class of re-scaling functions including those with slow growth, and the notion of asymptotic multifunction associated to a sequence of multifunctions rather to a single one. Based on these two concepts we establish new existence theorems for the MCP for a class of multifunctions larger than copositive without assuming positive (sub)homogeneity as in a previous work. In addition, some stability and sensitivity results, as well as a robustness property, are provided. Thus, in this regard, we unify and generalize some of the results previously established.

Keywords: Complementarity problem, copositive multifunction, asymptotically bounded multifunction, asymptotic analysis

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1. Introduction, notation and basic definitions

Given a multifunction $F : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ and $q \in \mathbb{R}^n$, the multivalued complementarity problem on the positive orthant consists in finding

$$\bar{x} \geq 0,\ \bar{y} \in F(\bar{x}) :\ \bar{y} + q \geq 0,\ \langle \bar{y} + q, \bar{x} \rangle = 0. \tag{1}$$

It is well documented that such a problem appears in many applications in Science and Engineering [12, 10, 2] and therefore was the object of many investigations in the last three decades. Apart from the case when F is pseudomonotone (in the Karamardian sense) or quasimonotone, which have been studied recently in [4, 5, 6, 1, 11], the work [10] considered the case when F is copositive and not necessarily positively

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homogeneous in a somewhat general framework. Complementarity problems under copositivity and positive (sub)homogeneity were studied in [9, 7]. The authors in [10] use asymptotic arguments by comparing the asymptotic behavior of F with that of the function $c(t) = t^\alpha$, for some $\alpha > 0$. More precisely, given $F : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$, they introduce the following asymptotic multifunction

$$\Gamma(v) \doteq \left\{ w \in \mathbb{R}^n : \ \lambda_m \uparrow +\infty, \ x^m \geq 0, \ \frac{x^m}{\lambda_m} \to v, \ y^m \in F(x^m), \ \frac{y^m}{\lambda_m^\gamma} \to w \right\}.$$

This mapping is well defined under the following property on F called Upper Limiting Homogeneity (ULH). First, we consider the set $\Omega \subseteq \mathbb{R}_+$ consisting of nonnegative scalars ω such that for every sequence $\{\lambda_m\}$ of positive scalars, every sequence $\{x^m\}$ of nonnegative vectors, and every sequence $\{y^m\}$ of vectors satisfying

$$\lambda_m \uparrow +\infty, \quad x^m/\lambda_m \text{ converges}, \ y^m \in F(x^m) \ \ \forall\, m,$$

the sequence $\{y^m/\lambda_m^\omega\}$ is bounded. Thus, we say that F posseses the ULH property if the set Ω is nonempty and the scalar $\gamma = \inf\{\omega : \ \omega \in \Omega\}$ is in Ω. This scalar being uniquely defined is called the ULH degree of the multifunction F. For such F, the mapping Γ is positively homogeneous of degree γ; i.e.,

$$\Gamma(tv) = t^\gamma \Gamma(v) \ \ \forall\, t > 0, \ \forall\, v \geq 0.$$

The class of mappings satisfying the above ULH property contains very interesting and important multifunctions as shown in [10]. However, if we consider the mapping

$$F_1(x) = \ln(|x|^\gamma + 1)H(x), \quad \text{or} \quad F_2(x) = \frac{1}{\ln(|x|^\gamma + 1)}H(x), \quad (\gamma \geq 1)$$

with H being a positive homogeneous multifunction of degree $p > 0$, that is, $H(tx) = t^p H(x)$ for all $t > 0$, $x \geq 0$, a comparison (at infinity) with any function of the form t^α provides no information. In these cases, a good choice for F_1 is $c_1(t) = t^p \ln(t^\gamma + 1)$, and $c_2(t) = \frac{t^p}{\ln(t^\gamma+1)}$ for F_2. We have the same situation if one deals with

$$F_3(x) = M_1 x + \ln(|x| + 1) M_2 x,$$

where M_i, $i = 1, 2$, is a real-matrix of order n. Here we use $c_3(t) = t\ln(t+1)$. Another example is

$$F_4(x) = (\langle M_1 x, x\rangle, \ln(|x| + 1)\langle M_2 x, x\rangle) \in \mathbb{R}^2,$$

where $x \geq 0$, $x \in \mathbb{R}^2$, and M_1, M_2 are matrices or order 2 with real entries. In this case $c(t) = t^2 \ln(t + 1)$ is useful. Our theory will be applicable to these functions.

Thus, the purpose of the present paper is, on one hand, to introduce a notion of asymptotic multifunction with respect to a class of re-scaling functions larger than t^α, and on the other hand, having in mind to discussing existence, sensibility, stability and also perturbation results, we introduce the notion of c-asymptotic multifunction associated to a sequence of multifunctions $\{F^k\}$ rather than to a single F. Here c belongs to a certain class of real functions containing those of the form t^α, $\alpha > 0$. Based on these two concepts we estalish new existence theorems for problem (1) for a class of multifunctions larger than copositive without assuming positive (sub)homogeneity as imposed for instance in the recent paper [7].

This approach allows us to unify and generalize some of the existence results in [7] and [10]: the authors in [7] extended the results in [10] to a class of mappings larger than copositive, but they required to satisfy a kind of positive homogeneity which is not assumed in [10]. This latter assumption will be not imposed here.

We end this section by stating some notation and basic definitions well known in Set-valued analysis. In Section 2 we introduce our main notion of c-asymptotic multifunction associated to a sequence of multifunctions $\{F^k\}$, and the property of c-Asymptotic boundedness. Some related results based on the previous two notions are described as well. Section 3 presents a preliminary lemma which describes the asymptotic behavior of some approximate solutions to (1), and establishes various equivalent conditions for the unboundedness of the solution set. In Section 4 we introduce two classes of asymptotically well-behaved multifunctions together with the main existence theorems. The notion of asymptotically regular mappings is introduced, and a relationship with a robustness property is discussed in Section 5. Finally, Section 6 presents some sensitivity results through a metric defined for positive homogeneous-type multifunctions.

Throughout this paper we will deal with multifunctions $F : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ such that $F(x) \neq \emptyset$ for all $x \geq 0$. The set $F(x)$ is called the *image* of x under F, or the *value* of F at x. As usual the set $\operatorname{gph} F \doteq \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n : \ y \in F(x)\}$ is termed the *graph* of F.

A multifunction $F : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ is said to be:

- *compact* (*convex/nonempty*) *valued* if, for each $x \geq 0$, the image $F(x)$ is a compact (convex/nonempty) subset of $\mathbb{R}^n$;
- *upper semicontinuous* (shortly usc) at $\bar{x} \geq 0$ if for any open set V containing $F(\bar{x})$, there is an open set U containing x such that $F(U \cap \mathbb{R}^n_+) \subseteq V$; F is upper semicontinuous (on $\mathbb{R}^n_+$) if it is at every $\bar{x} \geq 0$;
- *cusco* if it is upper semicontinuous and nonempty compact convex valued;
- *sequentially bounded* at $v \in \mathbb{R}^n_+$ if for any sequence $\{x^k\} \subseteq \mathbb{R}^n_+$ converging to v, any sequence $\{y^k\}$ with $y^k \in F(x^k)$ for all k, is bounded. This is equivalent to saying, if for some neighborhood U of v the set $F(U \cap \mathbb{R}^n_+)$ is bounded. Some authors also use the term *locally bounded* at v. Such a multifunction will be sequentially bounded on $\mathbb{R}^n_+$ if it is at every $v \in \mathbb{R}^n_+$;
- *uniformly bounded* if there exists a bounded set $C \subseteq \mathbb{R}^n$ such that $F(x) \subseteq C$ for all $x \geq 0$;
- *graph-closed* if its graph is closed.
  Furthermore, some algebraic notions are also needed. For any function $c : \mathbb{R}_+ \to \mathbb{R}_+$, the multifunction F is said to be:
- c-subhomogeneous if $F(\lambda x) \subseteq c(\lambda)F(x)\ \forall\ \lambda > 0,\ \forall\ x \geq 0$;
- c-homogeneous if $F(\lambda x) = c(\lambda)F(x)\ \forall\ \lambda > 0,\ \forall\ x \geq 0$;
- zero-subhomogeneous if $F(\lambda x) \subseteq F(x)\ \forall\ \lambda > 0,\ \forall\ x \geq 0$;
- c-Moré [7]: if $\forall\ x \geq 0,\ \forall\ \lambda > 0,\ \forall\ y \in F(\lambda x),\ \exists\ z \in F(x)$ such that

$$\langle y, x \rangle \geq c(\lambda)\langle z, x \rangle.$$

We set

$$\mathcal{X} \doteq \left\{F : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n : \ F \text{ is cusco}\right\}.$$

Given $A \subseteq \mathbb{R}^n$, we set

$$A^{\#} \doteq \{x \in \mathbb{R}^n : \ \langle u, x\rangle > 0 \ \ \forall\, u \in A, \ u \neq 0\},$$
$$A^{*} \doteq \{x \in \mathbb{R}^n : \ \langle u, x\rangle \geq 0 \ \ \forall\, u \in A\}.$$

Obviously $A^{\#} \subseteq A^*$, and since A^* is always a convex closed cone and $A \subseteq A^{**}$, we have int $A^* \subseteq A^{\#}$, where int A stands for the topological interior of A; ri A, co A, pos A, pos^+A mean, respectively, for the relative interior, the convex hull, the positive hull and the strictly positive hull of A, i.e., $\text{pos}^+A = \{tx : \ t > 0, \ x \in A\}$; $\mathcal{S}(q, F)$ stands for the solution set to problem (1).

2. Asymptotically bounded multifunctions and related properties

Set $\mathbb{R}_{++} = \ ]0, +\infty[$. Let us consider

$$\mathcal{C} \doteq \{c : \mathbb{R}_{++} \to \mathbb{R}_{++} : \ \lim_{t\to+\infty} c(t) = +\infty\}$$

and

$$\mathcal{C}_0 \doteq \left\{c \in \mathcal{C} : \ \lim_{t\to+\infty} \frac{c(\lambda t)}{c(t)} \in \mathbb{R} \text{ for all } \lambda > 0\right\}.$$

For $c \in \mathcal{C}_0$, set

$$c^\infty(\lambda) \doteq \lim_{t\to+\infty} \frac{c(\lambda t)}{c(t)}.$$

Thus, one immediately obtains that for all $c \in \mathcal{C}_0$,

$$c^\infty(1) = 1, \quad +\infty > c^\infty(t) > 0 \ \ \forall\, t > 0 \tag{2}$$

$$c^\infty(\xi)c^\infty(\lambda) = c^\infty(\xi\lambda), \ \xi > 0, \ \lambda > 0. \tag{3}$$

Although we would desire that (2) and (3) imply that for some $\alpha > 0$, $c^\infty(t) = t^\alpha$, $\forall\, t > 0$, we will need only the property

$$c^\infty(t)c^\infty(1/t) = 1 \ \ \forall\, t > 0 \tag{4}$$

in the next sections. Some examples for c are:

$$c_1(t) = t^p (p > 0), \ c_1^\infty(t) = t^p; \ c_2(t) = t^p \ln(t^\gamma + 1)(p > 0, \ \gamma \geq 1), \ c_2^\infty(t) = t^p;$$

$$c_3(t) = \frac{t^p}{\ln(t^\gamma + 1)}(p > 0, \gamma \geq 1), \ c_3^\infty(t) = t^p.$$

Given a sequence of multifunctions $F^k : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$, $k \in \mathbb{N}$, with nonempty values, and $c \in \mathcal{C}$, the *c-asymptotic* multifunction associated to $\{F^k\}$ is defined by

$$\limsup_k{}^{\infty,c} F^k(v)$$
$$\doteq \left\{w \in \mathbb{R}^n : \ \lambda_{k_m} \uparrow +\infty, \ x^{k_m} \geq 0, \ \frac{x^{k_m}}{\lambda_{k_m}} \to v, \ y^{k_m} \in F^{k_m}(x^{k_m}), \ \frac{y^{k_m}}{c(\lambda_{k_m})} \to w\right\}$$

When $c(t) = t^\gamma$ for some $\gamma > 0$, we denote simply $\limsup_k^{\infty,\gamma} F^k$ instead of $\limsup_k^{\infty,c} F^k$. Observe in this case, that if $|v| = 1$, one reduces to considering sequences x^{k_m} with $|x^{k_m}| \to +\infty$ such that $\frac{x^{k_m}}{|x^{k_m}|} \to v$, thus taking $\lambda_{k_m} = |x^{k_m}|$ suffices.

We can easily check that

$$\text{gph}(\limsup_k{}^{\infty,c} F^k) \text{ is closed and } \limsup_k{}^{\infty,c} F^k(v) \text{ is closed } \forall\ v \geq 0. \tag{5}$$

For a given multifunction $F : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ and $c \in \mathcal{C}$, we set

$$A_F(x) \doteq \sup_{y \in F(x)} |y|;$$

$$\begin{aligned} &\limsup_k{}^{\infty} A_{F^k,c}(v) \\ &\doteq \sup \left\{ \limsup_{m\to+\infty} \frac{1}{c(\lambda_{k_m})} A_{F^{k_m}}(\lambda_{k_m} x^{k_m}) : \ k_m \uparrow +\infty, \lambda_{k_m} \uparrow +\infty,\ x^{k_m} \to v \right\}. \end{aligned}$$

If we consider $F^k = F$ for all k, we set

$$F_c^\infty(v) \doteq \limsup_k{}^{\infty,c} F^k(v), \qquad A_{F,c}^\infty(v) \doteq \limsup_k{}^{\infty} A_{F^k,c}(v).$$

The finiteness of the latter function is related to the property *c-asymptotically bounded at v* (c-AB) which means: for every $k_m \to +\infty$ as $m \to +\infty$, every $\lambda_{k_m} \uparrow +\infty$, every $x^{k_m} \geq 0$, every $y^{k_m} \in F^{k_m}(x^{k_m})$, such that $\{x^{k_m}/\lambda_{k_m}\}$ converges to v, the sequence $\{y^{k_m}/c(\lambda_{k_m})\}$ is bounded. More precisely, one has the following lemma.

Lemma 2.1. *Let $F^k : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ with nonempty compact values, $c \in \mathcal{C}$ and $v \geq 0$. Then, the following assertions are equivalent:*

(a) $\limsup_k^\infty A_{F^k,c}(v) < +\infty$;

(b) $\{F^k\}$ *satisfies the c-asymptotically bounded property at v.*

Proof. $(a) \Longrightarrow (b)$: Take any sequence $k_m \to +\infty$ and sequences $\lambda_{k_m} \uparrow +\infty$, $x^{k_m} \geq 0$, $x^{k_m}/\lambda_{k_m} \to v$ and $y^{k_m} \in F^{k_m}(x^{k_m})$. Assume there exists a subsequence, still indexed by the same index, of y^{k_m} such that $|y^{k_m}/c(\lambda_{k_m})| \to \infty$ as $m \to +\infty$, then from the inequality

$$\left| \frac{y^{k_m}}{c(\lambda_{k_m})} \right| \leq \frac{1}{c(\lambda_{k_m})} A_{F^{k_m}}(x^{k_m}),$$

we reach a contradiction.

$(b) \Longrightarrow (a)$: If, to the contrary we suppose that $\limsup_k^\infty A_{F^k,c}(v) = +\infty$, (through a diagonalization procedure) there exist subsequences $k_m \to +\infty$, $\lambda_{k_m} \uparrow +\infty$, $x^{k_m} \to v$ as $m \to +\infty$ such that

$$+\infty = \limsup_k{}^{\infty} A_{F^k,c}(v) = \limsup_{m\to+\infty} \frac{1}{c(\lambda_{k_m})} A_{F^{k_m}}(\lambda_{k_m} x^{k_m}).$$

Since each F^{k_m} has compact values, there exists $y^{k_m} \in F^{k_m}(\lambda_{k_m} x^{k_m})$ satisfying $A_F(\lambda_{k_m} x^{k_m}) = |y^{k_m}|$. Assuming (b) holds, there exists $r > 0$ such that $|y^{k_m}/c(\lambda_{k_m})| \leq r$ $\forall\ m$, which contradicts a previous equality. $\square$

In order to illustrate the applicability of our notions to multifunctions that are not positively homogeneous, let us consider the following examples. More sophisticated instances involving multifunctions are exhibited in Example 2.11 by using the graphical convergence.

Example 2.2. Take any real matrices $M, M^k \in \mathbb{R}^{n\times n}$ satisfying $M^k \to M$, and $\rho \in \mathbb{R}$.

(a) $F^k(x) = |x|^\gamma M^k x - \rho x$ $(\gamma \geq 0)$, $c(t) = t^{\gamma+1} = c^\infty(t)$, then

$$\limsup_k{}^{\infty} A_{F^k,c}(v) = |v|^\gamma |Mv|, \qquad \limsup_k{}^{\infty,c} F^k(v) = |v|^\gamma Mv.$$

(b) $F^k(x) = \ln(|x|^\gamma + 1)M^k x - \rho x$ $(\gamma \geq 1)$, $c(t) = t\ln(t^\gamma + 1)$, then

$$\limsup_k{}^{\infty} A_{F^k,c}(v) = |Mv|, \qquad \limsup_k{}^{\infty,c} F^k(v) = Mv,\ c^\infty(t) = t.$$

(c) Take $\gamma > 0$ and $C^k \subseteq \mathbb{R}^n$ to be a sequence of nonempty closed sets converging (in the sense of Painlevé-Kuratowski) to a nonempty closed set $C \subseteq \mathbb{R}^n$, and consider

$$F^k(x) = |x|^\gamma C^k, \qquad \limsup_k{}^{\infty,\gamma} F^k(v) = |v|^\gamma C.$$

The following result, which is important by itself, will be used in subsequent sections.

Proposition 2.3. *Let $F^k : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ be any multifunctions with nonempty values, $c \in \mathcal{C}$ and $v \geq 0$. If $\limsup_k{}^{\infty} A_{F^k,c}(v) < +\infty$ then $\limsup_k{}^{\infty,c} F^k$ is locally bounded at v.*

Proof. Assume to the contrary that there are sequences $v_l \geq 0$, $w_l \in \limsup_k{}^{\infty,c} F^k(v_l)$, such that $v_l \to v$, $|w_l| \to +\infty$ as $l \to +\infty$. By definition, for every l, there are sequences $p^l_k \uparrow +\infty$, $\lambda_{p^l_k} \uparrow +\infty$ $(k \to +\infty)$, $y^{p^l_k} \in \mathbb{R}^n$, $x^{p^l_k} \geq 0$, satisfying

$$\frac{x^{p^l_k}}{\lambda_{p^l_k}} \to v_l,\ y^{p^l_k} \in F^{p^l_k}(x^{p^l_k}),\ \frac{y^{p^l_k}}{c(\lambda_{p^l_k})} \to w_l, \quad \text{as } k \to +\infty.$$

Through a diagonalization procedure, we can get sequences

$$\lambda_{p^l_{k_l}} \uparrow +\infty,\ \frac{x^{p^l_{k_l}}}{\lambda_{p^l_{k_l}}} \to v,\ y^{p^l_{k_l}} \in F^{p^l_{k_l}}(x^{p^l_{k_l}}),\ \left|\frac{y^{p^l_{k_l}}}{c(\lambda_{p^l_{k_l}})} - w_l\right| \to 0, \quad \text{as } l \to +\infty.$$

Therefore,

$$\left|\frac{y^{p^l_{k_l}}}{c(\lambda_{p^l_{k_l}})}\right| \to +\infty,$$

which contradicts our hypothesis. $\square$

Given a sequence of nonempty sets of $D^k \subseteq \mathbb{R}^p$ the following notion was introduced in [21, Chapter 4],

$$\limsup_k{}^{\infty} D^k \doteq \left\{ v \in \mathbb{R}^p :\ \lambda_{k_m} \uparrow +\infty,\ x^{k_m} \in D^{k_m},\ \frac{x^{k_m}}{\lambda_{k_m}} \to v \right\}.$$

When the sets $D^k \subseteq \mathbb{R}^n \times \mathbb{R}^n$ are the graphs of some multifunctions $F^k : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$, we need to consider different re-scaling in the domain and image spaces. More precisely, for a given $c \in \mathcal{C}$, we define

$$\limsup_k{}^{\infty,c}(\mathrm{gph}\, F^k)$$
$$\doteq \left\{(v,w) \in \mathbb{R}^n_+ \times \mathbb{R}^n :\ \lambda_{k_m} \uparrow +\infty,\ (x^{k_m}, y^{k_m}) \in \mathrm{gph}\, F^{k_m}, \left(\frac{x^{k_m}}{\lambda_{k_m}}, \frac{y^{k_m}}{c(\lambda_{k_m})}\right) \to (v,w)\right\}.$$

Then, $\limsup_k^{\infty,c} F^k$ is the multifunction having as its graph the set $\limsup_k{}^{\infty,c}(\mathrm{gph}\, F^k)$, that is,

$$\mathrm{gph}(\limsup_k{}^{\infty,c} F^k) = \limsup_k{}^{\infty,c}(\mathrm{gph}\, F^k).$$

Consequently the graph of $\limsup_k^{\infty,c} F^k$ is closed as already observed.

In case $F^k = F$ for all k, we denote

$$(\mathrm{gph}\, F)^{\infty,c} \doteq \limsup_k{}^{\infty,c}(\mathrm{gph}\, F^k),$$

and thus $(\mathrm{gph}\, F)^{\infty,c} = \mathrm{gph}\, F^\infty_c$.

If $c(t) = t$, we recover the usual notion of asymptotic cone: for any set $C \subseteq \mathbb{R}^n$, the asymptotic cone of C is the set

$$C^\infty \doteq \left\{u \in \mathbb{R}^n :\ \exists\ \lambda_k \uparrow +\infty,\ \exists\ x^k \in C,\ \frac{x^k}{\lambda_k} \to u\right\}.$$

Remark 2.4. (i) For F having nonempty values, one has $0 \in F^\infty_c(0)$, thus $0 \in \mathcal{S}(p, F^\infty_c)$ for all $p \geq 0$.

(ii) If $F^k = F$ for all k, a simple instance satisfying

$$A^\infty_{F,c}(v) < +\infty \quad \forall\ v \geq 0, \tag{6}$$

occurs when F is sequentially closed (e.g. cusco mapping) and c-subhomogeneous for $c \in \mathcal{C}$.

Another non-trivial instance satisfying (6) is when $F = F_1 + F_2$, where F_i, $i = 1, 2$, are cusco mappings, F_2 is c_2-subhomogeneous with $c_2(t) = t^\gamma$ for some $\gamma \geq 0$, F_1 satisfies the c_1-asymptotically bounded property at any $v \geq 0$, and $c = c_1 \in \mathcal{C}$, is such that

$$\lim_{t \to +\infty} \frac{t^\gamma}{c_1(t)} = 0.$$

The function F_3 of Section 1 is of the form above. A special realization of the mapping F_2 occurs when it is the classical (convex) subdifferential of a positively homogeneous function with degree $1 + \gamma$, $\gamma \geq 0$.

As mentioned in the introduction, the c-AB property was first introduced in [10] when $F^k = F$ and $c(t) = t^{\gamma'}$ for some $\gamma' > 0$. Such a property is called Upper Limiting Homogeneity (ULH) by the authors in [10]. They also introduced the ULH degree as

γ to be the minimum of those γ' for which the ULH property holds. In this case γ is uniquely defined. Thus, for a multifunction F having the ULH property with degree γ one obtains $F^{\infty}_{\gamma}(tv) = t^{\gamma}F^{\infty}_{\gamma}(v)$ for all $t > 0$, $v \geq 0$.

In our setting, the choice of c is a matter of testing by taking into account the behavior of F^k. For instance, if $F^k = F$ for all k and $F(tx) = t^{\gamma}F(x)$ $\forall\ t > 0,\ x \geq 0$, then we must take $c(t) = t^{\gamma}$. The fact of taking $c(t) = t^{\gamma'}$ with $\gamma' > \gamma$ or $\gamma' < \gamma$ becomes useless since it provides no information.

Theorem 2.5. *Let $F^k : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ be any sequence of multifunctions with nonempty compact values and $c \in \mathcal{C}$. The following assertions hold.*

(a) *If $v \geq 0$, $\limsup^{\infty}_k A_{F^k,c}(v) < +\infty$ then*

$$\limsup_k{}^{\infty,c} F^k(v) \neq \emptyset, \qquad \limsup_k{}^{\infty} A_{F^k,c}(v) = \sup\left\{|w| :\ w \in \limsup_k{}^{\infty,c} F^k(v)\right\}.$$

As a consequence $\limsup_k^{\infty,c} F^k(v)$ is compact.

(b) *If $c \in \mathcal{C}_0$ then*

$$\limsup_k{}^{\infty,c} F^k \quad \text{is } c^{\infty}\text{−homogeneous.}$$

(c) *If each F^k is c-subhomogeneous, then*

$$\limsup_k{}^{\infty,c} F^k \quad \text{is } c-\text{subhomogeneous.}$$

Thus $\limsup_k^{\infty,c} F^k$ is c−homogeneous provided $c(\lambda)c(1/\lambda) = 1\ \forall\ \lambda > 0$.

Proof. (a): For any m, consider $x^m = mv$. By assumption, given any $y^m \in F^m(x^m)$ there exists y^{k_m} such that $\frac{y^{k_m}}{c(k_m)}$ converges, proving the first part.

Let $w \in \limsup_k^{\infty,c} F^k(v) \neq \emptyset$. Then, there are sequences $\lambda_{k_m} \uparrow +\infty$, $x^{k_m} \geq 0$, $y^{k_m} \in F^{k_m}(x^{k_m})$ such that $\frac{x^{k_m}}{\lambda_{k_m}} \to v$, $\frac{y^{k_m}}{c(\lambda_{k_m})} \to w$. Since

$$\frac{1}{c(\lambda_{k_m})}|y^{k_m}| \leq \frac{1}{c(\lambda_{k_m})} A_{F^{k_m}}\left(\lambda_{k_m}\frac{x^{k_m}}{\lambda_{k_m}}\right),$$

it follows that $|w| \leq \limsup^{\infty}_k A_{F^k,c}(v)$, proving that

$$\sup\{|w| :\ w \in \limsup_k{}^{\infty,c} F^k(v)\} \leq \limsup_k{}^{\infty} A_{F^k,c}(v).$$

To prove the other inequality, assume there is $t \in \mathbb{R}$ such that

$$\sup\{|w| :\ w \in \limsup_k{}^{\infty,c} F^k(v)\} < t < \limsup_k{}^{\infty} A_{F^k,c}(v).$$

Then, we can choose sequences $\lambda_{k_m} \uparrow +\infty$, $x^{k_m} \to v$, $x^{k_m} \geq 0$ such that

$$t < \limsup_{m\to+\infty} \frac{1}{c(\lambda_{k_m})} A_{F^{k_m}}(\lambda_{k_m} x^{k_m}) < +\infty.$$

Since F^{k_m} has compact values, for every m there exists $y^{k_m} \in F^{k_m}(\lambda_{k_m} x^{k_m})$ such that $A_{F^{k_m}}(\lambda_{k_m} x^{k_m}) = |y^{k_m}|$. Hence, up to further subsequences, $t < |w| = \lim_{m\to+\infty} \frac{1}{c(\lambda_{k_m})}|y^{k_m}|$ for some $w \in \limsup_k^{\infty,c} F^k(v)$, a contradiction.

(b): Let $\lambda > 0$, $v \geq 0$ and take $w \in \limsup_k^{\infty,c} F^k(\lambda v)$. Then, $\exists\ \lambda_{k_m} \uparrow +\infty$, $\exists\ x^{k_m} \geq 0$, $\frac{x^{k_m}}{\lambda_{k_m}} \to \lambda v$, $\exists\ y^{k_m} \in F^{k_m}(x^{k_m})$, $\frac{y^{k_m}}{c(\lambda_{k_m})} \to w$. We write

$$\frac{y^{k_m}}{c(\lambda\lambda_{k_m})} = \frac{c(\lambda_{k_m})}{c(\lambda\lambda_{k_m})}\frac{y^{k_m}}{c(\lambda_{k_m})} \to \frac{w}{c^\infty(\lambda)} \in \limsup_k{}^{\infty,c} F^k(v).$$

This proves $\limsup_k^{\infty,c} F^k(\lambda v) \subseteq c^\infty(\lambda) \limsup_k^{\infty,c} F^k(v)$. We use (4) to deduce the reverse inclusion.

(c): We proceed as in (b) to get the subsequences λ_{k_m}, x^{k_m}, y^{k_m}. By the c-subhomogeneity of F^k, we obtain

$$\frac{y^{k_m}}{c(\lambda)} \in F^{k_m}\left(\frac{x^{k_m}}{\lambda}\right),$$

and since $\frac{1}{\lambda_{k_m}}\frac{x^{k_m}}{\lambda} \to v$, we obtain

$$\frac{1}{c(\lambda_{k_m})}\frac{y^{k_m}}{c(\lambda)} \to \frac{w}{c(\lambda)} \in \limsup_k{}^{\infty,c} F^k(v).$$

$\square$

Proposition 2.6. *Let $F^k, G^k : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$, $k \in \mathbb{N}$, be any multifunctions with nonempty values, and $c \in \mathcal{C}$.*

(a) *If $c \in \mathcal{C}_0$ then*

$$\limsup_k{}^\infty A_{F^k,c}(\lambda v) = c^\infty(\lambda) \limsup_k{}^\infty A_{F^k,c}(v) \quad \forall\ \lambda > 0,\ v \geq 0,$$

i.e., $\limsup_k{}^\infty A_{F^k,c}$ is c^∞-homogeneous.

(b) $\limsup_k^\infty A_{F^k+G^k,c}(v) \leq \limsup_k^\infty A_{F^k,c}(v) + \limsup_k^\infty A_{G^k,c}(v) \quad \forall\ v \geq 0.$

Proof. (a): Take any $\lambda > 0$, $\lambda_m \to +\infty$, $x^m \geq 0$, $x^m \to \lambda v$. We write

$$A_m = \frac{c(\lambda_m\lambda)}{c(\lambda_m)}\frac{1}{c(\lambda_m\lambda)} A_{F^m}(\lambda_m x^m)$$

to infer that

$$\limsup_{m\to+\infty} A_m \leq c^\infty(\lambda) \limsup_{m\to+\infty} \frac{1}{c(\lambda_m\lambda)} A_{F^m}(\lambda_m x^m) \leq c^\infty(\lambda) \limsup_k{}^\infty A_{F^k,c}(v).$$

This implies

$$\limsup_k{}^\infty A_{F^k,c}(\lambda v) \leq c^\infty(\lambda) \limsup_k{}^\infty A_{F^k,c}(v).$$

The reverse inequality is obtained by taking into account (4).

(b): It follows directly from the inequality

$$A_{F^k+G^k}(x) \leq A_{F^k}(x) + A_{G^k}(x).$$

$\square$

The proof of the next proposition is straightforward.

Proposition 2.7. *Let $F^k, G^k : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ be multifunctions with nonempty values for $k \in \mathbb{N}$; $c \in \mathcal{C}$, $v \geq 0$. If*

$$\limsup_k{}^{\infty} A_{G^k,c}(v) = 0, \tag{7}$$

then

$$\limsup_k{}^{\infty,c} H^k(v) = \limsup_k{}^{\infty,c} F^k(v),$$

where $H^k = F^k + G^k$.

The multifunctions $G^k(x) = \partial\sigma_{C^k}(x)$, $k \in \mathbb{N}$, where C^k is any sequence of nonempty compact convex sets in $\mathbb{R}^n$ which converges (in the sense of Painlevé-Kuratowski) to the nonempty compact convex set C, is a typical example satisfying (7) for every $c \in \mathcal{C}$. In fact, due to the convergence of C^k, these sets remain in a fixed bounded set.

Having in mind to discuss some kind of sensitivity and perturbation results, we need a good notion of convergence for multifunctions. The graphical convergence had proved to be very useful in this regard as shown in [7]. An exhaustive study of such a notion may be found in [21, Chapter 5].

Given a sequence of multifunctions $F^k : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$, the definition of graphical convergence of $\{F^k\}$ is given in terms of the graphical outer limit, denoted by $g - \limsup_k F^k$, which is the multifunction having as its graph the set $\limsup_k(\operatorname{gph} F^k)$, and the graphical inner limit, denoted by $g - \liminf_k F^k$, which is the multifunction having as its graph the set $\liminf_k(\operatorname{gph} F^k)$. Then, F^k converges graphically to F, if and only if, the outer and inner limits agree, in such case we write $F^k \xrightarrow{g} F$, or $F = g - \lim_k F^k$. However, when $F^k \in \mathcal{X}$ the previous convergence can be induced by a metric. To this end, given a nonempty set $C \subseteq \mathbb{R}^n$, let us denote by $d_C(x) \doteq d(C, x)$ the distance from x to C. Let $A, B \subseteq \mathbb{R}^n$ be two nonempty sets, the *integrated set distance* between them is defined by

$$\mathrm{d\!l}(A, B) \doteq \int_0^\infty \mathrm{d\!l}_\rho(A, B) e^{-\rho} d\rho.$$

where for $\rho \geq 0$,

$$\mathrm{d\!l}_\rho(A, B) \doteq \max_{|x| \leq \rho} |d_A(x) - d_B(x)|.$$

The expression $\mathrm{d\!l}$ gives a metric on $\mathrm{cl} - \mathrm{sets}_{\neq\emptyset}(\mathbb{R}^n)$-the space of all nonempty closed subsets of $\mathbb{R}^n$, and characterizes the ordinary set convergence in the sense of Painlevé-Kuratowski, i.e. $C^k \to C \iff \mathrm{d\!l}(C^k, C) \to 0$.

On $\mathcal{X}$ we consider the metric (we shall denote also by $\mathrm{d\!l}$)

$$\mathrm{d\!l}(F^1, F^2) \doteq \mathrm{d\!l}(\operatorname{gph} F^1, \operatorname{gph} F^2).$$

Then ([21, Theorem 5.50]),

$$F^k \xrightarrow{g} F \iff \mathrm{d\!l}(F^k, F) \to 0.$$

Remark 2.8. Another instance more general than that exhibited in Remark 2.4 for which

$$\limsup_k{}^{\infty} A_{F^k,c}(v) < +\infty \;\; \forall\, v \geq 0,$$

is given when each F^k is cusco and c-subhomogeneous (hence F is also c-subhomogeneous by Proposition 6.6 of [7] provided c is continuous) for $c \in \mathcal{C}$, and $F^k \xrightarrow{g} F$. This results from the definition of $\limsup_k^{\infty,c} F^k$ and the property of the uniformity in graphical convergence of F^k (see [21, Exercise 5.34]).

Theorem 2.9. *Let $F^k, F : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ be multifunctions in $\mathcal{X}$ for $k \in \mathbb{N}$; $c \in \mathcal{C}$, $v \geq 0$. Assume that $F^k \xrightarrow{g} F$, then*

$$F_c^{\infty}(v) \subseteq \limsup_k{}^{\infty,c} F^k(v). \tag{8}$$

If, in addition, each F^k is c-subhomogeneous, then

$$\limsup_k{}^{\infty,c} F^k(v) \subseteq F(v).$$

Proof. Let $w \in F_c^{\infty}(v)$. Then, there exist sequences $\lambda_m \uparrow +\infty$, $x^m \geq 0$, $y^m \in F(x^m)$ such that $\frac{x^m}{\lambda_m} \to v$ and $\frac{y^m}{c(\lambda_m)} \to w$. We apply Theorem 5.37 in [21] to obtain $x_l^m \geq 0$, $y_l^m \in F^{i_l^m}(x_l^m)$ satisfying $x_l^m \to x^m$, $y_l^m \to y^m$ and $i_l^m \to +\infty$ as $l \to +\infty$. By a diagonalization procedure, we get

$$y_{l_m}^m \in F^{i_{l_m}^m}(x_{l_m}^m),\; |y_{l_m}^m - y^m| \to 0,\; |x_{l_m}^m - x^m| \to 0,\; i_{l_m}^m \to +\infty \;\text{ as } m \to +\infty.$$

Thus,

$$\frac{y_{l_m}^m}{c(\lambda_m)} \to w, \qquad \frac{x_{l_m}^m}{\lambda_m} \to v.$$

Hence $w \in \limsup_k^{\infty,c} F^k(v)$.

We now prove the second inclusion. Take any $w \in \limsup_k^{\infty,c} F^k(v)$. Then, there exist subsequences $\lambda_{k_m} \uparrow +\infty$, $x^{k_m} \geq 0$, $y^{k_m} \in F^{k_m}(x^{k_m})$ such that $\frac{x^{k_m}}{\lambda_{k_m}} \to v$ and $\frac{y^{k_m}}{c(\lambda_{k_m})} \to w$. By c-subhomogeneity and the uniformity in graphical convergence of F^k, we obtain $w \in F(v)$. □

Corollary 2.10. *Let $F^k, F : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ be multifunctions in $\mathcal{X}$ for $k \in \mathbb{N}$; $c \in \mathcal{C}$, $v \geq 0$. Assume that $F^k \xrightarrow{g} F$.*

(*a*) *If $F^k(x) \subseteq F(x) + G^k(x)$ for all k, all $x \geq 0$ with $\limsup_k^{\infty} A_{G^k,c}(v) = 0$, then*

$$\limsup_k{}^{\infty,c} F^k(v) = F_c^{\infty}(v).$$

(*b*) *If each F^k is c-subhomogeneous and F is c-homogeneous, then*

$$\limsup_k{}^{\infty,c} F^k(v) = F_c^{\infty}(v) = F(v).$$

Proof. (a): By setting $H^k = F + G^k$, we have

$$\limsup_k{}^{\infty,c} F^k(v) \subseteq \limsup_k{}^{\infty,c} H^k(v) = F_c^\infty(v)$$

by Proposition 2.7. We then apply the previous theorem.

(b): We need to prove $F(v) \subseteq F_c^\infty(v)$. Let $w \in F(v)$. By setting $x^m = mv$, $y^m = c(m)w$, the c-homogeneity of F imply that $y^m \in F(x^m)$ and obviously $\frac{y^m}{c(m)} \to w$. Therefore $w \in F_c^\infty(v)$. We then apply the previous theorem. $\square$

Example 2.11. (i) We consider $F^k(x) = \ln(|x|^\gamma + 1)M^k x$, $\gamma \geq 1$, $M^k \to M$, and $G^k(x) = \ln(|x|^\gamma + 1)(M^k - M)x$. Take $c(t) = t\ln(t^\gamma + 1)$. Then, $F^k \xrightarrow{g} F$, $F(x) = \ln(|x|^\gamma + 1)Mx$ by [21, Theorem 5.40]. Thus

$$(F^k)_c^\infty(v) = M^k v, \; F_c^\infty(v) = Mv \quad \text{and} \quad \limsup_k{}^{\infty,c} F^k(v) = F_c^\infty(v).$$

(ii) More generally, take any sequence of multifunctions $H^k \in \mathcal{X}$ which is asymptotically equi-osc and converges pointwise to $H \in \mathcal{X}$ (See Sections E and F of Chapter 5 in [21]). By [21, Theorem 5.40], $H^k \xrightarrow{g} H$. In addition, assume that for some $p > 0$,

$$H^k(tx) = t^p H^k(x) \;\; \forall\, t > 0, \; \forall\, x \geq 0.$$

Then $H(tx) = t^p H(x)\; \forall\, t > 0$, $\forall\, x \geq 0$. By setting $F^k(x) = \ln(|x|^\gamma + 1)H^k(x)$, $(\gamma \geq 1)$, by [21, Theorem 5.40], we obtain $F^k \xrightarrow{g} F$ with $F(x) = \ln(|x|^\gamma + 1)H(x)$, and

$$(F^k)_c^\infty(v) = H^k(v), \;\; \limsup_k{}^{\infty,c} F^k(v) = F_c^\infty(v) = H(v),$$

with $c(t) = t^p \ln(t^\gamma + 1)$. Thus, $(F^k)_c^\infty \xrightarrow{g} F_c^\infty$.

A particular sequence for H^k with $p = 1$ is given by

$$H^k(x) = \{M^k y : \;\; Ax + By \leq 0\},$$

where $M^k \in \mathbb{R}^{n\times n}$, $A, B \in \mathbb{R}^{m\times n}$.

Remark 2.12. Some other sufficient conditions on F^k such that (with $c(t) = t$)

$$F^k \xrightarrow{g} F \Longrightarrow \limsup_k{}^{\infty,c} F^k(v) = F_c^\infty(v) \;\; \forall\, v \geq 0, \tag{9}$$

are exhibited in [21], Theorem 4.25. They are related to the notion of total convergence ("$\xrightarrow{t}$") introduced in the same reference. In that spirit we could define the following notion of convergence:

$$F^k \xrightarrow{t} F \Longleftrightarrow F^k \xrightarrow{g} F, \;\; \limsup_k{}^{\infty,c} F^k(v) \subseteq F_c^\infty(v) \;\; \forall\, v \geq 0. \tag{10}$$

By Theorem 2.9 the inclusion in (10) is actually an equality.

3. Asymptotic analysis and unboundedness of solution sets

We recall briefly some useful notations:

- $\mathrm{ri}A$ is the relative interior of A, that is, the interior with respect to its affine hull;
- given $J \subseteq I \doteq \{1, \ldots, n\}$ and $d > 0$ (component-wise), we set $\Delta_J = \Delta_J(d) \doteq \mathrm{co}\{\frac{1}{d_i}e^i : \ i \in J\}$, where e^i is the i-th column of the identity matrix in $\mathbb{R}^{n\times n}$. However, sometimes we will omit the dependence on d when no confusion arises; denote $\Delta_d \doteq \{x \geq 0 : \ \langle d, x\rangle = 1\} = \Delta_I$;
- given $x \in \mathbb{R}^n$, we set $\mathrm{supp}\{x\} \doteq \{i \in I : \ x_i \neq 0\}$;
- given $k \in \mathbb{N}$, $\sigma_k > 0$ and $d > 0$ (component-wise), we set

$$D_k \doteq \{x \geq 0 : \ \langle d, x\rangle \leq \sigma_k\}.$$

The following lemma has its origin in [7] where the assumption of subhomogeneity was imposed. Its proof is almost the same, but it is presented here for reader's convenience. This lemma will be the basis for deriving existence as well as sensitivity results.

Lemma 3.1. *Let $d > 0$, $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$; $q, q^k \in \mathbb{R}^n$, $q^k \to q$; $F, F^k, G^k \in \mathcal{X}$, and $\{(x^k, y^k, r^k)\}$ be a sequence of solutions to*

$$\text{find } x^k \in D_k : \quad y^k \in F^k(x^k),\ r^k \in G^k(x^k),\ \langle y^k + r^k + q^k, x - x^k\rangle \geq 0 \ \ \forall\, x \in D_k. \quad (11)$$

such that $\langle d, x^k\rangle = \sigma_k$ and $\frac{x^k}{\sigma_k} \to v$ as $k \to +\infty$. Then, there exist subsequences $\{\sigma_{k_m}\}$ and $\{(x^{k_m}, y^{k_m}, r^{k_m})\}$, numbers k_0, $m_0 \in \mathbb{N}$, and an index set $\emptyset \neq J_v \subseteq I$ such that

(*a*) *for all $k \geq k_0$, $x^k - \frac{\sigma_k}{2}v \geq 0$ and $0 < \langle d, x^k - \frac{\sigma_k}{2}v\rangle < \langle d, x^k\rangle$;*
(*b*) *for all $m \geq m_0$, $\frac{1}{\sigma_{k_m}}x^{k_m} \in \mathrm{ri}(\Delta_{J_v})$, thus $\mathrm{supp}\{x^{k_m}\} = J_v$ (hence $\mathrm{supp}\{v\} \subseteq J_v$);*
(*c*) *for all $m \geq m_0$, $z \in \Delta_{J_v}$: $\langle y^{k_m} + r^{k_m} + q^{k_m}, \sigma_{k_m}z - x^{k_m}\rangle = 0$. Moreover, for a given $c \in \mathcal{C}$,*
(*d*) *if*

$$\limsup_k {}^{\infty}A_{F^k,c}(v) < +\infty \quad \textit{and} \quad \limsup_k {}^{\infty}A_{G^k,c}(v) = 0,$$

then the subsequences $\{y^{k_m}\}$, σ_{k_m} may be chosen in such a way that there is a vector w such that $\frac{1}{c(\sigma_{k_m})}y^{k_m} \to w \in \limsup_k^{\infty,c} F^k(v)$, $\langle w, v\rangle \leq 0$, $\langle w, y\rangle \geq \langle d, y\rangle\langle w, v\rangle$ for all $y \geq 0$, and $\langle w, z\rangle = \langle w, v\rangle$ for all $z \in \Delta_{J_v}$;
(*e*) *if each F^k is c-Moré, $F^k \xrightarrow{g} F$ and $\limsup_k^{\infty} A_{G^k,c}(v) = 0$, then there exist a vector w, a sequence $\{w^{k_m}\}$ such that $w^{k_m} \in F^{k_m}(\frac{x^{k_m}}{\sigma_{k_m}})$, $w^{k_m} \to w \in F(v)$ and $\langle w, v\rangle \leq 0$.*

Proof. (a): As $\frac{1}{\sigma_k}x^k \to v$, for $\varepsilon = \min\{\frac{v_i}{2} : v_i > 0\} > 0$ there exists k_0 such that for all $k \geq k_0$, $\sum_{i=1}^n |\frac{x_i^k}{\sigma_k} - v_i| < \varepsilon$. This implies $\frac{v_i}{2} < \frac{x_i^k}{\sigma_k}$ for $i \in \mathrm{supp}\{v\}$. Thus $0 \neq x^k - \frac{\sigma_k}{2}v \geq 0$, and then ($a$) holds.

(b): Clearly $\Delta_d = \Delta_I = \mathrm{co}\{\frac{1}{d_i}e_i : i \in I\}$ and it may be written as the disjoint union of the relative interior of its extreme faces. More precisely, if we denote its extreme faces by $\Delta_{J_1}, \Delta_{J_2}, \ldots, \Delta_{J_{2^n-1}}$, then

$$\Delta_d = \bigcup_{i=1}^{2^n-1} \mathrm{ri}(\Delta_{J_i}).$$

As $\frac{1}{\sigma_k}x^k \in \Delta_d$, $k \in \mathbb{N}$, there exist an $i_0 \in \{1, 2, \dots, 2^n - 1\}$, m_0, and a subsequence $\{x^{k_m}\}$ such that $\frac{1}{\sigma_{k_m}}x^{k_m} \in \mathrm{ri}(\Delta_{J_{i_0}})$ for all $m \geq m_0$. By setting $J_v \doteq J_{i_0}$, one obtains $\mathrm{supp}\{x^{k_m}\} = J_v$ and $\mathrm{supp}\{v\} \subseteq J_v$.

(c): We analyze two cases, whether J_v is a singleton or not. In the first case, we have $\frac{1}{\sigma_{k_m}}x^{k_m} = v$ for all $m \geq m_0$ because of $\mathrm{ri}(\Delta_{J_v}) = \Delta_{J_v}$, which proves (c). In the second case, by (a) we have that for all $z \in \Delta_{J_v}$ and all $m \geq m_0$, there exists $\varepsilon_z > 0$ such that

$$\frac{1}{\sigma_{k_m}}x^{k_m} + t\left(z - \frac{1}{\sigma_{k_m}}x^{k_m}\right) \in \Delta_{J_v} \ \ \forall\ t,\ |t| < \varepsilon_z.$$

Because of the choice of x^{k_m}, we have

$$\left\langle y^{k_m} + r^{k_m} + q^{k_m}, \sigma_{k_m}\left(\frac{x^{k_m}}{\sigma_{k_m}} + t\left(z - \frac{x^{k_m}}{\sigma_{k_m}}\right)\right) - x^{k_m}\right\rangle \geq 0, \quad \forall\ |t| < \varepsilon_z.$$

Then

$$\langle y^{k_m} + r^{k_m} + q^{k_m}, t(\sigma_{k_m}z - x^{k_m})\rangle \geq 0, \quad \forall\ |t| < \varepsilon_z.$$

Hence

$$\langle y^{k_m} + r^{k_m} + q^{k_m}, \sigma_{k_m}z - x^{k_m}\rangle = 0, \quad \forall\ z \in \Delta_{J_v}.$$

(d): Since $\{F^k\}$ is c-asymptotically bounded, we may also assume that $\frac{y^{k_l}}{c(\sigma_{k_l})} \to w \in \limsup_k^{\infty,c} F^k(v)$. Moreover, taking into account the assumption $\limsup_k{}^{\infty} A_{G^k,c}(v) = 0$, and after dividing the inequality in (11) by $c(\sigma_{k_l})\sigma_{k_l}$ and letting $l \to +\infty$ for $x = 0$ and $x = \sigma_{k_l}\frac{y}{\langle y,d\rangle}$ with $0 \neq y \geq 0$ respectively, we obtain $\langle w, v\rangle \leq 0$ and $\langle w, y\rangle \geq \langle d, y\rangle\langle w, v\rangle$ for all $y \geq 0$. Dividing (c) by $c(\sigma_{k_l})\sigma_{k_l}$ and letting $l \to +\infty$ we obtain the last part of (d).

(e): By assumption, $\langle y^k, \frac{x^k}{\sigma_k}\rangle \geq c(\sigma_k)\langle w^k, \frac{x^k}{\sigma_k}\rangle$ for some $w^k \in F^k(\frac{x^k}{\sigma_k})$. By the property of the uniformity in graphical convergence of F^k (see [21, Exercise 5.34]), we may suppose up to subsequences that $w^k \to w$ and $w \in F(v)$. On dividing (11) (for $x = 0$) by $c(\sigma_k)$, we get

$$-\left\langle \frac{r^k + q^k}{c\,(\sigma_k)}, \frac{x^k}{\sigma_k}\right\rangle \geq \left\langle \frac{y^k}{c(\sigma_k)}, \frac{x^k}{\sigma_k}\right\rangle \geq \left\langle w^k, \frac{x^k}{\sigma_k}\right\rangle.$$

Taking the limit we obtain $\langle w, v\rangle \leq 0$. □

Remark 3.2. In the previous lemma we actually get $\langle v, d\rangle = 1$. Additionally, by choosing $y = e^i$, $i = 1, \dots, n$, in (d), and setting $z \doteq w - \langle w, v\rangle d \geq 0$, we obtain $\langle z, v\rangle = 0$. Therefore, if $\tau \doteq -\langle w, v\rangle \geq 0$, then

$$J_v \neq \emptyset,\ \ v \in \Delta_{J_v},\ \ v \neq 0, \quad v \in \mathcal{S}(\tau d, \limsup_k{}^{\infty,c} F^k). \tag{12}$$

The following result provides an important estimate for the cone $(\mathcal{S}(q, F + G))^\infty$ as a consequence of Lemma 3.1.

Proposition 3.3. *Let $d > 0$, $q \in \mathbb{R}^n_+$, $c \in \mathcal{C}$, and $F, G \in \mathcal{X}$ such that*

$$A^\infty_{F,c}(v) < +\infty \quad \textit{and} \quad A^\infty_{G,c}(v) = 0 \ \ \forall\ v \in \Delta_d.$$

Then,

$$(\mathcal{S}(q, F + G))^\infty \subseteq \mathcal{S}(0, F^\infty_c).$$

Proof. Let $v \in [\mathrm{S}(q, F+G)]^\infty$. Then, there exists $x^k \in \mathcal{S}(q, F+G)$ such that $\langle x^k, d\rangle \to +\infty$ and $\frac{x^k}{\langle x^k, d\rangle} \to v$. Moreover, there exist $y^k \in F(x^k)$ and $r^k \in G(x^k)$ such that $y^k + r^k + q \geq 0$ and $\langle y^k + r^k + q, x^k\rangle = 0$ for all k. Clearly, $\sigma_k \doteq \langle d, x^k\rangle \to +\infty$ and $\frac{x^k}{\sigma_k} \to v$ as $k \to +\infty$. Thus, Lemma 3.1 (for $F^k = F$, $G^k = G$, and $q^k = q$ for all k) implies the existence of $w \in F_c^\infty(v)$ and $\emptyset \neq J_v \subseteq I$, such that (d) of that lemma holds. Dividing $y^k + r^k + q \geq 0$ (resp. $\langle y^k + r^k + q, x^k\rangle = 0$) by $c(\sigma_k)$ (resp. $c(\sigma_k)\sigma_k$) and taking the limit we obtain $w \geq 0$, $\langle w, v\rangle = 0$, and $w_{J_v} = 0$. Hence $v \in \mathcal{S}(0, F_c^\infty)$. □

We next present various equivalent conditions for the unboundedness of the solution set.

Theorem 3.4 (General existence theorem). *Let $d > 0$ be a positive vector, $\{\sigma_k\}$ be an increasing sequence of positive numbers converging to $+\infty$ and $q \in \mathbb{R}^n$; let $F \in \mathcal{X}$, and $\{(x^k, y^k)\}$ be a sequence of solutions to*

$$\text{find } x^k \in D_k : \quad y^k \in F(x^k), \quad \langle y^k + q, x - x^k\rangle \geq 0 \quad \forall\, x \in D_k. \tag{13}$$

such that $\langle d, x^k\rangle = \sigma_k$ and $\frac{x^k}{\sigma_k} \to v$ as $k \to +\infty$. Then the following assertions are equivalent:

(a) *there exist m_0 and a subsequence $\{x^{k_m}\}$ such that $\langle y^{k_m} + q, v\rangle \geq 0$ for all $m \geq m_0$;*

(b) *there exist m_0 and a subsequence $\{x^{k_m}\}$ such that for all $m \geq m_0$ there is $u^{k_m} \geq 0$, $0 < \langle d, u^{k_m}\rangle < \langle d, x^{k_m}\rangle$ and $\langle y^{k_m} + q, u^{k_m} - x^{k_m}\rangle \leq 0$.*

(c) *there exist m_0 and a subsequence $\{x^{k_m}\}$ such that $x^{k_m} \in \mathcal{S}(q, F)$ for all $m \geq m_0$;*

(d) *there exist m_0 and a subsequence $\{x^{k_m}\}$ such that $\langle y^{k_m} + q, v\rangle = 0$ for all $m \geq m_0$.*

Proof. $(a) \Longrightarrow (b)$: By taking $u^{k_m} = x^{k_m} - \frac{\sigma_{k_m}}{2} v$, (a) of Lemma 3.1 (with $F^k = F$, $G^k = 0$, $q^k = q$ for all k) implies the desired result.

$(b) \Longrightarrow (c)$: suppose to the contrary, that there exist $m \geq m_0$ and $y_0 \in \mathbb{R}^n_+ \setminus D_{k_m}$ such that $\langle y^{k_m} + q, y_0 - x^{k_m}\rangle < 0$. As $0 < \langle d, u^{k_m}\rangle < \langle d, x^{k_m}\rangle$, there is $t \in\ ]0,1[$ such that $z_t \doteq t u^{k_m} + (1-t) y_0 \in D_{k_m}$. Thus $\langle y^{k_m} + q, z_t - x^{k_m}\rangle \geq 0$, and then $\langle y^{k_m} + q, y_0 - x^{k_m}\rangle \geq 0$, leading to a contradiction.

$(c) \Longrightarrow (d)$: if (c) holds then, for all $x \geq 0$ and $m \geq m_0$, we have $\langle y^{k_m} + q, x - x^{k_m}\rangle \geq 0$. By taking $x = x^{k_m} + v \geq 0$ (resp. $x = x^{k_m} - \frac{\sigma_{k_m}}{2} v \geq 0$ for k_m such that $k_m \geq k_0$, k_0 as in (a) of Lemma 3.1) we obtain $\langle y^{k_m} + q, v\rangle \geq 0$ (resp. $\langle y^{k_m} + q, v\rangle \leq 0$). Hence $\langle y^{k_m} + q, v\rangle = 0$. $(d) \Longrightarrow (a)$: It is straightforward. □

Example 3.5. This example is discussed in [7] in a different context. Let $d = (1,1)^\top$, $\sigma_k \geq 1$ for all k such that $\sigma_{k+1} > \sigma_k$ and $\sigma_k \to +\infty$. Consider $F(x_1, x_2) = [-x_1, x_1] \times [-x_2, x_2]$, $q = (0, -1)^\top$. Then, the sequence $\{(x^k, y^k)\}$ with $y^k = (0,1)$ and $x^k = (0, \sigma_k)$, satisfies the assumptions of Theorem 3.4 Obviously $\langle y^k + q, v\rangle = 0$, so (d) of such a theorem holds. Hence, $(0, \sigma_k) \in \mathcal{S}(q, F)$ for all k. Indeed, $\mathcal{S}(q, F) = \{(x_1, x_2) : \ x_1 \geq 0, x_2 \geq 1\}$.

4. Asymptotically well-behaved mappings and existence theorems

In the existence theory of multivalued complementarity problems several classes of multifunctions arise as natural extensions of the linear case. In what follows some of them are recalled. Let $F : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ be a multifunction with nonempty values. We say that F is a:

(i) *copositive* mapping if $\langle x, y\rangle \geq 0 \ \forall\ (x, y) \in \operatorname{gph} F$;
(ii) *strictly copositive* mapping if $\langle x, y\rangle > 0 \ \forall\ (x, y) \in \operatorname{gph} F$, $x \neq 0$;
(iii) (assuming $0 \in F(0)$) *semimonotone* mapping if $\mathcal{S}(p, F) = \{0\} \ \forall\ p > 0$;
(iv) (assuming $0 \in F(0)$ and given $p > 0$) $\mathbf{G}(p)$*-mapping* or shortly $F \in \mathbf{G}(p)$ if $\mathcal{S}(\tau p, F) = \{0\} \ \forall\ \tau > 0$. In the case when F is c-homogeneous with $c(t) = t^\gamma$, $\gamma > 0$, we get
$$F \in \mathbf{G}(p) \Longleftrightarrow \mathcal{S}(p, F) = \{0\},$$
since $v \in \mathcal{S}(\tau p, F) \Longleftrightarrow \tau^{-1/\gamma} v \in \mathcal{S}(p, F)$.

Motivated by the asymptotic analysis carefully carried out in Lemma 3.1, we introduce the following definitions, which are extensions and improvements of the ones discussed in [7].

We recall that (see the end of Section 1)
$$\mathcal{X} \doteq \{F : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n : \ F \text{ is cusco } \},$$
and (see Section 2)
$$A^\infty_{F,c}(v) \doteq \sup\left\{\limsup_{m\to+\infty} \frac{1}{c(\lambda_{k_m})}\Big(\sup_{y\in F(\lambda_{k_m} x^{k_m})} |y|\Big) : \ k_m \uparrow +\infty, \lambda_{k_m} \uparrow +\infty, \ x^{k_m} \to v \right\}.$$
$$F^\infty_c(v) \doteq \left\{ w \in \mathbb{R}^n : \ \lambda_{k_m} \uparrow +\infty, \ x^{k_m} \geq 0, \ \frac{x^{k_m}}{\lambda_{k_m}} \to v, \ y^{k_m} \in F(x^{k_m}), \ \frac{y^{k_m}}{c(\lambda_{k_m})} \to w \right\}.$$

Definition 4.1. Given $d >$ and $c \in \mathcal{C}$, we say that $F \in \mathcal{X}$ is:

(i) *asymptotically well-behaved* $\mathbf{T}$*-mapping* (shortly $F \in \mathbf{T}$) if $A^\infty_{F,c}(v) < +\infty \ \forall\ v \in \Delta_d$, and for any index set $\alpha \subseteq I$, one has
$$\left.\begin{array}{l} v \geq 0, \ w \geq 0, \ w \in F^\infty_c(v) \\ \alpha \neq \emptyset, \ v \in \Delta_\alpha, \ w_\alpha = 0 \end{array}\right\} \Longrightarrow v \in (F(\operatorname{pos}^+\Delta_\alpha))^*. \tag{14}$$

(ii) *asymptotically well-behaved* $\tilde{\mathbf{T}}$*-mapping* (shortly $F \in \tilde{\mathbf{T}}$) if $A^\infty_{F,c}(v) < +\infty \ \forall\ v \in \Delta_d$, and for any index set $\alpha \subseteq I$, one has
$$\left.\begin{array}{l} v \geq 0, \ w \geq 0, \ w \in F^\infty_c(v) \\ \alpha \neq \emptyset, \ v \in \Delta_\alpha, \ w_\alpha = 0 \end{array}\right\} \Longrightarrow \ \langle x, y\rangle \geq 0 \ \forall\ x \in \operatorname{pos}^+\Delta_\alpha, \ \forall\ y \in F(x). \tag{15}$$

Remark 4.2. (i) Since we usually will impose $c \in \mathcal{C}_0$ which implies the c^∞-homogeneity of F^∞_c (Theorem 2.5), we obtain the independence on d in the definition of $\mathbf{T}$-mapping. This is a consequence of the following two facts ($\alpha \neq \emptyset$):
$$\operatorname{pos}^+\Delta_\alpha(d) = \operatorname{pos}^+\Delta_\alpha(d') \ \forall\ d' > 0; \ v \in \Delta_\alpha(d) \Longleftrightarrow \langle d, v\rangle = 1 \text{ and } \operatorname{supp}\{v\} \subseteq \alpha.$$

(ii) It is clear that when $A^\infty_{F,c}(v) < +\infty \ \forall\ v \in \Delta_d$, and F is copositive, then F is asymptotically well-behaved $\tilde{\mathbf{T}}$-mapping and F^∞_c is also copositive; and like in the linear case, we should expect that it must be semimonotone as well. This is true if additionally $0 \in \mathcal{S}(p, F_c^\infty)$ for all $p \geq 0$, which holds if $F \in \mathcal{X}$ in view of Remark 2.4 since in this case we have $0 \in F_c^\infty(0)$. Hence, $F_c^\infty \in \mathbf{G}(d)\ \forall\ d > 0$.

(iii) If $A_{F,c}^\infty(v) < +\infty\ \forall\ v \in \Delta_d$ and $\mathcal{S}(0, F^\infty) = \{0\}$, then F is asymptotically well-behaved $\mathbf{T}$ and $\tilde{\mathbf{T}}$-mappings, since (14) and (15) hold vacuously.

Remark 4.3. When F is c-subhomogeneous the following notion of T-mapping was introduced in [7]. For any index set $\alpha \subseteq I$, one has

$$\left.\begin{array}{l} v \geq 0,\ w \geq 0,\ w \in F(v) \\ \alpha \neq \emptyset,\ v \in \Delta_\alpha,\ w_\alpha = 0 \end{array}\right\} \Longrightarrow v \in (F(\mathrm{pos}^+\Delta_\alpha))^*.$$

This class of mappings is strictly included in that of Definition 4.1 by Theorem 2.9 and (ii) of Remark 2.4.

Example 4.4. Take

$$M = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$$

and consider $F(x) = \ln(|x| + 1)Mx$, $c(t) = t\ln(t + 1)$. Then $c^\infty(\lambda) = \lambda$ and $F_c^\infty(v) = Mv$, $v \geq 0$. In this case $\mathcal{S}(0, F_c^\infty) = \{(x_1, 0) : x_1 \geq 0\}$. It is clear that F is asymptotically well-behaved $\mathbf{T}$-mapping. Notice that F is not copositive.

Notice that under assumptions of Theorem 4.5 below,

$$0 \in F_c^\infty(0) \quad \text{and} \quad \mathcal{S}(0, F_c^\infty) \text{ is a cone,}$$

and therefore $\mathcal{S}(0, F_c^\infty) \neq \emptyset$. If, in addition, F_c^∞ is sequentially bounded, then $\mathcal{S}(0, F_c^\infty)$ is also closed, and hence by [21, Exercise 6.22], we obtain

$$\mathrm{int}\ [\mathcal{S}(0, F_c^\infty)]^* = [\mathcal{S}(0, F_c^\infty)]^\#.$$

Set

$$\mathrm{fr}\,([\mathcal{S}(0, F_c^\infty)]^*) = [\mathcal{S}(0, F_c^\infty)]^* \setminus [\mathcal{S}(0, F_c^\infty)]^\#,$$

the boundary of $[\mathcal{S}(0, F_c^\infty)]^*$.

The next two theorems are extensions and improvements of some results in [7].

Theorem 4.5. *Let $d > 0$, $c \in \mathcal{C}_0$ and let $F \in \mathcal{X}$ be an asymptotically well-behaved $\mathbf{T}$-mapping such that $F_c^\infty \in \mathbf{G}(d)$.*

- (a) *If $q \in \mathrm{int}\ [\mathcal{S}(0, F_c^\infty)]^*$ then $\mathcal{S}(q, F + G)$ is nonempty and compact for all $G \in \mathcal{X}$ copositive and zero-subhomogeneous.*
- (b) *If $q \in \mathrm{fr}\,([\mathcal{S}(0, F_c^\infty)]^*)$ then $\mathcal{S}(q, F + G)$ is nonempty and compact for all $G \in \mathcal{X}$ strictly copositive and zero-subhomogeneous.*
- (c) *If $q \in \mathrm{fr}\,([\mathcal{S}(0, F_c^\infty)]^*)$ then $\mathcal{S}(q, F)$ is nonempty (possibly unbounded).*

Proof. Let σ_k be a increasing sequence of positive real numbers converging to $+\infty$. For fixed $q \in \mathbb{R}^n$ and every k, we consider the problem

$$\text{find } x^k \in D_k : \quad y^k \in F(x^k),\ r^k \in G(x^k),\ \langle y^k + r^k + q, x - x^k\rangle \geq 0\ \ \forall\ x \in D_k. \quad (16)$$

Because of the assumptions on F, G, we conclude that such a sequence $\{(x^k, y^k, r^k)\}$ does exists (see Lemma 4.1 in [22] for instance). We distinguish two cases: either $\sup_k\langle d, x^k\rangle < +\infty$ or $\sup_k\langle d, x^k\rangle = +\infty$. When $\sup_k\langle d, x^k\rangle < +\infty$ occurs, we may suppose that $x^k \to \bar{x}$, $r^k \to r \in G(\bar{x})$, and since $F \in \mathcal{X}$, we obtain $y^k \to y \in F(\bar{x})$. Take any $x \geq 0$ and let k_0 such that $\langle d, x\rangle < \sigma_{k_0}$ and $\langle d, \bar{x}\rangle < \sigma_{k_0}$. Then

$$\langle y^k + r^k + q, x - x^k\rangle \geq 0 \quad \forall\, k \geq k_0.$$

Hence $\langle y+r+q, x-\bar{x}\rangle \geq 0$, since $x \geq 0$ was arbitrary, we conclude that $\bar{x} \in \mathcal{S}(q, F+G)$. We now suppose that $\langle d, x^k\rangle \to +\infty$. By re-defining σ_k if necessary, we assume that $\langle d, x^k\rangle = \sigma_k$ for all k. Thus, we are in the situation of Lemma 3.1, and hence (a), (b) and (c) of the same lemma hold. In particular,

$$\langle y^{k_m} + r^{k_m} + q, v\rangle = \left\langle y^{k_m} + r^{k_m} + q, \frac{x^{k_m}}{\sigma_{k_m}}\right\rangle (\leq 0). \tag{17}$$

Moreover, the assumptions in (d) of Lemma 3.1 also hold. Thus $w \in F_c^\infty$, and by Remark 3.2, $v \in \mathcal{S}(-\langle w, v\rangle d, F_c^\infty)$. Since $F_c^\infty \in \mathbf{G}(d)$, $\langle w, v\rangle = 0$. By (14) $v \in (F(\mathrm{pos}^+\Delta_\alpha))^*$ for all m sufficiently large. On the other hand, we also have $r^{k_m} \in G(x^{k_m}) \subseteq G(\frac{x^{k_m}}{\sigma_{k_m}})$, and thus (up to a subsequence) $r^{k_m} \to r \in G(v)$. From (16) we get $\langle r^{k_m} + q, v\rangle \leq \langle y^{k_m} + r^{k_m} + q, v\rangle \leq 0$. By taking the limit, we obtain $\langle r + q, v\rangle \leq 0$, which in turn implies that $\langle q, v\rangle \leq 0$, reaching a contradiction if $q \in [\mathcal{S}(0, F_c^\infty)]^\#$ in case (a). In the case (b), we get $\langle r, v\rangle > 0$, which gives $\langle q, v\rangle < 0$, reaching a contradiction if $q \in [\mathcal{S}(0, F_c^\infty)]^*$. This proves, in both cases, that $\sup_k\langle d, x^k\rangle < +\infty$, and hence any limit point of $\{x^k\}$ belongs to $\mathcal{S}(q, F + G)$. The same reasoning also shows the boundedness of the solution set. In the case (c), we get

$$0 \leq \langle q, v\rangle \leq \langle y^{k_m} + q, v\rangle = \left\langle y^{k_m} + q, \frac{x^{k_m}}{\sigma_{k_m}}\right\rangle \leq 0.$$

Hence, $\langle y^{k_m} + q, v\rangle = 0$. We apply Theorem 3.4 to conclude that $\mathcal{S}(q, F) \neq \emptyset$. □

The next paragraph exhibits four instances showing that (a) in the previous theorem may be false, if either $F \notin \mathbf{T}$, or $q \notin [\mathcal{S}(0, F_c^\infty))]^\#$ or F_c^∞ is not a $\mathbf{G}$-mapping. The fourth example shows that the strict copositivity of G or the condition $q \in [\mathcal{S}(0, F_c^\infty)]^\#$ cannot be avoided to obtain the boundedness of the solution set.

Example 4.6 ([7]). (i) Let

$$M = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad q = (1, -1, 1)^\top, \quad \text{and}$$

$$F(x) = F_c^\infty(x) = Mx, \quad \text{for } c(t) = t.$$

It is clear that $F \in \mathbf{G}(d)\ \forall\, d > 0$ and $F \notin \mathbf{T}$. Further, one obtains

$$\mathcal{S}(0, F_c^\infty) = \left\{(x_1, 0, x_3)^\top : x_1 \geq 0, x_3 \geq 0, x_1x_3 = 0\right\}.$$

Thus, $q \in [\mathcal{S}(0, F_c^\infty)]^\#$ but $\mathcal{S}(q, F) = \emptyset$.

(ii) We now consider

$$M = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}, \qquad q = (-1,1)^\top.$$

If $F(x) = Mx$, then, for $c(t) = t$, one obtains $F = F_c^\infty \in \mathbf{T} \cap \mathbf{G}(d)\ \forall\ d > 0$. Moreover, $\mathcal{S}(0,F) = \{(x_1,0)^\top :\ x_1 \geq 0\}$, $q \notin [\mathcal{S}(0,F_c^\infty)]^\#$ and $\mathcal{S}(q,F) = \emptyset$.

(iii) Take $F(x) = F_c^\infty(x) = Mx$ for $c(t) = t$ and

$$M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Clearly F is not in $\mathbf{G}(d)$ for any $d > 0$, but $F \in \mathbf{T}$. If $q = (-1,0)^\top$, $\mathcal{S}(0,F_c^\infty) = \{(0,0)^\top\}$ and $\mathcal{S}(q,F) = \emptyset$.

(iv) This example shows, on one hand, the necessity of the strict copositivity of G in (b) of the previous theorem, and on the other hand, the necessity of $q \in [\mathcal{S}(0,F_c^\infty)]^\#$ in (a), to get the boundedness of the solution set. Consider the same mapping as in (ii) with $q = (0,\lambda)^\top$, $\lambda \in \mathbb{R}$, and $G \equiv 0$. Then

$$\mathcal{S}(q,F) = \{(x_1,0) :\ x_1 \geq \max\{0,-\lambda)\},$$

which is unbounded. Observe that $q \in [\mathcal{S}(0,F_c^\infty)]^* \setminus [\mathcal{S}(0,F_c^\infty)]^\#$.

The next theorem generalizes and extends Theorem 2 (and so also Corollary 2) of [10]. Indeed, in that paper is assumed $c(t) = t^\gamma > 0$, $\gamma > 0$, $F^k = F$, $A^\infty_{F,c}(v) < +\infty\ \forall\ v \geq 0$, $v \neq 0$, and F is copositive. Thus, $F \in \tilde{\mathbf{T}}$. Moreover, by (ii) of Remark 4.2, F_c^∞ is copositive as well and therefore it is in $\mathbf{G}(d)\ \forall\ d > 0$. Example 4.8 below exhibits an instance in which Theorem 4.7 is applicable but not Theorem 2 in [10].

Theorem 4.7. *Let $d > 0$, $c \in \mathcal{C}_0$ and let $F \in \mathcal{X}$ be an asymptotically well-behaved $\tilde{\mathbf{T}}$-mapping such that $F_c^\infty \in \mathbf{G}(d)$. Then $\mathcal{S}(q,F+G)$ is nonempty and compact for all $q \in [\mathcal{S}(0,F_c^\infty)]^\#$ and all $G \in \mathcal{X}$ copositive such that $A^\infty_{G,c}(v) = 0\ \forall\ v \in \Delta_d$.*

Proof. We proceed as in the previous theorem to infer, in particular, that $\langle y^{k_m} + r^{k_m} + q, \frac{x^{k_m}}{\sigma_{k_m}}\rangle \leq 0$. As before $\langle w,v\rangle = 0$, and by (15), $\langle x,y\rangle \geq 0\ \ \forall\ x \in \mathrm{pos}^+\Delta_\alpha,\ \forall\ y \in F(x)$. This along with the copositivity of G yield

$$\left\langle q, \frac{x^{k_m}}{\sigma_{k_m}} \right\rangle \leq \left\langle r^{k_m} + q, \frac{x^{k_m}}{\sigma_{k_m}} \right\rangle \leq \left\langle y^{k_m} + r^{k_m} + q, \frac{x^{k_m}}{\sigma_{k_m}} \right\rangle \leq 0. \tag{18}$$

It follows that $\langle q,v\rangle \leq 0$, a contradiction if $q \in [\mathcal{S}(0,F_c^\infty)]^\#$, proving that $\mathcal{S}(q,F+G) \neq \emptyset$. This reasoning also shows the boundedness of such a set. $\square$

Example 4.8. Let M be the matrix of Example 4.4 and consider $F(x) = \ln(|x|+1)Mx$. Then, $F \in \mathcal{X} \cap \tilde{\mathbf{T}}$ and F is not copositive, and by taking $c(t) = t\ln(t+1)$, we obtain $F_c^\infty(v) = Mv$, $v \geq 0$, with $F_c^\infty \in \mathbf{G}(d)\ \forall\ d > 0$. Thus, given $d > 0$, $\mathcal{S}(q,F+G)$ is nonempty and compact for all $q \in [\mathcal{S}(0,F_c^\infty)]^\#$ and all $G \in \mathcal{X}$ copositive such that $A^\infty_{G,c}(v) = 0\ \forall\ v \in \Delta_d$. Notice that a comparison (at infinity) with any function of the form t^γ, $\gamma > 0$, provides no information.

We now show that one cannot avoid the copositivity of G in Theorem 4.7 Let $|\cdot|$ be the Euclidean norm and consider [10]

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad q = (2, -2)^\top, \quad \text{and}$$

$$F(x) = |x| M x, \qquad G(x) = -2\sqrt{2}x.$$

Clearly, by taking $c(t) = t^2$, $A^\infty_{G,c}(v) = 0 \ \forall\ v \geq 0$, $F = F^\infty_c$ is copositive. Further $\mathcal{S}(0, F^\infty_c) = \mathbb{R}_+ \times \{0\}$, so that $[\mathcal{S}(0, F^\infty_c)]^*$ is the closed right half-plane and $q \in [\mathcal{S}(0, F^\infty_c)]^\#$. In [10] it is proved that $\mathcal{S}(q, F + G) = \emptyset$.

In what follows, we set

$$U_0(F) \doteq \{v \geq 0 :\ w \in F(v),\ \langle w, v\rangle = 0\}.$$

Theorem 4.9. *Let $d > 0$ be a positive vector and $c \in \mathcal{C}$; let $F \in \mathcal{X}$ be c-Moré.*

(a) *If F is strictly copositive, then $\mathcal{S}(q, F + G)$ is nonempty and compact for all $q \in \mathbb{R}^n$ and all $G \in \mathcal{X}$ satisfying $A^\infty_{G,c}(v) = 0 \ \forall\ v \in \Delta_d$;*

(b) *If F is copositive but not strictly copositive then $\mathcal{S}(q, F + G)$ is nonempty and compact for all $q \in [U_0(F)]^\#$ and all $G \in \mathcal{X}$ copositive satisfying $A^\infty_{G,c}(v) = 0$ $\forall\ v \in \Delta_d$.*

Proof. We proceed as in Theorem 4.5, but in this case we use (e) of Lemma 3.1 to prove (a). In the case (b), we obtain $\langle w, v\rangle = 0$, and from 17 along with the copositivity of F and G, we get

$$\left\langle q, \frac{x^{k_m}}{\sigma_{k_m}} \right\rangle \leq \left\langle y^{k_m} + r^{k_m} + q, \frac{x^{k_m}}{\sigma_{k_m}} \right\rangle \leq 0.$$

It follows that $\langle q, v\rangle \leq 0$, which is impossible if $q \in [U_0(F)]^\#$ □

Part (a) is a generalization of a result of [23] when it is specified to $\mathbb{R}^n_+$ and where the single-valued case and $G \equiv 0$ is only considered; see also [13].

The next theorem is an immediate consequence of Lemma 3.1

Theorem 4.10. *Let $d > 0$ be a positive vector and $c \in \mathcal{C}$ such that $c^\infty(\lambda) > 0 \ \ \forall\ \lambda > 0$; let $F \in \mathcal{X}$. If $A^\infty_{F,c}(v) < +\infty \ \forall\ v \in \Delta_d$ and F^∞_c is strictly copositive, then $\mathcal{S}(q, F+G)$ is nonempty and compact for all $q \in \mathbb{R}^n$ and all $G \in \mathcal{X}$ satisfying $A^\infty_{G,c}(v) = 0 \ \forall\ v \in \Delta_d$.*

5. Asymptotically regular mappings and a robustness property

Let $d > 0$, $c \in \mathcal{C}$ and $F \in \mathcal{X}$ such that $A^\infty_{F,c}(v) < +\infty \ \forall\ v \in \Delta_d$. The system

$$v \geq 0, \langle d, v\rangle = 1, w \in F^\infty_c(v), \langle w, v\rangle \leq 0, w - \langle w, v\rangle d \geq 0, \tag{19}$$

found in Lemma 3.1 (for $G^k = G$, and $q^k = q$ for all k), plays a fundamental role in characterizing the nonemptiness and boundedness of $\mathcal{S}(q, F+G)$ for all $q \in \mathbb{R}^n$. When F^∞_c is c^∞-subhomogeneous (this is true whenever $c \in \mathcal{C}_0$) the inconsistency of (19) is equivalent to the inconsistency of the following system

$$0 \neq v \geq 0, z \in F^\infty_c(v),\ \tau \geq 0, z + \tau d \geq 0, \langle z + \tau d, v\rangle = 0. \tag{20}$$

When $F(x) = Mx$ with M being a real matrix and d to be the vector of ones the previous system was introduced in [14], giving rise to *regular matrices* which proved to be very useful in the development of some algorithms in linear complementarity problems as shown in [8]. Such a system was further developed in [15, 19, 20] for functions that are positively homogeneous of some degree. Afterwards, the set-valued version was introduced in [10] for $c(t) = t^\gamma$, $\gamma > 0$. The following definition is a natural extension of regularity originally introduced for matrices. Notice that when $F \in \mathcal{X}$, $0 \in F_c^\infty(0)$, and so $0 \in \mathcal{S}(p, F_c^\infty)$ for all $p \geq 0$.

Definition 5.1. Given $d >$ and $c \in \mathcal{C}_0$, we say that $F \in \mathcal{X}$ is *asymptotically* (*regular*) $\mathbf{R}(d)$*-mapping* if $A^\infty_{F,c}(v) < +\infty \ \forall\, v \in \Delta_d$, and

$$\mathcal{S}(\tau d, F_c^\infty) = \{0\} \quad \forall\, \tau \geq 0.$$

The next theorem unifies and generalizes Corollary 2 and Theorem 3 of [10], and therefore also generalizes Theorem 3.1 in [14] and Theorem 3.1 in [19]. In addition, it also extends and improves Theorem 5.12 in [7], showing the existence of some kind of robustness property with respect to certain classes of perturbations, and provides characterizations of our new notion of regular mappings related to the membership of $\mathbf{G}(d)$ as expected.

Theorem 5.2. *Let $d > 0$, $c \in \mathcal{C}_0$ and $F \in \mathcal{X}$ such that $A^\infty_{F,c}(v) < +\infty \ \forall\, v \in \Delta_d$. Consider the statements*

- (a) *the system* (19) *is inconsistent;*
- (b) $F_c^\infty \in \mathbf{G}(d)$ *and $\mathcal{S}(q, F + G)$ is nonempty and compact for all $q \in \mathbb{R}^n$ and all $G \in \mathcal{X}$ copositive satisfying $A^\infty_{G,c}(v) = 0 \ \ \forall\, v \in \Delta_d$;*
- (c) $F_c^\infty \in \mathbf{G}(d)$ *and $\mathcal{S}(q, F + G)$ is nonempty and compact for all $q \in \mathbb{R}^n$ and all $G \in \mathcal{X}$ copositive and uniformly bounded;*
- (d) $F_c^\infty \in \mathbf{G}(d)$ *and $\mathcal{S}(q, F + G)$ is nonempty and compact for all $q \in \mathbb{R}^n$ and all $G \in \mathcal{X}$ copositive and zero-subhomogeneous;*
- (e) $F_c^\infty \in \mathbf{G}(d)$ *and $\mathcal{S}(q, F)$ is nonempty and compact for all $q \in \mathbb{R}^n$;*
- (f) *F is asymptotically $\mathbf{R}(d)$-mapping.*

The following implications hold:

$$(f) \iff (a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d) \Longrightarrow (e).$$

Moreover, if each F is c-subhomogeneous, then all the statements are equivalent.

Proof. (a) $\Rightarrow$ (b): We first prove that $F_c^\infty \in \mathbf{G}(d)$. Let $\tau > 0$ and $x \in \mathcal{S}(\tau d, F_c^\infty)$. Then there is $y \in F_c^\infty(x)$ such that $y + \tau d \geq 0$ and $\langle y + \tau d, x\rangle = 0$. If $\langle y, x\rangle = 0$ then $\langle d, x\rangle = 0$, which implies $x = 0$. If $\langle y, x\rangle < 0$ then for $v = x/\langle d, x\rangle$ we get $w \doteq y/c^\infty(\langle d, x\rangle) \in F_c^\infty(v)$ and since $\tau\langle d, x\rangle = -\langle y, x\rangle$, clearly (19) holds, a contradiction. The previous reasoning also shows that $\mathcal{S}(0, F_c^\infty) = \{0\}$, and thus $F \in \tilde{\mathbf{T}}$ by (iii) of Remark 4.2. By Theorem 4.7 we conclude that $\mathcal{S}(q, F + G)$ is nonempty and compact for all $q \in \mathbb{R}^n$ and all $G \in \mathcal{X}$ copositive satisfying $A^\infty_{G,c}(v) = 0 \ \forall\, v \in \Delta_d$. ($b$) $\Longrightarrow$ (c), (c) $\Longrightarrow$ (d) and (d) $\Longrightarrow$ (e) are obvious. The equivalence (a) $\iff$ (f) is a consequence of the equivalence between (19) and (20).

$(e) \Longrightarrow (f)$: Assume there exists $0 \neq v \in \mathcal{S}(0, F_c^\infty)$. Then, by c^∞-homogeneity of F_c^∞ (see (b) of Theorem 2.5, $tv \in \mathcal{S}(0, F_c^\infty)$ for all $t > 0$. From Theorem 2.9 it follows that

$$F_c^\infty(tv) \subseteq F(tv).$$

Hence, $tv \in \mathcal{S}(0, F)$ for all $t > 0$, contradicting the boundedness of $\mathcal{S}(0, F)$. □

By re-writing the previous theorem, one obtains the next corollary.

Corollary 5.3. *Let $d > 0$, $c \in \mathcal{C}_0$ and $F \in \mathcal{X}$ be c-subhomogeneous such that $A_{F,c}^\infty(v) < +\infty \ \forall \ v \in \Delta_d$. Assume in addition that $F_c^\infty \in \mathbf{G}(d)$. The following assertions are equivalent:*

- (a) *$\mathcal{S}(q, F)$ is nonempty and compact for all $q \in \mathbb{R}^n$;*
- (b) *$\mathcal{S}(q, F + G)$ is nonempty and compact for all $q \in \mathbb{R}^n$ and all $G \in \mathcal{X}$ copositive satisfying $A_{G,c}^\infty(v) = 0 \ \ \forall \ v \in \Delta_d$;*
- (c) *$\mathcal{S}(q, F + G)$ is nonempty and compact for all $q \in \mathbb{R}^n$ and all $G \in \mathcal{X}$ copositive and uniformly bounded;*
- (d) *$\mathcal{S}(q, F + G)$ is nonempty and compact for all $q \in \mathbb{R}^n$ and all $G \in \mathcal{X}$ copositive and zero-subhomogeneous;*
- (e) *$\mathcal{S}(0, F_c^\infty) = \{0\}$.*

By taking into account Remark 2.12 the preceding two results may admit some variants.

6. Some sensitivity results

Now, our interest is establishing some sensitivity results. For that purpose we need to measure the difference between two c^∞-homogeneous multifunctions, i.e., multifunctions $H : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ satisfying

$$H(tx) = c^\infty(t)H(x) \ \ \forall \ t > 0, \ \forall \ x \geq 0,$$

with c^∞ satisfying (2) and (3). By $\mathcal{H}$, we denote the family of multifunctions with nonempty values having this property. For $H \in \mathcal{H}$ and $d > 0$, let us consider the *outer norm* ([21], Chapter 9, Section D])

$$|H|_d^+ \doteq \sup\{|y| : \ x \in \Delta_d, \ y \in H(x)\}.$$

Set

$$\mathcal{H}_0 \doteq \{H \in \mathcal{H} : \ H \text{ is locally bounded on } \Delta_d, \ H(0) = \{0\}\}.$$

Then, $|H|_d^+ < +\infty$ for all $H \in \mathcal{H}_0$. On $\mathcal{H}_0$,

$$|H_1 - H_2|_d^+ \doteq \sup\{|y_1 - y_2| : \ y_1 \in H_1(x), \ y_2 \in H_2(x), \ x \in \Delta_d\}$$

becomes a metric due to (4).

Furthermore, given $c \in \mathcal{C}_0$, set

$$\mathcal{X}_0 \doteq \left\{F \in \mathcal{X} : \ F_c^\infty(0) = \{0\}, \ A_{F,c}^\infty(v) < +\infty, \ \forall \ v \in \Delta_d\right\}.$$

By virtue of Proposition 2.3 and Theorem 2.5, if $F \in \mathcal{X}_0$ then $F_c^\infty \in \mathcal{H}_0$.

The following proposition shows the importance of the metric associated to the outer norm for the first time.

Proposition 6.1. *Let $d > 0$, $c \in \mathcal{C}_0$, $q^0 \in \mathbb{R}^n$, and $F^0 \in \mathcal{X}_0$. If $q^0 \in (\mathcal{S}(0,(F^0)^\infty_c))^\#$, then there exists $\varepsilon > 0$ such that for all $q \in \mathbb{R}^n$ and all $F \in \mathcal{X}_0$ satisfying*

$$|q - q^0| + |F^\infty_c - (F^0)^\infty_c|^+_d < \varepsilon,$$

one has $q \in (\mathcal{S}(0, F^\infty_c))^\#$.

Proof. Suppose on the contrary, that there exist sequences $q^k \in \mathbb{R}^n$, $F^k \in \mathcal{X}_0$, $v^k \geq 0$ satisfying $q^k \to q^0$, $|(F^k)^\infty_c - (F^0)^\infty_c|^+_d \to 0$, $0 \neq v^k \in \mathcal{S}(0,(F^k)^\infty_c)$, and $\langle q^k, v^k\rangle \leq 0$. Since $(F^k)^\infty_c$ is c^∞-homogeneous, we may assume that $|v^k| = 1$. Thus, up to subsequences, $v^k \to v \neq 0$ and there exits $w^k \in (F^k)^\infty_c(v^k)$, $w^k \geq 0$, $\langle w^k, v^k\rangle = 0$. Take $u^k \in (F^0)^\infty_c(v^k)$. Since $(F^0)^\infty_c$ is locally bounded (see Proposition 2.3), we may also assume that $u^k \to u$ with $u \in (F^0)^\infty_c(v)$ because of the closedness of the graph of $(F^0)^\infty_c$. Thus,

$$|w^k - u^k| \leq |(F^k)^\infty_c - (F^0)^\infty_c|^+_d \to 0 \ \text{ as } k \to +\infty.$$

Therefore, $w^k \to u$. Then, $u \geq 0$ and $\langle u, v\rangle = 0$. Hence $0 \neq v \in \mathcal{S}(0,(F^0)^\infty_c)$ and $\langle q^0, v\rangle \leq 0$, a contradiction. □

Theorem 6.2. *Let $d > 0$, $c \in \mathcal{C}_0$, $q^0 \in \mathbb{R}^n$, and $F^0 \in \mathcal{X}_0$. If $q^0 \in (\mathcal{S}(0,(F^0)^\infty_c))^\#$, then there exists $\varepsilon > 0$ such that for all $q \in \mathbb{R}^n$ and all $F \in \mathcal{X}_0$, $F \in \mathbf{T} \cup \tilde{\mathbf{T}}$, $F^\infty_c \in \mathbf{G}(d)$ satisfying*

$$|q - q^0| + |F^\infty_c - (F^0)^\infty_c|^+_d < \varepsilon,$$

one has $\mathcal{S}(q, F)$ is non-empty and compact.

Proof. It is a consequence of Theorems 4.5 and 4.7 together with Proposition 6.1 □

Conclusion.

In this paper the basic notion of asymptotically bounded (Section 2, asymptotically well-behaved (Definition 4.1 and asymptotically regular (Definition 5.1 mappings are introduced, and several related properties are established as well. These notions are defined for a sequence of multifunctions instead for a single one, and compare their asymptotic behavior at infinity with respect to a class of re-scaling functions that strictly contains those of the form t^α, $\alpha > 0$. The existence theorems extend, generalize and unify all the existence results proved by Gowda and Pang in [10] which are valid only in the context of copositivity or regularity. Some of the results in [7] are also extended. Some sensitivity results based on a metric (outer norm according to [21]) defined for positive homogeneous-type multifunctions, are also established. Finally, we point out that our results provide new ones even in the linear case.

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